LECTURE 6

First Order Linear Differential Equations

A *linear first order ordinary differential equation* is a differential equation of the form

$$(1) \qquad a(x)y' + b(x)y = c(x) \quad .$$

Here y represents the unknown function, y' its derivative with respect to the variable x, and $a(x), b(c)$ and $c(x)$ are certain prescribed functions of x (the precise functional form of $a(x)$, $b(x)$, and $c(x)$ will fixed in any specific example). So long as $a(x) \neq 0$, this equation is equivalent to a differential equation of the form

$$(2) \qquad y' + p(x)y = g(x)$$

where

$$\begin{aligned} p(x) &= \frac{b(x)}{a(x)}, \\ g(x) &= \frac{c(x)}{a(x)}. \end{aligned}$$

We shall refer to a differential equation (2) as the **standard form** of differential equation (1). (In general, we shall say that an ordinary linear differential equation is in **standard form** when the coefficient of the highest derivative is 1.)

Our goal now is to develop a formula for the general solution of (2). To accomplish this goal, we shall first construct solutions for several special cases. Then with the knowledge gained from these simpler examples, we will develop a general formula for the solution of **any** differential equation of the form (2).

1. The Homogeneous Case

Let's begin with the case when the function $g(x)$ on the right hand side of (2) is identically 0.

$$((3)) \qquad y' + p(x)\,y = 0 \quad .$$

The solution of this special case will actually play a fundamental role in our solution of the general case later on. A similar phenomenon will happen when we look at second order linear differential equations. Because of this, we have a special nomenclature for the two basic possibilities for y-independent terms of a linear differential equation. We say that

$y' + p(x)\,y = 0$ is a **homogeneous** 1^{st} order linear differential equation

and

$y' + p(x)\,y = g(x)$ is a **nonhomogeneous** 1^{st} order linear differential equation whenever $g(x) \neq 0$.

OK. Let's see if we can solve (3). This is in fact relatively easy since (3) can be written in separable form:

$$y' + p(x)\,y = 0 \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = -p(x)$$

Applying the technique we developed for Separable Equations:

$$\begin{aligned}
\frac{1}{y}dy &= -p(x)\,dx \quad \Rightarrow \quad \int \frac{1}{y}dy = -\int p(x)\,dx + C \\
&\Rightarrow \quad \ln|y| = -\int p(x)\,dx + C \\
&\Rightarrow \quad y = \exp\left(-\int p(x)\,dx + C\right) \\
&\Rightarrow \quad y = \exp\left(-\int p(x)\,dx\right)\exp(C) \\
&\Rightarrow \quad y = A\exp\left(-\int p(x)\,dx\right) \qquad \text{where } A = \exp(C) \text{ is an arbitrary constant}
\end{aligned}$$

Example 6.1. Integrating both sides of the latter equation (the left hand side with respect to y and the right hand side with respect to x) yields

$$\ln(y) = -\int^x p(x')\,dx' \quad + \quad C$$

or, exponentiating both sides

$$y = \exp\left[-\int^x p(x')\,dx' \quad + \quad C\right]$$

$$\begin{aligned}
y &= \exp\left[-\int^x p(x')\,dx' \quad + \quad C\right] \\
&= e^C \exp\left[-\int^x p(x')\,dx'\right] \\
&= A\exp\left[-\int^x p(x')\,dx'\right]
\end{aligned}$$

In the last step we have simply replaced the constant e^C, which is arbitrary since C is arbitrary, by another arbitrary constant A. There is nothing tricky here; the point is that in the general solution the numerical factor in front of the exponential function is arbitrary and so rather than writing this factor as e^C we use the simpler form A. Thus, the general solution of

$$y' + p(x) = 0$$

is

$$y = A\exp\left[-\int^x p(x')\,dx'\right] \quad .$$

Example 6.2. $y' + \frac{3}{x}y = 0$

- First, we recognize this differential equation as a *separable* differential equation:

$$y' + \frac{3}{x}y = 0 \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = -\frac{3}{x}$$

  We next apply the technique for solving separable equations:

$$\begin{aligned}
\frac{1}{y}dy &= -3xdx \quad \Rightarrow \quad \int \frac{1}{y}dy = \int -\frac{3}{x}\ dx + C \\
&\Rightarrow \quad \ln|y| = -3\ln|x| + C \\
&\Rightarrow \quad y = \exp(-3\ln|x| + C) \\
&\Rightarrow \quad y = \exp(-3\ln|x|)\exp(C) \\
&\Rightarrow \quad y = A\exp(-3\ln|x|) \qquad \text{where } A = \exp(C) \text{ is an arbitrary constant}
\end{aligned}$$

Digression: (a very common and useful identity) Let me begin by recalling three basic facts:

(i) The exponential function $\exp(x) \equiv e^x$ where $e = 2.718281...$ is the natural logarithmic base.
(ii) The logarithm function is the functional inverse of the exponential function: $\exp(\ln|x|) = x = \ln(\exp(x))$.
(iii) Powers of powers of a number obey $\left(A^b\right)^c = A^{bc}$.

