1. Find the first four terms of the Taylor expansion about $x = 0$ of the solution of

(1) \quad y' = y^2

(2) \quad y(0) = 1

- The Taylor expansion of the solution $y(x)$ about $x = 0$ is

\[
y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(0) x^n
\]

\[
= y(0) + y'(0)x + \frac{1}{2} y''(0)x^2 + \frac{1}{6} y'''(0)x^3 + \cdots
\]

To write down the first four terms of this expansion explicitly, we need to calculate $y(0)$, $y'(0)$, $y''(0)$, and $y'''(0)$. The initial condition $y(0) = 1$ already gives us $y(0)$. To calculate $y'(0)$, we simply evaluate the differential equation at $x = 0$:

(3) \quad y'(0) = y^2\big|_{x=0} = (y(0))^2 = (1)^2 = 1.

To calculate $y''(0)$ we first differentiate the differential equation

(4) \quad y''(x) = \frac{d}{dx} y'(x) = \frac{d}{dx} (y^2(x)) = 2y(x)y'(x)

Evaluating this equation at $x = 0$ yields

(5) \quad y''(0) = 2y(0)y'(0) = 2(1)(1) = 2

where we have used (2) and (3) to reduce the right hand side to a pure number. To calculate $y'''(0)$ we differentiate (4) and evaluate the right hand side at $x = 0$ using (2), (3), and (5):

\[
y'''(0) = \frac{d}{dx} (2y(x)y'(x))\big|_{x=0}
\]

\[
= (2y'(x)y'(x) + 2y(x)y''(x)\big|_{x=0}
\]

\[
= 2 (y'(0))^2 + 2y(0)y'''(0)
\]

\[
= 2(1)^2 + 2(1)(2)
\]

\[
= 6
\]

Hence,

\[
y(x) = y(0) + y'(0)x + \frac{1}{2} y''(0)x^2 + \frac{1}{6} y'''(0)x^3 + \cdots
\]

\[
= 1 + x + \frac{1}{2}(2)x^2 + \frac{1}{6}(6)x^3 + \cdots
\]

\[
= 1 + x + x^2 + x^3 + \cdots
\]

2. Find the first four terms of the Taylor expansion about $x = 1$ of the solution of

(6) \quad y' = x^2

\quad y(1) = 1

- In this problem we seek a Taylor expansion of our solution $y(x)$ about $x = 1$, and so we set

\[
y(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x - 1)^n
\]

\[
y(1) + y'(1)(x - 1) + \frac{1}{2} y''(1)(x - 1)^2 + \frac{1}{6} y'''(1)(x - 1)^3 + \cdots
\]
and try to calculate $y(1)$, $y'(1)$, $y''(1)$, and $y'''(1)$. The initial condition gives us
\begin{equation}
(7) \quad y(1) = 1.
\end{equation}
Evaluating the differential equation at $x = 1$ gives us
\begin{equation}
(8) \quad y'(1) = x^2\bigg|_{x=1} = (1)^2 = 1.
\end{equation}
Differentiating the differential equation and evaluating the result at $x = 1$ yields
\begin{equation}
(9) \quad y''(1) = \frac{d}{dx} y'(x)\bigg|_{x=1} = \frac{d}{dx} x^2\bigg|_{x=1} = 2x\bigg|_{x=1} = 2.
\end{equation}
Finally, differentiating the differential equation twice and evaluating the result at $x = 1$ yields
\begin{equation}
(10) \quad y'''(1) = \frac{d^2}{dx^2} y'(x)\bigg|_{x=1} = \frac{d^2}{dx^2} x^2\bigg|_{x=1} = 2|_{x=1} = 2.
\end{equation}
Plugging (7), (8), (9), and (10) into (6) yields
\begin{equation}
\begin{split}
y(x) &= 1 + (1)(x - 1) + \frac{1}{2}(2)(x - 1)^2 + \frac{1}{6}(2)(x - 1)^3 + \cdots \\
&= 1 + (x - 1) + (x - 1)^2 + \frac{1}{3}(x - 1)^3 + \cdots
\end{split}
\end{equation}
3. Solve the following differential equation using Separation of Variables.
\begin{equation}
\frac{dy}{dx} = x e^y
\end{equation}
- We can explicitly separate the $x$-dependence from the $y$-dependence in this equation by multiplying both sides by $e^{-y} dx$:
\begin{equation}
e^{-y} dx \left( \frac{dy}{dx} = x e^y \right) \Rightarrow e^{-y} dy = x dx
\end{equation}
Integrating both sides of the resulting equation yields
\begin{equation}
-e^{-y} = \int e^{-y} dy = \int x dx = \frac{1}{2} x^2 + C
\end{equation}
or
\begin{equation}
e^{-y} = C' - \frac{1}{2} x^2
\end{equation}
or
\begin{equation}
-y = \ln \left| C' - \frac{1}{2} x^2 \right|
\end{equation}
or
\begin{equation}
y(x) = - \ln \left| C' - \frac{1}{2} x^2 \right|.
\end{equation}
4. Solve the following differential equation using Separation of Variables.
\begin{equation}
\frac{dx}{dt} = t x e^t
\end{equation}
- Multiplying both sides of this equation by $\frac{1}{x} dt$ yields
\begin{equation}
\frac{dx}{x} = t e^t dt
\end{equation}
and so the equation is separable. Integrating both sides we get
\begin{equation}
\ln |x| = \int \frac{dx}{x} = \int t e^t dt = \int \frac{1}{2} e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^t + C
\end{equation}
where we have used the substitutions \( u = t^2, \ du = 2tdt \), to carry out the integration over \( t \). Solving the extreme sides of this equation for \( x \) yields

\[
x(t) = \exp \left( \frac{1}{2}e^{t^2} + C \right) = \tilde{C} \exp \left( \frac{1}{2}e^{t^2} \right)
\]

5. Solve the following differential equation using Separation of Variables.

\[
x^2y' + e^y = 0
\]

- Taking the \( e^y \) term to the right hand side and then multiplying by \( x^{-2}e^{-y} \dx \) yields

\[
e^{-y}dy = -\frac{dx}{x^2}
\]

Integrating both sides of this equation yields

\[
-e^{-y} = \int e^{-y}dy = \int -\frac{dx}{x^2} = - \left( -\frac{1}{x} \right) + C
\]

Solving this equation for \( y \) yields

\[
y(x) = \ln \left| C' - \frac{1}{x} \right|.
\]

6. Solve the following differential equation using Separation of Variables.

\[
xy' = e^x
\]

- Multiplying both sides by \( \dx \) yields

\[
y \dy = e^x \dx.
\]

Integrating both sides of this equation produces

\[
\frac{1}{2}y^2 = \int y \dy = \int e^x \dx = e^x + C
\]

Solving the extreme sides of this equation for \( y \) yields

\[
y(x) = \pm \sqrt{2e^x + C'}.
\]