Math 2233
Homework Set 1

1. 

(a) Plot the direction field for the differential equation

\[ y' = y^{4/5}. \]

- Below is the Maple plot produced by the commands
  1. `with(DEtools);`
  2. `dfieldplot(diff(y(x)=y^(4/5),[y],x=0..4, y=-2..2);`

(b) Sketch the solution that satisfies \( y(0) = 2 \)
(c) Sketch the solution that satisfies $y(0) = 1$. 

2. Use Maple to generate direction fields for the following differential equations on the given interval.

(a) $y' = 2y; -2 \leq x \leq 2, 0 \leq y \leq 6$. 

- The plot below was produced via the Maple commands 
  > with(DEtools);
  > dfieldplot(diff(y(x)=2*y,[y],x=-2..2, y=0..6); 

(b) $y' = 3y(1 - y); -2 \leq x \leq 3, -3 \leq y \leq 4$. 

- The plot below was produced using the Maple commands 
  > with(DEtools);
  > dfieldplot(diff(y(x)=3*y*(1-y),[y],x=-2..3, y=-3..4);
3. For the differential equation in Problem 2(b), what can you say about the behavior of solutions as $x \to \infty$?

- By virtue of the differential equation

$$y' = 3y(1 - y)$$

we see that the value of $y$ determines whether a solution $y(x)$ is increasing, decreasing, or constant (that is to say, when $y'(x)$ is positive, negative, or zero). $< 0$ are decreasing.

1. If $y > 1$, then $y' < 0$, and so solutions in the region $y > 1$ are always decreasing.
2. If $y = 1$, then $y' = 0$, and so solutions for which $y = 1$ are constant and so stick to the line $y = 1$.
3. If $0 < y < 1$, then $y' > 0$, and so solutions in the region $0 < y < 1$ are increasing.
4. If $y = 0$, then $y' = 0$, and so solutions for which $y = 0$ are constant, and so stick to the line $y = 0$.
5. If $y < 0$, then $y' < 0$, and so solutions in the region $y$.

We thus have four basic classes of solutions. The solutions in the region $y > 1$ are always decreasing, but they cannot pass through the line $y = 1$, since that corresponds to a constant solution. These solutions asymptotically approach the line $y = 1$ as $x \to \infty$. The solutions in the region $0 < y < 1$ are always increasing, but they cannot increase past the line $y = 1$ because again $y(x) = 1$ is a constant solution. These solutions must also asymptotically approach the line $y = 1$ (from below). The solutions in the region $y < 0$ are always decreasing, these solutions must tend to $-\infty$ as $x \to \infty$.

4. Using the Euler Method, find an approximate value for $y(1)$ for the following initial value problem (take $h = \Delta x = 0.02$):

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

- We'll do this problem by hand. In accordance with the initial condition $y(0) = 1$ we set $x_0 = 0$ and $y_0 = 1$. To get the next pair of points on the solution curve we use the fact that the slope of the best straight line fit to the solution curve at $(x_0, y_0) = (0, 1)$ must be

$$m_0 = \left. \frac{dy}{dx} \right|_{(x_0, y_0)} = x_0 + y_0 = 0 + 1 = 1.$$  

Setting

$$x_1 = x_0 + \Delta x = 0 + 0.2 = 0.2$$
we get an approximate value for \( y_1 = y(x_1) \) using the formula \( \Delta y = m \Delta x \); (for the case at hand, this formula implies \( y_1 = y_0 + m_0 \Delta x \))

\[
y_1 = y_0 + m_0 \Delta x \\
= 1 + (1)(0.2) \\
= 1.2
\]

Thus the next pair of points on the solution curve should be \((x_1, y_1) = (0.2, 1.2)\). Now we calculate the slope of the best straight line fit to the solution that passes through the point \((x_1, y_1)\):

\[
m_1 = \frac{dy}{dx}\bigg|_{(x_1, y_1)} = x_1 + y_1 = 0.2 + 1.2 = 1.4
\]

