LECTURE 9

Integrating Factors

Recall that a differential equation of the form

$$M(x,y) + N(x,y)y' = 0 \tag{1}$$

is said to be **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad , \tag{2}$$

and that in such a case, we could always find an implicit solution of the form

$$\psi(x,y) = C \tag{3}$$

with

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= M(x,y) \\ \frac{\partial \psi}{\partial y} &= N(x,y) \quad . \end{aligned} \tag{4}$$

Even if (1) is not exact, it is sometimes possible multiply it by another function of x and/or y to obtain an equivalent equation which is exact. That is, one can sometimes find a function $\mu(x,y)$ such that

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0 \tag{5}$$

is exact. Such a function $\mu(x,y)$ is called an ntegrating factor. If an integrating factor can be found, then the original differential equation (1) can be solved by simply constructing a solution to the equivalent exact differential equation (5).

EXAMPLE 9.1. Consider the differential equation

$$x^2y^3 + x\left(1+y^2\right)\frac{dy}{dx} = 0 \quad .$$

This equation is not exact; for

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2y^3\right) &= 3x^2 \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(1+y^2\right)\right) &= 1+y^2 \quad . \end{aligned}$$

and so

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad .$$

However, if we multiply both sides of the differential equation by

$$\mu(x,y) = \frac{1}{xy^3}$$

we get

$$x + \frac{1+y^2}{y^3}\frac{dy}{dx} = 0$$

which is not only exact, it is also separable. The general solution is thus obtained by calculating

$$\begin{aligned} H_1(x) &= \int x\,dx = \tfrac{1}{2}x^2 \\ H_2(y) &= \int \tfrac{1+y^2}{y^3}\,dy = \tfrac{1}{2y^2} + \ln|y| \end{aligned}$$

and then demanding that y is related to x by

$$H_1(x) + H_2(y) = C$$

or

$$\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln|y| = C \quad .$$

Now, in general, the problem of finding an integrating factor $\mu(x, y)$ for a given differential equation is very difficult. In certain cases, it is rather easy to find an integrating factor.

0.1. Equations with Integrating Factors that depend only on x. Consider a general first order differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad . \tag{6}$$

We shall suppose that there exists an integrating factor for this equation that depends only on x:

$$\mu = \mu(x) \quad . \tag{7}$$

If μ is to really be an integrating factor, then

$$\mu(x)M(x, y) + \mu(x)N(x, y)\frac{dy}{dx} \tag{8}$$

must be exact; i.e.,

$$\frac{\partial}{\partial y}\left(\mu(x)M(x, y)\right) = \frac{\partial}{\partial x}\left(\mu(x)N(x, y)\right) \quad . \tag{9}$$

Carrying out the differentiations (using the product rule, and the fact that $\mu(x)$ depends only on x), we get

$$\mu\frac{\partial M}{\partial y} = \frac{d\mu}{dx}N + \mu\frac{\partial N}{\partial x}$$

or

$$\frac{d\mu}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu \qquad . \tag{10}$$

Now if μ is depends only on x (and not on y), then necessarily $\frac{d\mu}{dx}$ depends only on x. Thus, the self-consistency of equations (7) and (10) requires the right hand side of (10) to be a function of x alone. We presume this to be the case and set

$$p(x) = -\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

so that we can rewrite (10) as

$$\frac{d\mu}{dx} + p(x)\mu = 0 \quad . \tag{11}$$

This is a first order linear differential equation for μ hat we can solve! According to the formula developed in Section 2.1, the general solution of (11) is

$$\mu(x) = A\exp\left[\int -p(x)dx\right] = A\exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial y}\right)dx\right] \quad . \tag{12}$$

The formula (12) thus gives us an integrating factor for (6) so long as

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

depends only on x.

