LECTURE 13

Second Order Linear Equations with Constant Coefficients

We shall now begin to investigate how to actually solve linear ODE's of degree 2. We shall begin with differential equations of a particularly simple type; equations of the form

$$y'' + py' + qy = 0 \tag{1}$$

1where p and q are constant.

A clue as to how one might construct a solution to (1) comes from the observation that (1) implies that y'', y' and y are related to one another by multiplicative constants. There is one class of functions for which is certainly true: the exponential functions; i.e., functions of the form

$$y(x) = e^{\lambda x} \quad . \tag{2}$$

We will therefore look for solutions of (1) having the form (2).

Plugging (2) into (1) yields

$$0 = \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + qe^{\lambda x} = \left(\lambda^2 + p\lambda + q\right) e^{\lambda x} \quad . \tag{13.1}$$

Since the exponential function $e^{\lambda x}$ never vanishes we must have

$$\lambda^2 + p\lambda + q = 0 \quad . \tag{3}$$

Equation (3) is called the **characteristic equation** for (1) since for any λ satisfying (3) we will have a solution $y(x) = e^{\lambda x}$ of (1).

Now because (3) is a quadratic equation we can employ the Quadratic Formula to find all of its roots:

$$\lambda^2 + p\lambda + q = 0 \qquad \Rightarrow \qquad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \quad . \tag{4}$$

Note that a root λ of (3) need not be a real number. Indeed, if $p^2 - 4q < 0$, then in order to compute λ via the Quadratic Formula we have to take the square root of a negative number and that forces us into the realm of complex numbers. We shall postpone temporarily the case when a root λ is complex and first discuss the case when the roots of (3) are all real. This requires $p^2 - 4q \geq 0$.

If $p^2 - 4q$ is positive, $\sqrt{p^2 - 4q}$ is a positive real number and

$$\begin{aligned} \lambda_+ &= \tfrac{-p+\sqrt{p^2-4q}}{2} \\ \lambda_- &= \tfrac{-p-\sqrt{p^2-4q}}{2} \end{aligned} \tag{5}$$

are distinct real roots of (3). Thus,

$$\begin{aligned} y_1(x) &= e^{\lambda_+ x} \\ y_2(x) &= e^{\lambda_- x} \end{aligned}$$

will both be solutions of (1). If we multiply these solutions py arbitrary constants c_1 and c_2, the resulting functions will still be solutions of (1). In fact, we can take arbitrary linear combinations of $y_1(x)$ and $y_2(x)$; sy,

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

and the result will still be a solution of (1).

For suppose $y_1(x)$ and $y_2(x)$ are the two solutions of (1) given above. Then if we set

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

we have

$$\begin{aligned} y'' + py' + qy &= a\frac{d^2}{dx^2}(c_1y_1 + c_2y_2) + b\frac{d}{dx}(c_1y_1 + c_2y_2) \\ &\quad +c(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0 \end{aligned}$$

Example

$$y'' + 3y' + 2y = 0 \quad .$$

Setting

$$y(x) = e^{\lambda x}$$

and plugging into the differential equation we get

$$\begin{aligned} 0 &= \lambda^2 e^{\lambda x} + 3\lambda e^{\lambda x} + 2e^{\lambda x} \\ &= e^{\lambda x}\left(\lambda^2 + 3\lambda + 2\right) \\ &= e^{\lambda x}(\lambda + 1)(\lambda + 2) \end{aligned}$$

Since $e^{\lambda x}$ never vanishes (for any finite x), we must have

$$\lambda = -1 \quad \text{or} \quad \lambda = -2 \quad .$$

We thus find two distinct solutions

$$\begin{aligned} y_1(x) &= e^{-x} \\ y_2(x) &= e^{-2x} \quad . \end{aligned}$$

The general solution is thus

$$y(x) = c_1e^{-x} + c_2e^{-2x} \quad .$$

0.1. Constant Coefficients: Particular Subcases. We have just seen that one can construct solutions of the differential equation of the form

$$y'' + py' + qy = 0 \quad , \quad p, q \text{ constants} \tag{6}$$

by looking for solutions of the form

$$y(x) = e^{\lambda x} \quad . \tag{7}$$

Let us review that construction. Plugging (7) into (6) yields

$$0 = \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + qe^{\lambda x} = \left(\lambda^2 + p\lambda + q\right)e^{\lambda x} \quad .$$

Since the exponential function $e^{\lambda x}$ never vanishes we must have

$$\lambda^2 + p\lambda + q = 0 \quad . \tag{8}$$

Equation (8) is called the **characteristic equation** for (6) since for any λ satisfying (8) we will have a solution $y(x) = e^{\lambda x}$ of (6).