Taking \(x_2 = x_1 + \Delta x = 0.4\), we calculate \(y_2\)

\[
y_2 = y_1 + m_1 \Delta x \\
= 1.2 + (1.4)(0.2) \\
= 1.48
\]

We continue in this manner:

\[
m_2 = x_2 + y_2 = 0.4 + 1.48 = 1.88 \\
x_3 = x_2 + \Delta x = 0.4 + 0.2 = 0.6 \\
y_3 = y_2 + m_2 \Delta x = 1.48 + (1.88)(0.2) = 1.856 \\
m_3 = x_3 + y_3 = 0.6 + 1.856 = 2.456 \\
x_4 = x_3 + \Delta x = 0.6 + 0.2 = 0.8 \\
y_4 = y_3 + m_3 \Delta x = 1.856 + (2.456)(0.2) = 2.3472 \\
m_4 = x_4 + y_4 = 0.8 + 2.3472 = 3.1472 \\
x_5 = x_4 + \Delta x = 0.8 + 0.2 \\
y_5 = y_4 + m_4 \Delta x = 2.3472 + (3.1472)(0.2) = 2.9766
\]

Thus \(y(1) = y(x_5) = y_5 = 2.9766\).

5. Using the Euler Method, find an approximate value for \(y(1)\) for the following initial value problem (take \(h = \Delta x = 0.1\)):

\[
\frac{dy}{dx} = xe^y \quad , \quad y(0) = 0
\]

- For this problem we’ll resort to Maple. The algorithm used in the previous problem can generalized and applied to this problem as follows.

  (i) In accordance with the initial condition \(y(0) = 0\) set \(x_0 = 0\) and \(y_0 = 0\).

  (ii) Set \(\Delta x = 0.1\).

  (iii) We divide the interval \([0, 1]\) into \(10 = \frac{1-0}{\Delta x}\) subintervals by setting

\[
x_n = 0 + n \Delta x \quad n = 0, 1, \ldots, 10
\]

  (iv) We set

\[
y_n = y(x_n) \quad , \quad n = 0, 1, \ldots, 10
\]

  Our goal is to calculate \(y_{10} = y(x_{10}) = y(1)\).

  (v) The slope of the solution passing through the point \((x_i, y_i)\) is determined by the right hand side of the differential equation evaluated at \((x_i, y_i)\):

\[
m_i = \frac{dy}{dx}\bigg|_{(x_i, y_i)} = xe^y\bigg|_{(x_i, y_i)} = x_i e^{y_i}
\]
(vi) Given a point \((x_i, y_i)\) on the solution curve we can approximate \(y_{i+1}\) by using the formula

\[ m_i = \frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{\Delta x} \]

or, after solving for \(y_{i+1}\) is equivalent to

\[ y_{i+1} = y_i + m_i \Delta x = y_i + (x_i e^{y_i}) \Delta x \]

Thus, \(y_{i+1}\) is completely determined by the preceding values \(x_i\) and \(y_i\) of, respectively, \(x\) and \(y\).

(vii) Now we can explicitly state the algorithm by which we will calculate \(y(1) = y_{10}\).
We set \(x_0 = 0\), \(y_0 = 0\) and \(\Delta x = 0.1\). For each \(i\) from 0 to 9 we will successively calculate

\[ x_{i+1} = 0 + (i + 1) \Delta x \]
\[ y_{i+1} = y_i + (x_i e^{y_i}) \Delta x. \]

Below is the Maple routine that accomplishes this.

```maple
> x[0] := 0;
> y[0] := 0;
> dx := 0.1;
> for i from 0 to 9 do
> x[i+1] := (i+1)*dx;
> y[i+1] := y[i] + x[i]*exp(y[i]):
> od:
> y[10];
```

The result calculated by Maple is \(y_{10} = 0.5653922980\).