Now consider the identity

$$\begin{aligned} x &= \exp(\ln|x|) \qquad \text{by (ii)} \\ &\Rightarrow \quad x = e^{\ln|x|} \qquad \text{by (i)} \\ &\Rightarrow \quad x^a = \left(e^{\ln(x)}\right)^a = e^{a\ln|x|} \qquad \text{by (iii)} \\ &\Rightarrow \quad \exp(a\ln|x|) = x^a \qquad \text{by (i)} \end{aligned}$$

Thus, we have an identity

$$\exp(a\ln|x|) = x^a \tag{4}$$

Using this identity, we can simply our expression for the general solution of $y' + \frac{3}{x}y = 0$:

$$y = A\exp(-3\ln|x|) \quad \Rightarrow \quad y(x) = Ax^{-3}$$

EXAMPLE 6.3. (Example 1 Reloaded) In the preceding example, we solved $y' + \frac{3}{x}y = 0$ by repeating the steps by which we solved the general case $y' + p(x)y = 0$. However, the main reason for treating the general case, was to get a general formula for that case. Indeed, dropping the intermediary steps we have from our initial discussion

$$y' + p(x)y = 0 \quad \Rightarrow \quad y(x) = A\exp\left(-\int p(x)\,dx\right) \quad . \tag{5}$$

So another way of attacking $y' + \frac{3}{x}y = 0$ is to recognize it as a particular instance of the case governed by the formula (5). Thus, we can also proceed as follows:

- $y' + \frac{3}{x}y = 0$ is of the form $y' + p(x)y = 0$ when we take $p(x) = \frac{3}{x}$.
- Plugging $p(x) = \frac{3}{x}$ into the formula (5) for the solution of $y' + p(x)y = 0$ yields

$$\begin{aligned} y(x) &= A\exp\left(-\int \frac{3}{x}dx\right) \\ &= A\exp(-3\ln|x|) \\ &= Ax^{-3} \qquad \text{(applying identify (4))} \end{aligned}$$

Thus, recovering the same answer but a little more efficiently.

2. A simple nonhomogenous case

We are still not prepared to attack differential equations of the form

$$\frac{dy}{dx} + p(x)y = g(x)$$

when $g(x) \neq 0$. However, we'll continue to pursue our strategy of developing solution techniques by introducing complications in a very controlled way. Having handled the case when $g(x) = 0$, the next simple case we might consider is when $g(x) \neq 0$ but at least the function $p(x)$ is only a constant; say $p(x) = a$ for all x. So now we'll try to solve

$$\frac{dy}{dx} + ay = g(x) \quad \text{where } a \text{ is a constant.} \tag{6}$$

To solve this equation we'll employ a trick.[1] Let's multiply both sides of this equation by e^{ax}:

$$e^{ax}y' + ae^{ax}y = e^{ax}g(x).$$

Noticing that the right hand side is $\frac{d}{dx}\left(e^{ax}y\right)$ (via the product rule for differentiation), we have

$$\frac{d}{dx}\left(e^{ax}y\right) = e^{ax}g(x) \quad .$$

We now take anti-derivatives of both sides to get

$$e^{ax}y = \int^{x} e^{ax'}g(x')\,dx' \quad + \quad C$$

or

$$y(x) = \frac{1}{e^{ax}}\int^{x} e^{ax'}g(x')\,dx' \quad + \quad Ce^{-ax} \quad .$$

Example 6.4.

$$y' - 2y = x^2e^{2x}$$

This equation is of type (iii) with

$$\begin{aligned} p &= -2 \\ g(x) &= x^2e^{2x} \quad . \end{aligned}$$

So we multiply both sides by e^{-2x} to get

$$\frac{d}{dx}\left(e^{-2x}y\right) = e^{2x}\left(y' - 2y\right) = e^{-2x}\left(x^2e^{2x}\right) = x^2$$

Integrating both sides with respect to x, and employing the Fundamental Theorem of Calculus on the left yields

$$e^{-2x}y = \frac{1}{3}x^3 + C$$

or

$$y = \frac{1}{3}x^3e^{2x} + Ce^{2x} \quad .$$

Let us now confirm that this is a solution

$$\begin{aligned} y' &= x^2e^{2x} + \frac{2}{3}x^3e^{2x} + 2Ce^{2x} \\ -2y &= -\frac{2}{3}x^3e^{2x} - 2Ce^{2x} \end{aligned}$$

so

$$y' - 2y = x^2e^{2x}$$

□

3. The General Case

We are now prepared to handle the case of a general first order linear differential equation; i.e., differential equations of the form

$$y' + p(x)y = g(x) \tag{7}$$

with $p(x)$ and $g(x)$ are arbitrary functions of x.