Now because (4) is a quadratic equation we can employ the Quadratic Formula to find all of its roots:

$$\lambda^2 + p\lambda + q = 0 \qquad \Rightarrow \qquad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \quad .$$

Note that a root λ of (4) need not be a real number. Indeed, if $p^2 - 4q < 0$, then in order to compute λ via the Quadratic Formula we have to take the square root of a negative number and that forces us into the realm of complex numbers. We shall postpone temporarily the case when a root λ is complex and first discuss the case when the roots of (4) are all real. This requires $p^2 - 4q \geq 0$.

Case (i): $p^2 - 4q > 0$

Because $p^2 - 4q$ is positive, $\sqrt{p^2 - 4q}$ is a positive real number and

$$\begin{aligned} \lambda_+ &= \frac{-p+\sqrt{p^2-4q}}{2} \\ \lambda_- &= \frac{-p-\sqrt{p^2-4q}}{2} \end{aligned} \tag{9}$$

are distinct real roots of (4). Thus,

$$\begin{aligned} y_1 &= e^{\lambda_+ x} \\ y_2 &= e^{\lambda_- x} \end{aligned}$$

will both be solutions of (2). Noting that

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_1' y_2 \\ &= \lambda_- e^{\lambda_+ x} e^{\lambda_- x} - \lambda_+ e^{\lambda_+ x} e^{\lambda_- x} \\ &= (\lambda_- - \lambda_+)\, e^{(\lambda_+ + \lambda_-)x} \\ &= \frac{\sqrt{p^2-4q}}{a} e^{-\frac{b}{a}x} \end{aligned}$$

is non-zero, we conclude that if $p^2 - 4q \neq 0$, then the roots (9) furnish two linearly independent solutions of (6) and so the general solution is given by

$$y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x} \quad .$$

Case (ii): $p^2 - 4q = 0$

If $p^2 - 4q = 0$, however, this construction only gives us one distinct solution; because in this case $\lambda_+ = \lambda_-$. To find a second fundamental solution we must use the method of Reduction of Order.

So suppose $y_1(x) = e^{-\frac{p}{2}x}$ is the solution corresponding to the root

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{-p \pm 0}{2} = \frac{-p}{2}$$

of

$$\lambda^2 + p\lambda - q = 0 \ \ , \ \ p^2 - 4q = 0.$$

Then the Reduction of Order formula gives us a second linearly independent solution

$$y_2(x) = y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp\left[\int^s -p(t)dt\right] ds$$

gives us a second linearly independent solution. Plugging in $y_1(x) = e^{-\frac{p}{2}x}$ and $p(t) = p$, yields

$$\begin{aligned} y_2(x) &= e^{-\frac{p}{2}x} \int^x \frac{1}{\left(e^{-\frac{p}{2}s}\right)^2} \exp\left[\int^s -pdt\right] ds \\ &= e^{-\frac{p}{2}x} \int^x \frac{1}{e^{-ps}} \exp\left[-ps\right] ds \\ &= e^{-\frac{p}{2}x} \int^x e^{ps} e^{-ps} ds \\ &= e^{-\frac{p}{2}x} \int^x ds \\ &= x e^{-\frac{p}{2}x} \\ &= x y_1(x) \end{aligned}$$

In summary, for the case when $p^2 - 4q = 0$, we only have one root of the characterisitic equation, and so we get only one distinct solution $y_1(x)$ of the original differential equation by solving the characteristic equation for λ. To get a second linearly solution we must use the Reduction of Order formula; however, the result will always be the same: **the second linearly independent solution will always be x times the solution** $y_1(x) = e^{-\frac{p}{2}x}$. Thus, the general solution in this case will be

$$y(x) = c_1 e^{-\frac{p}{2}x} + c_2 x e^{-\frac{p}{2}x} \quad , \quad \text{if } p^2 - 4q = 0.$$

We now turn to the third and last possibility.

