LECTURE 2

Solutions and Classification of Differential Equations

1. Solutions of Ordinary Differential Equations

DEFINITION 2.1. *A **solution** of an (ordinary) differential equation*

$$F\left(x, f, f', f'', \ldots, f^{(n)}\right) = 0$$

on the interval $[a,b]$ is a function ϕ such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist for all $x \in [a,b]$ and

$$F\left(x, \phi(x), \phi'(x), \phi''(x), \ldots, \phi^{(n)}(x)\right) = 0$$

for all $x \in [a,b]$.

EXAMPLE 2.2. A solution of

$$\frac{df}{dx} = -\lambda f$$

on the open interval $(-\infty, +\infty)$ is the function

$$\phi = Ae^{-\lambda x} \quad .$$

EXAMPLE 2.3. A solution of

$$\frac{d^2 f}{dx^2} + \omega^2 f = 0$$

is

$$\phi_1 = A \sin \omega x \quad .$$

But note that

$$\phi_2 = B \cos \omega x$$

is also a solution.

2. Classification of Differential Equations

The purpose of this course is to teach you some basic techniques for "solving" differential equations and to study the general properties of the solutions of differential equations. I put the word solving in quotes because we will rarely find the solutions of a differential equation by systematically manipulating an equation until the unknown function is isolated. More often we shall find the solutions by first constructing, or even guessing, some solutions and then applying some general theorems to verify that every solution can be found in this manner. Indeed, there exists no general algorithm for solving differential equations. However, if one knows how to construct a solution for one differential equation, one can often generalize the technique to a whole class of differential equations. For this reason the first step in solving a differential equation is to identify its general class.

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2.1. Partial vs. Ordinary Differential Equations. An **ordinary differential equation** is an equation relating a function and its derivatives with respect to a single variable.

A **partial differential equation** is one relating a function of more than one variable to its partial derivatives.

EXAMPLE 2.4.

$$\frac{d\phi}{dx}(x) + g(x)\phi(x) = 0$$

is an ordinary differential equation, while

$$\frac{\partial \Phi}{\partial x}(x,y) + G(x,y)\Phi(x,y) = 0$$

is a partial differential equation.

In this course we shall deal almost exclusively with ordinary differential equations.

2.2. The Order of a Differential Equation. The **order** of a differential equation is the highest number of derivatives appearing in the equation. Thus,

$$2x^2 \frac{df}{dx} + \sqrt{x}\frac{d^3f}{dx^3} + e^x = 2$$

is an third order, ordinary, differential equation, while

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial t^2} = 0$$

is a second order, partial, differential equation.

2.3. Linear vs Non-Linear Differential Equations. An ordinary or partial differential equation is said to be **linear** if it is linear in the "unknowns" f , $\frac{df}{dx}$, $\frac{d^2f}{dx^2}$, etc.. Thus, a general, linear, ordinary, n^{th} order, differential equation would be one of the form

$$a_n(x)\frac{d^nf}{dx^n}(x) + a_{n-1}(x)\frac{d^{n-1}f}{dx^{n-1}}(x) + \cdots + a_1(x)\frac{df}{dx}(x) + f(x) = g(x) \quad .$$

It is important to note that the functions $a_n(x), \ldots, a_1(x), g(x)$ need not be linear functions of x. The following two examples should convey the general idea.

EXAMPLE 2.5.

$$x^2 \frac{\partial f}{\partial x} + z\frac{\partial^2 f}{\partial y^2} = e^{zxy}$$

is a 2nd order, linear, partial, differential equation.

EXAMPLE 2.6.

$$\frac{d^3f}{dx^3} + x^2\frac{df}{dx} + f^2 = 1$$

is a non-linear, ordinary, differential equation of order 3. The equation is non-linear arises because of the presence of the term f^2 which is a quadratic function of the unknown function f.

3. Systems of Differential Equations

A system of algebraic equations is simply a set of equations

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\begin{aligned}
a_1 x + a_2 y + \cdots + a_n z &= a_0 \\
b_1 x + b_2 y + \cdots + b_n z &= b_0 \\
&\vdots \\
s_1 x + s_2 y + \cdots + s_n z &= s_0
\end{aligned}
$$

which are all presumed to hold simultaneously.

Similarly, one can consider systems of differential equations; for example,

$$
\begin{aligned}
f'' + f'g + x^2 g &= 0 \\
g' + x^3 &= h'' \\
\sqrt{x}\,f' + h' - x^2 g &= 0
\end{aligned}
$$

and look for a set of functions f,g,h which satisfy all these equations simultaneously.