Note: This case includes all the preceding cases.

[1]According to Richard Feynman, if you employ a trick more than once, it becomes a *method*. Indeed, shortly, the trick introduced here will lead to a bone-fide method - the integrating factors method.

We shall construct a solution of this equation in a manner similar to case when $p(x)$ is a constant; that is, we will try find a function $\mu(x)$ satisfying

$$\mu(x)\left(y' + p(x)y\right) = \frac{d}{dx}\left(\mu(x)y\right) \tag{8}$$

Multiplying (7) by $\mu(x)$, we could then obtain

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x)$$

which when integrated yields

$$\mu(x)y = \int^x \mu(x')g(x')\,dx' + C$$

or

$$y = \frac{1}{\mu(x)}\int^x \mu(x')g(x')\,dx' \;+\; \frac{C}{\mu(x)} \tag{9}$$

It thus remains to find a suitable function $\mu(x)$; i.e., we need to find a function $\mu(x)$ so that

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)y' + p\left(x\right)\mu(x)y$$

This will certainly be true if

$$\frac{d}{dx}\mu(x) = p(x)\mu(x) \quad . \tag{10}$$

For then

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)y' + \left(\frac{d}{dx}\mu\left(x\right)\right)y = \mu(x)y' + p\left(x\right)\mu(x)y$$

Thus, we have to solve another first order, linear, differential equation of type (iii). As before we re-write (10) in terms of differentials to get

$$\frac{d\mu}{\mu} = p(x)dx \quad ,$$

and then integrate both sides; yielding

$$\ln(\mu) = \int^x p(x')\,dx' + A \quad .$$

Exponentiating both sides of this relation yields

$$\mu = \exp\left(\int^x p(x')dx' + A\right)$$

$$\begin{aligned} \mu &= \exp\left(\int^x p(x')dx' + A\right) \\ &= A'\exp\left(\int^x p(x')\,dx'\right) \end{aligned}$$

So a suitable function $\mu(x)$ is

$$\mu(x) = A'\exp\left(\int^x p(x')dx'\right)$$

Inserting this expression for $\mu(x)$ into our formula (9) for y yields

$$y(x) = \frac{1}{A'\exp\left(\int^x p(x'')dx''\right)}\left[\int^x A'\exp\left(\int^{x'} p(x')dx''\right)g(x')dx' \quad + \quad C\right]$$

It is easily see that the constant A' in the denominator is irrelevant to the final answer. This is because it can be canceled out by the A' within the integral over x', and it can be absorbed into the arbitrary constant C in the second term. Thus, the general solution to a first order linear equation

$$y' + p(x)y = g(x)$$

is given by

$$y(x) = \frac{1}{\mu(x)} \int^x \mu(x')g(x')dx' + \frac{C}{\mu(x)} \tag{11a}$$

where

$$\mu(x) = \exp\left(\int^x p(x')dx'\right) \tag{11b}$$

EXAMPLE 6.5.

$$xy' + 2y = \sin(x) \tag{12}$$

Putting this equation in standard form requires we set

$$\begin{aligned} p(x) &= \tfrac{2}{x} \\ g(x) &= \tfrac{\sin(x)}{x} \end{aligned}$$

Now

$$\int p(x)\,dx = \int \frac{2}{x}dx = 2\ln(x) = \ln\left(x^2\right),$$

so

$$\begin{aligned} \mu(x) &= \exp\left[\textstyle\int^x p(x')\,dx'\right] \\ &= \exp\left[\ln\left(x^2\right)\right] \\ &= x^2 \end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int^x \mu(x')g(x')\,dx' + \frac{C}{\mu(x)} \\ &= \frac{1}{x^2} \int^x (x')^2 \frac{\sin(x')}{x'}\,dx' + \frac{C}{x^2} \\ &= \frac{1}{x^2} \int^x x' \sin(x')\,dx' + \frac{C}{x^2} \end{aligned}$$

Now

$$\int x\sin(x)\,dx$$

can be integrated by parts. Set

$$u = x \qquad , \qquad dv = \sin(x)dx$$

Then

$$du = dx \qquad , \qquad v = \int dv = -\cos(x)$$

and the integration by parts formula,

$$\int udv = uv - \int vdu \quad ,$$

tells us that

$$\begin{aligned} \int x\sin(x)\,dx &= -x\cos(x) + \int \cos(x)\,dx \\ &= -x\cos(x) + \sin(x) \quad . \end{aligned}$$

Therefore, we have as a general solution of (12),

$$\begin{aligned} y(x) &= \frac{1}{x^2}\left(-x\cos(x)+\sin(x)\right)+\frac{C}{x^2} \\ &= \frac{1}{x^2}\sin(x)-\frac{1}{x}\cos(x)+\frac{C}{x^2} \quad . \end{aligned}$$

□