Case (iii): $p^2 - 4q < 0$

In this case

$$\sqrt{p^2 - 4q}$$

will be undefined unless we introduce complex numbers. But when we set

$$\sqrt{-1} = i$$

we have

$$\sqrt{p^2 - 4q} = \sqrt{(-1)(4q - p^2)} = \sqrt{-1}\sqrt{4q - p^2} = i\sqrt{4q - p^2} \quad .$$

The square root on the right hand side is well-defined since $4q - p^2$ is a positive number. Thus,

$$\lambda_\pm = \frac{-p \pm i\sqrt{4q - p^2}}{2} = \alpha \pm i\beta$$

where

$$\alpha = -\frac{b}{2} \quad , \quad \beta = \frac{\sqrt{4q - p^2}}{2} \quad ,$$

will be a complex solution of (8) and

$$y(x) = c_1 e^{\alpha x + i\beta x} + c_2 e^{\alpha x - i\beta x}$$

would be a solution of (6) if we could make sense out the notion of an exponential function with a complex argument.

Thus, we must address the problem of ascribing some meaning to

$$e^{\alpha x + i\beta x}$$

as a function of x. To ascribe some sense to this expression we considered the Taylor series expansion of e^x

(10)
$$\begin{aligned} e^x &= 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \\ &= \textstyle\sum_{i=0}^{\infty} \tfrac{1}{i!}x^i \end{aligned}$$

Now although we do not yet understand what $e^{\alpha x + i\beta x}$ means, we can nevertheless substitute $\alpha x + i\beta$ for x on the right hand side of (6), and get a well defined series with values in the complex numbers. One can show that this series converges for all α, β and x. We thus take

(11)
$$e^{\alpha x + i\beta} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i!}(\alpha x + i\beta x)^i$$

which agrees with (6) when $\beta = 0$.

One can also show that

$$e^{\alpha x + i\beta x} = e^{\alpha x} e^{i\beta x}.$$

Thus, when $p^2 - 4q = 0$, we have two complex valued solutions to (6)

$$y_1(x) = e^{\alpha x} e^{i\beta x} \quad \text{and} \quad y_2(x) = e^{\alpha x} e^{-i\beta x} \quad ,$$

where

$$\alpha = \frac{-p}{2} \quad , \quad \beta = \frac{\sqrt{4q - p^2}}{2} \quad .$$

A general solution of (6) would then be

$$y(x) = c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{i\beta x}.$$

However, this is rarely the form in which one wants a solution of (6). One would prefer solutions that are **real-valued functions of** x rather that complex-valued functions of x. But these can be had as well, since if $z = x + iy$ is a complex number, then

$$\begin{aligned} Re(z) &= \tfrac{1}{2}(z + \bar{z}) = x \\ Im(z) &= \tfrac{1}{2i}(z - \bar{z}) = y \end{aligned}$$

are both real numbers. Applying the Superposition Principle, it is easy to see that if

$$y(x) = e^{\alpha x} e^{i\beta x}$$

and

$$\bar{y}(x) = e^{\alpha x} e^{-i\beta x}$$

are two complex-valued solutions of (6), then

$$y_r(x) = \frac{1}{2}(y(x) + \bar{y}(x)) = e^{\alpha x}\left(\frac{e^{i\beta x} + e^{-i\beta x}}{2}\right)$$

and

$$y_i(x) = \frac{1}{2i}(y(x) - \bar{y}(x)) = e^{\alpha x}\left(\frac{e^{i\beta x} - e^{-i\beta x}}{2i}\right)$$

are both real-valued solutions of (6).

Let us now compute the series expansion of

$$\frac{e^{ix} + e^{-ix}}{2}$$

and

$$\frac{e^{ix} - e^{-ix}}{2i} \quad .$$

$$\begin{aligned} \tfrac{1}{2}\left(e^{ix} + e^{-ix}\right) &= \tfrac{1}{2}\left(1 + (ix) + \tfrac{1}{2!}(ix)^2 + \tfrac{1}{3!}(ix)^3 + \cdots\right) \\ &\quad + \tfrac{1}{2}\left(1 + (-ix) + \tfrac{1}{2!}(-ix)^2 + \tfrac{1}{3!}(-ix)^3 + \cdots\right) \\ &= \left(1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \cdots\right) \end{aligned}$$

The expression on the right hand side is readily identified as the Taylor series expansion of $\cos(x)$. We thus conclude

(12) $$\boxed{\cos(x) = \frac{e^{ix} + e^{-ix}}{2}} \quad .$$

Similarly, one can show that

(13) $$\boxed{\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}} \quad .$$

On the other hand, if one adds (8) to i times (9) one gets

$$\cos(x) + i\sin(x) = \frac{e^{ix} + e^{-ix}}{2} + i\frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{ix} + e^{-ix} + e^{ix} - e^{-ix}}{2} = e^{ix}$$

or

$$\boxed{e^{ix} = \cos(x) + i\sin(x)} \tag{14}$$

Thus, the real part of e^{ix} is $\cos(x)$, while the pure imaginary part of e^{ix} is $\sin(x)$.

