LECTURE 5

Separable Equations

Recall that our main problem at this point of the course is to solve first order differential equations

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

Last time we considered the special case when the function $F(x, y)$ on the right hand side depended only on x. Another relatively easy case is when the right hand side depends only on the unknown function y:

$$\frac{dy}{dx} = F(y) \tag{2}$$

However, although the differential equation is similar we can not use the preceding result

$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad y(x) = \int f(x)\,dx + C$$

to solve (2). You see, even if we think of solving (2) by integrating both sides

$$y(x) = \int F(y)\,dx + C$$

we have a fundament problem: since we don't know what $y(x)$ is, we can't actually plug it into the function $F(y)$ and integrate with respect to x.

So how do we solve (2)? The basic idea will be to find an algebraic relationship between y and x that's equivalent to the original differential equation. (In latter lectures there will be more instance of this basic technique).

Suppose we knew x as a function of y :

$$x = H(y) \tag{3}$$

If we solved this equation for y we'd end up with a particular function $y(x)$ of x such that

$$x = H(y(x)) \tag{4}$$

would be a mathematical identity. We could also differentiate both sides of (4) with respect to x (employing the Chain Rule on the right) to get

$$1 = \frac{dH}{dy}\frac{dy}{dx}$$

Or

$$\frac{dy}{dx} = \frac{1}{H'(y)} \tag{5}$$

Notice that the functional form of (5) is just like our differential-equation-of-the-day (2):

$$\frac{dy}{dx} = \text{some function of } y \text{ alone}$$

Indeed, to solve (2), we will just reverse the steps that led us to (5). Thus, expecting a relationship of the form (3), we set

$$\frac{dy}{dx} = F(y) = \frac{1}{H'(y)}$$

This tells us that

$$H'(y) = \frac{1}{F(y)}$$

But this means that $H(y)$ has to be an anti-derivative of $\frac{1}{F(y)}$. Thus,

$$H(y) = \int \frac{1}{F(y)} dy + C$$

Furthermore, we can conclude that the solution $y(x)$ we want can be obtained by computing $H(y)$ and then solving

$$x = \int \frac{1}{F(y)} dy + C \tag{6}$$

for y in terms of x.

0.1. Mneumonic Method. Here's a quick way of recovering the formula (6) directly from the differential equation (2). Ok, we start with

$$F(y) = \frac{dy}{dx}$$

Multiply both sides by dx to get

$$F(y)\, dx = dy$$

Divide both sides by $F(y)$

$$dx = \frac{1}{F(y)} dy$$

Integrate both sides adding an arbitrary constant of integration to one side or the other

$$\int dx = \int F(y)\, dy + C \qquad \Rightarrow \qquad x = \int F(y)\, dy + C$$

0.2. Example.

EXAMPLE 5.1. Solve

$$\frac{dy}{dx} = \cos^2(y).$$

- We'll employ the mneumonic method:

$$\begin{aligned} dy &= \cos^2(y)\, dx \\ \frac{1}{\cos^2(y)} dy &= dx \\ \int dx &= \int \sec^2(y)\, dy + C \\ x &= \tan(y) + C \end{aligned}$$

  We now know that the solution y to the differential equation is related to x via and equation of the form

$$\tan(y) = x - C \tag{7}$$

  Solving (algebraically) for y we get

$$y(x) = \tan^{-1}(x - C) \tag{6}$$

1. Implicit Solutions and Explicit Solutions

Let me recap the basic method for solving differential equations of the form

$$\frac{dy}{dx} = F(y)$$

- We think of such an equation as *arising from an algebraic equation* of the form
$$x = H(y) .$$
- To get this algebraic equation we can employ the *mneumonic method* to transform the differential equation to the relation
$$x = \int \frac{dy}{F(y)} + C$$
- Once we compute the integral on the right hand side we get an explcit function of y on the right. The resulting equation is called the *implicit solution* of the differential equation. It is not really the complete solution because we have not yet isolated y as a function of x (rather we have x as a function of y).
- We then solve, algebraically, the implicit solution to get y as a function of x. This function of x is what we call the *explicit solution* the differential equation.

For example, in the preceding example,

$$x = \tan(y) + C$$

is the implicit solution of $\frac{dy}{dx} = \cos^2(x)$; and

$$y = \tan^{-1}(x - C)$$

is the explicit solutionl

2. Separable Equations

The technique we used to solve

$$\frac{dy}{dx} = F(y)$$

is readily generalized to solve an even wider class of differential equations.

Definition 5.2. *A first order differential equation is said to be* ***separable*** *if it can be written in the form*

$$N(y)\frac{dy}{dx} = M(x) \quad . \tag{9}$$

Note that the first term depends only on x and the second term depends only on y and y'. In other words, a differential equation is separable if we can separate the x-dependent terms from the y-dependent terms.

