Reduction of Order

Recall that the general solution of a second order homogeneous linear differential equation

$$(1) \qquad\qquad L[y] = y'' + p(x)y' + q(x)y = 0$$

is given by

$$(2) \qquad\qquad y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 and y_2 are any two solutions such that

$$(3) \qquad\qquad W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0 \quad .$$

In this section we shall assume that we have already found one solution y_1 of (1) and that we are seeking to find another solution y_2 so that we can write down the general solution as in (2).

So suppose we have one non-trivial solution $y_1(x)$ of (1) and suppose there is another solution of the form

$$y_2(x) = v(x)y_1(x) \quad .$$

Then

$$
\begin{aligned}
W[y_1, y_2] &= y_1 y_2' - y_1' y_2 \\
&= y_1(v'y_1 + vy_1') - y_1'(vy_1) \\
&= (y_1)^2 v' \\
&\neq 0
\end{aligned}
$$

unless $v' = 0$. Thus, any solution we construct by multiplying our given solution $y_1(x)$ by a non-constant function $v(x)$ will give us another linearly independent solution.

The question we now wish to address is: how does one find an appropriate function $v(x)$?

Certainly, we want to choose $v(x)$ so that $y_2(x) = v(x)y_1(x)$ satisfies (1). So let us insert $y(x) = v(x)y_1(x)$ into (1):

$$
\begin{aligned}
0 &= \frac{d^2}{dx^2}(vy_1) + p(x)\frac{d}{dx}(vy_1) + q(vy_1) \\
&= v''y_1 + 2v'y_1' + vy_1'' + p(x)v'y_1 + p(x)vy_1' + qvy_1 \\
&= v(y_1'' + p(x)y_1' + q(x)y_1) + v''y_1 + (2y_1' + p(x)y_1)v'
\end{aligned}
$$
$$(4)$$

The first term vanishes since y_1 is a solution of (1), so $v(x)$ must satisfy

$$0 = y_1 v'' + (2y_1' + p(x)y_1)v'$$

or

$$(5) \qquad\qquad v'' + \left(p(x) + \frac{2y_1'}{y_1}\right)v' = 0 \quad .$$

Now set

$$(6) \qquad\qquad u(x) = v'(x) \quad .$$

Then we have

$$(7) \qquad\qquad u' + \left(p(x) + \frac{2y_1'(x)}{y_1(x)}\right)u = 0 \quad .$$

This is a first order linear differential equation which we know how to solve. Its general solution is

(8)
$$
\begin{aligned}
u(x) &= C \exp\left[-\int^x \left(p(t) + \frac{2y_1'(t)}{y_1(t)}\right) dt\right] \\
&= C \exp\left[-\int^x (p(t))\, dt - 2\int^x \frac{y_1'(t)}{y_1(t)} dt\right]
\end{aligned}
$$

Now note that

$$
\frac{d}{dt} \ln[y_1(t)] = \frac{y_1'(t)}{y_1(t)} \quad,
$$

so

$$
\begin{aligned}
\exp\left[-2\int^x \frac{y_1'(t)}{y_1(t)} dy\right] &= \exp\left[-2\int^x \frac{d}{dt}(\ln[y_1(t)])\, dt\right] \\
&= \exp[-2\ln[y_1(x)]] \\
&= \exp\left[\ln\left[(y_1(x))^{-2}\right]\right] \\
&= \frac{1}{(y_1(x))^2} \quad.
\end{aligned}
$$

Thus, (8) can be written as

$$
u(x) = \frac{C}{(y_1(x))^2} \exp\left[-\int^x p(t)dt\right] \quad.
$$

Now recall from (6) that $u(x)$ is the derivative of the factor v(x) which we originally sought out to find. So

$$
\begin{aligned}
v(x) &= \int^x u(t)\, dt + D \\
&= \int^x \left[\frac{C}{(y_1(t))^2} \exp\left[-\int^t p(t')dt'\right]\right] + D
\end{aligned}
$$

It is not too difficult to convince oneself that it is not really necessary to carry along the constants of integration C and D. For the constant D can be absorbed into the constant c_1 of the general solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$, while the factor C can be absorbed into the constant c_2. Thus, without loss of generality, we can take $C = 1$ and $D = 0$. So given one solution $y_1(x)$ of (1), a second solution $y_2(x)$ of (1) can be formed by computing

$$
v(x) = \int^x u(t)\, dt
$$

where

$$
u(x) = \frac{1}{(y_1(x))^2} \exp\left[-\int^x p(t)dt\right]
$$

and then setting

$$
y_2(x) = v(x)y_1(x) \quad.
$$

The general solution of (1) is then

$$
y(x) = c_1 y_1(x) + c_2 v(x)y_1(x) \quad.
$$

This technique for constructing the general solution from single solution of a second order linear homogeneous differential equation is called **reduction of order**.

For those of you who like nice tidy formulae we can write

$$
\boxed{y_2(x) = y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp\left[-\int^s p(t)dt\right] ds}
$$

for the second solution.

EXAMPLE 12.1.

$$
y_1(x) = e^{-x}
$$

is one solution of

$$
y'' + 2y' + y = 0 \quad.
$$

Find another linearly independent solution.

Well, $p(x) = 2$, so

$$\begin{aligned}
u(x) &= \frac{C}{(y_1(x))^2}\exp\left[-\int^x p(t)dt\right]\\
&= \frac{C}{e^{-2x}}\exp\left[-2x\right]\\
&= Ce^{2x}e^{-2x}\\
&= C
\end{aligned}$$

So

$$v(x) = \int^x u(t)\,dt = \int^x C\,dt = Cx.$$

Thus,

$$y_2(x) = v(x)y_1(x) = Cxe^{-x}\quad.$$

EXAMPLE 12.2.

$$y_1(x) = x$$

is a solution of

$$x^2 y'' + 2xy' - 2y = 0\quad.$$

Use reduction of order to find the general solution.

Well, we first put the differential equation in standard form:

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0\quad.$$

Thus,

$$p(x) = \frac{2}{x}\quad,\quad q(x) = \frac{-2}{x^2}\quad.$$

We first compute $u(x)$.

$$\begin{aligned}
u(x) &= \frac{C}{(y_1(x))^2}\exp\left[-\int^x p(t)dt\right]\\
&= \frac{C}{x^2}\exp\left[\int^x -\frac{2}{t}dt\right]\\
&= \frac{C}{x^2}\exp\left[-2\ln[x]\right]\\
&= \frac{C}{x^2}x^{-2}\\
&= Cx^{-4}\quad.
\end{aligned}$$

So

$$\begin{aligned}
v(x) &= \int^x u(t)dt\\
&= \int^x Ct^{-4}dt\\
&= -\frac{C}{3}x^{-3}
\end{aligned}$$

and

$$y_2(x) = v(x)y_1(x) = -\frac{C}{3x^3}x = C'x^{-2}\quad.$$

The general solution of the original differential equation is thus

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2 x^{-2}\quad.$$