We now have a means of interpreting the function

$$e^{\alpha x + i\beta x}$$

in terms of elementary functions (rather than as a power series); namely,

$$e^{\alpha x + i\beta x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right) .$$

Thus,

$$\boxed{\begin{array}{rcll} Re\left[e^{\alpha x + i\beta x}\right] & = & e^{\alpha x}\cos(\beta x) & , \\ Im\left[e^{\alpha x + i\beta x}\right] & = & e^{\alpha x}\sin(\beta x) & . \end{array}} \tag{15}$$

I now want to show how (12) and (13) allow us to write down the general solution of a differential equation of the form

$$y'' + py' + qy = 0 \quad , \quad p^2 - 4q < 0 \tag{16}$$

as a linear combination of real-valued functions.

Now when $p^2 - 4q < 0$, then

$$\lambda_{\pm} = \frac{-p \pm i\sqrt{4q - p^2}}{2} = \alpha \pm i\beta$$

are the (complex) roots of the characteristic equation

$$\lambda^2 + p\lambda + q = 0$$

corresponding to (16) and

$$y_{\pm}(x) = e^{\alpha x \pm i\beta}$$

are two (complex-valued) solutions of (16). But since (16) is linear, since y_+ and y_- are solutions so are

$$\begin{array}{rcl} y_1(x) & = & \frac{1}{2}\left(y_+(x) + y_-(x)\right) \\ & = & \frac{1}{2}\left(e^{\alpha x + i\beta x} + e^{\alpha x - i\beta x}\right) \\ & = & e^{\alpha x}\left(\frac{e^{i\beta x} + e^{-i\beta x}}{2}\right) \\ & = & e^{\alpha x}\cos(\beta x) \end{array}$$

and

$$\begin{array}{rcl} y_2(x) & = & \frac{1}{2i}\left(y_+(x) - y_-(x)\right) \\ & = & \frac{1}{2i}\left(e^{\alpha x + i\beta x} - e^{\alpha x - i\beta x}\right) \\ & = & e^{\alpha x}\left(\frac{e^{i\beta x} - e^{-i\beta x}}{2i}\right) \\ & = & e^{\alpha x}\sin(\beta x) \qquad . \end{array}$$

Note that y_1 and y_2 are both **real-valued functions**.

We conclude that if the characteristic equation corresponding to

$$y'' + py' + qy = 0$$

has two complex roots

$$\lambda = \alpha \pm i\beta$$

then the general solution is

$$\boxed{y(x) = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x) \quad .} \tag{17}$$

EXAMPLE 13.1. The differential equation

$$y'' - 2y' - 3y$$

has as its characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0 \quad .$$

The roots of the characteristic equation are given by

$$\begin{aligned} \lambda &= \tfrac{2 \pm \sqrt{4+12}}{2} \\ &= 3, -1 \quad . \end{aligned}$$

These are distinct real roots, so the general solution is

$$y(x) = c_1 e^{3x} + c_2 e^{-x} \quad .$$

EXAMPLE 13.2. The differential equation

$$y'' + 4y' + 4y = 0$$

has

$$\lambda^2 + 4\lambda + 4 = 0$$

as its characteristic equation. The roots of the characteristic equation are given by

$$\begin{aligned} \lambda &= \tfrac{-4 \pm \sqrt{16-16}}{2} \\ &= -2 \quad . \end{aligned}$$

Thus we have a double root and the general solution is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} \quad .$$

EXAMPLE 13.3. The differential equation

$$y'' + y' + y = 0$$

has

$$\lambda^2 + \lambda + 1 = 0$$

as its characteristic equation. The roots of the characteristic equation are

$$\begin{aligned} \lambda &= \tfrac{-1 \pm \sqrt{1-4}}{2} \\ &= -\tfrac{1}{2} \pm i \tfrac{\sqrt{3}}{2} \end{aligned}$$

and so the general solution is

$$y(x) = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) \quad .$$