Our method for constructing solutions of (9) will be a natural extension of the method we used to solve (2). Again the main idea is to replace the differential equation with an equivalent algebraic relation (the so-called implicit solution) and then solve this algebraic equation to get y as a function of x. The mnemonic method for getting this form of the implicit solution works in the present case just as in the last case:

- Multiply both sides of (9) by dx to get a relationship between differentials
$$N(y)\,dy = M(x)dx \tag{10}$$
- Integrate both sides of this relation while simultaneously introducing a constant of integration to one side
$$\int N(y)\,dx = \int M(x)\,dx + C \tag{11}$$

- Compute

$$H_1(x) = \int M(x)\,dx \tag{12a}$$

$$H_2(y) = \int N(y)\,dy \tag{12b}$$

- Plugging replacing the integrals in (11) with the results of the computations in (12a) and (12b); we end up the relation

(13) $$H_2(y) = H_1(x) + C$$

This algebraic equation will be the *implicit solution.*

- The (explicit) solution to the differential equation is finally obtained by solving the implicit solution to get y as an explicit function of x.

2.1. Examples.

EXAMPLE 5.3.

(14) $$x^2 + y\frac{dy}{dx} = 1$$

- First, we need to get this equation in **exactly** the same form as (9). So we move the 1 on the right hand side to the left hand side.

(15) $$\left(x^2 - 1\right) + (y)\frac{dy}{dx} = 0$$

In equation (15), we have collected the x-dependent terms

$$M(x) = -x^2 + 1$$

from the y-dependent terms

$$N(y)\frac{dy}{dx} = y\frac{dy}{dx}$$

(Notice that the 1 that originally appeared on right side could not be brought into the function $N(y)$ since it is not being mulitplied by $\frac{dy}{dx}$; it has to go into the $M(x)$ term.)

Now we multiply both sides of (15) by dx to get the differential relation

(16) $$ydy = \left(1 - x^2\right)dx$$

Integrating both sides of (16) and adding in by hand a constant of integration to one side we get

$$\int ydy = \int \left(1 - x^2\right)dx + C$$

or

$$\frac{1}{2}y^2 = x - \frac{1}{3}x^3 + C$$

or

(17) $$y^2 = \frac{2}{3}x^3 - 2x + 2C$$

Equation (17) is our **implicit solution** to (14). To get the **explicit solution** we now solve (17) for y:

(18) $$y = \sqrt{\frac{2}{3}x^3 - 2x + 2C}$$

Finally, since C is an arbitrary constant, so is $2C$. So we may as well express the explicit solution as

$$y = \sqrt{\frac{2}{3}x^3 - 2x + C}$$

EXAMPLE 5.4.

$$p(x)y + \frac{dy}{dx} = 0$$

- The first thing we need to do is to get this differential into explicitly separable form. Dividing both sides by y we get

$$p(x) + \frac{1}{y}\frac{dy}{dx} = 0$$

This is nearly in separated form

$$N(y)\frac{dy}{dx} = M(x)$$

with

$$\begin{aligned} M(x) &= -p(x) \\ N(y) &= \frac{1}{y} \end{aligned}$$

We precede as before:

$$\begin{aligned} &\Rightarrow \quad \frac{1}{y}dy = -p(x)\,dx \\ &\Rightarrow \quad \int \frac{1}{y}dy = \int -p(x)\,dx + C \\ &\Rightarrow \quad \ln|y| = -\int p(x)\,dx + C \\ &\Rightarrow \quad y = \exp\left(-\int p(x)\,dx + C\right) \end{aligned}$$

Note that in this example, our method went through even though the function $p(x)$ was not stated explicitly.

Example 5.5.

$$y' = \frac{y^2}{x} \tag{5.1}$$

After multiplying both sides by $\frac{x}{y^2}$, this equation can also be rewritten as

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x} \quad ;$$

or

$$\frac{dy}{y^2} = \frac{dx}{x} \quad .$$

Integrating the left hand side with respect to x and the right hand side with respect to y yields

$$\int \frac{dy}{y^2} = \int \frac{dx}{x} + C$$

or

$$-\frac{1}{y} = \ln|x| + C$$

$$\frac{1}{y} = -\ln|x| - C$$

or

$$y(x) = \frac{1}{-C - \ln|x|} \quad .$$

or even

$$y(x) = \frac{1}{C - \ln|x|}$$

(N.B. $-C$ is just as much an arbitrary constant as C itself - in the last step, we have just chosen the simplest way to express the arbitrariness in the solution.) The equation above represents the general solution of (5.1). □