0.2. Equations with Integrating Factors that depend only on y. Consider again the general first order differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad . \tag{13}$$

We shall suppose that there exists an integrating factor for this equation that depends only on y:

$$\mu = \mu(y) \quad . \tag{14}$$

If μ is to really be an integrating factor, then

$$\mu(y)M(x,y) + \mu(y)N(x,y)\frac{dy}{dx} \tag{15}$$

must be exact; i.e.,

$$\frac{\partial}{\partial y}\left(\mu(y)M(x,y)\right) = \frac{\partial}{\partial x}\left(\mu(y)N(x,y)\right) \quad . \tag{16}$$

Carrying out the differentiations (using the product rule, and the fact that $\mu(y)$ depends only on y), we get

$$\frac{d\mu}{dy}M + \mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x}$$

or

$$\frac{d\mu}{dy} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu \quad . \tag{17}$$

Now since μ is depends only on y (and not on x), then necessarily $\frac{d\mu}{dy}$ depends only on y. Thus, the self-consistency of equations (14) and (17) requires the right hand side of (10) to be a function of y alone. We presume this to be the case and set

$$p(y) = -\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

so that we can rewrite (10) as

$$\frac{d\mu}{dy} + p(y)\mu = 0 \quad . \tag{18}$$

According to the formula developed in Section 2.1, the general solution of (18) is

$$\mu(y) = A\exp\left[\int -p(y)dx\right] = A\exp\left[\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\right] \quad . \tag{19}$$

The formula (19) thus gives us an integrating factor for (13) so long as

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

depends only on y.

0.3. Summary: Finding Integrating Factors. Suppose that

$$M(x,y) + N(x,y)y' = 0 \tag{20}$$

is not exact.

A. If

$$F_1 = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \tag{21}$$

depends only on x then

$$\mu(x) = \exp\left(\int F_1(x)dx\right) \tag{22}$$

will be an integrating factor for (20).

B. If

$$F_2 = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \tag{23}$$

depends only on y then

$$\mu(y) = \exp\left(\int F_2(y)dy\right) \tag{24}$$

will be an integrating factor for (20).

C. If neither A nor B is true, then there is little hope of constructing an integrating factor.

EXAMPLE 9.2.

$$(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0 \tag{25}$$

Here

$$\begin{aligned} M(x,y) &= 3x^2y + 2xy + y^3 \\ N(x,y) &= x^2 + y^2 \quad . \end{aligned}$$

Since

$$\frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2 \neq 2x = \frac{\partial N}{\partial x}$$

this equation is not exact.

We seek to find a function μ such that

$$\mu(x,y)(3x^2y + 2xy + y^3)dx + \mu(x,y)(x^2 + y^2)dy = 0$$

is exact. Now

$$\begin{aligned} F_1 &\equiv \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = \frac{3(x^2 + y^2)}{x^2 + y^2} = 3 \\ F_2 &\equiv \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2x - 3x^2 - 2x - 3y^2}{3x^2y + 2xy + y^3} = \frac{-3\left(x^2 + y^2\right)}{3x^2y + 2xy + y^3} \end{aligned}$$

Since F_2 depends on both x and y, we cannot construct an integrating factor depending only on y from F_2. However, since F_1 does not depend on y, we can consistently construct an integrating factor that is a function of x alone. Applying formula (22) we get

$$\mu(x) = \exp\left(\int F_1(x)dx\right) = \exp\left[\int 3dx\right] = e^{3x} \quad .$$

We can now employ this $\mu(x)$ as an integrating factor to construct a general solution of

$$e^{3x}(3x^2 + 2x + 3y^2) + e^{3x}(x^2 + y^2)y' = 0$$

which, by construction, must be exact. So we seek a function ψ such that

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= e^{3x}(3x^2y + 2xy + y^3) \\ \frac{\partial \psi}{\partial y} &= e^{3x}(x^2 + y^2) \quad . \end{aligned} \tag{26}$$

Integrating the first equation with respect to x and the second equation with respect to y yeilds

$$\begin{aligned} \psi(x,y) &= x^2ye^{3x} + \tfrac{1}{3}y^3e^{3x} + h_1(y) \\ \psi(x,y) &= x^2ye^{3x} + \tfrac{1}{3}y^3e^{3x} + h_2(x) \quad . \end{aligned}$$

Comparing these expressions for $\psi(x,y)$ we see that we msut take $h_1(y) = h_2(x) = C$, a constant. Thus, function ψ satisfying (26) must be of the form

$$\psi(x,y) = e^{3x}x^2y + e^{3x}y^3 + C \quad .$$

Therefore, the general solution of (25) is found by solving

$$e^{3x}x^2y + e^{3x}y^3 = C$$

for y.