LECTURE 11

Change of Variable

We will now discuss one last technique for solving non-linear first order differential equations, where the basic idea is to make a change of variables in order to recast the differential equation into one of the cases we have already solved. We'll begin with a special case of this method.

1. Homogeneous Equations of Degree Zero

The special form to be considered here is when the given differential equation

$$y' = f(x,y) \tag{1}$$

can be expressed as

$$y' = F\left(\frac{y}{x}\right) \quad . \tag{2}$$

In such a case, we say the differential equation (1) is **homogeneous of degree zero**.

EXAMPLE 11.1.

$$y' = \frac{y^2 + 2xy}{x^2} = \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)^{-1} = F\left(\frac{y}{x}\right) \text{ when } F(v) \equiv v^2 + \frac{2}{v}$$

EXAMPLE 11.2.

$$y' = \ln(x) - \ln(y) = -ln\left(\frac{y}{x}\right) = F\left(\frac{y}{x}\right) \text{ when } F(v) \equiv -\ln(v)$$

EXAMPLE 11.3.

$$y' = \frac{x+y}{y-x} = \frac{1+\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)-1} = F\left(\frac{y}{x}\right) \text{ when } F(v) \equiv \frac{1+v}{v-1}$$

To solve such equations we introduce a new variable which we will denote by v, to represent the ratio of y to x:

$$v = \frac{y}{x} \quad . \tag{3}$$

We then have

$$\frac{dv}{dx} = -\frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx}$$

or

$$\frac{dy}{dx} = x\left(\frac{y}{x^2} + \frac{dv}{dx}\right) = v + x\frac{dv}{dx} \quad . \tag{4}$$

Hence equation (2) becomes

$$x\frac{dv}{dx} + v = F(v) \tag{5}$$

or

$$xdv = (F(v) - v)\,dx$$

or

$$\frac{dx}{x} = \frac{dv}{F(v) - v} \quad . \tag{6}$$

Integrating both sides of this equation yields

$$\ln|x| = \int \frac{dv}{F(v) - v} + C \quad . \tag{7}$$

Suppose now that the integral on the right hand side has been carried out so

$$H(v) = \int \frac{dv}{F(v) - v}$$

is some explicit function of v. Equation (7) becomes

$$\ln|x| - H(v) = C$$

or

$$\ln|x| - H\left(\frac{y}{x}\right) = C \quad . \tag{8}$$

We now have an equation specifying y as an implicit function of x. An explicit solution of the differential equation (1) is obtained whenever equation (9) can be solved for y in terms of x and C.

In summary, the change of variables $y \to v$ allows us to transform a differential equation of the form (2) to a separable differential equation (5) which can be solved by integrating both sides of (6) and then solving for v in terms of x and then substituting $\frac{y}{x}$ for v and solving for y in terms of x.

Example 11.4.

$$y' = \frac{x+y}{x}$$

This equation is homogeneous, since we can re-write it as

$$y' = 1 + \frac{y}{x} \quad .$$

We thus take

$$F(v) = 1 + v$$

and try to solve

$$\frac{dx}{x} = \frac{dv}{F(v) - v} = \frac{dv}{1 + v - v} = dv \quad .$$

Integrating both sides yields

$$\ln(x) = v + C = \frac{y}{x} + C$$

or

$$y = x\left(\ln(x) - C\right) \quad .$$

2. Other Substitutions

The substitution method can also be applied in other situations; however, in such cases there usually isn't a clean litmus test that will tell you whether or not a given substitution will help solve the differential equation. Rather one has to resort to trial and error in order to find an appropriate substitution. There is at least a guiding principle though: you want to look for a substitution that will end up simplifying the differential equation. Here are some examples.

Example 11.5.

$$\frac{dy}{dx} = (x + y + 1)^2 + 3 \tag{9}$$

In order to simplify the quadratic expression on the right hand side we'll try the following substitution:

$$z = x + y + 1 \quad \Rightarrow \quad \frac{dz}{dx} = 1 + \frac{dy}{dx} + 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dz}{dx} - 1$$

Substituting z for $x + y + 1$ on the right hand side of (9) and $z' - 1$ for y' on the left hand side of (9) we obtain the following equivalent differential equation:

$$\frac{dz}{dx} - 1 = z^2 + 3$$

or

$$\frac{dz}{dx} = z^2 + 4$$

or

$$\frac{dz}{z^2 + 4} = dx$$

This equation is separable, and integrating both sides yields

$$\frac{1}{2} \arctan \frac{1}{2} z = \int \frac{dz}{z^2 + 4} = \int dx + C = x + C$$

or

$$\arctan \left(\frac{z}{2} \right) = 2x + C'$$

or

$$z = 2 \tan \left(2x + C' \right)$$

or, recalling that $z = x + y + 1$,

$$x + y + 1 = 2 \tan \left(2x + C' \right)$$

or, solving for y,

$$y = 2 \tan \left(2x + C' \right) - x - 1$$