LEC TUR E 28

The L a p l ace Tra n sfo rm

Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a “nice” (to be qualified later) function of \( x \). The Laplace transform \( \mathcal{L}[f] \) of \( f \) is the function from \( \mathbb{R} \) to \( \mathbb{R} \) defined by

\[
(28.1) \quad \mathcal{L}[f](s) = \int_{0}^{\infty} e^{-sx} f(x) \, dx .
\]

We note that in the formula above, \( s \) is the variable upon which the Laplace transform \( \mathcal{L}[f] \) depends.

Example 28.1. If

\[
(28.2) \quad f(x) = ax \quad \text{then}
\]

then

\[
(28.3) \quad \mathcal{L}[f](s) = \int_{0}^{\infty} ax e^{-sx} \, dx = \lim_{N \to \infty} \left( \int_{0}^{N} e^{-sx} ax \, dx \right)
\]

Note that this result really only makes sense for \( s > 0 \); for \( x \leq 0 \) the integral does not converge.

Example 28.2. If

\[
(28.4) \quad f(x) = \sin(ax) \quad \text{then, integrating by twice by parts,}
\]

then, integrating by twice by parts,

\[
(28.5) \quad \mathcal{L}[f](s) = \int_{0}^{\infty} \sin(ax) e^{-sx} \, dx = \lim_{N \to \infty} \left( e^{-sx} \frac{1}{a} \cos(ax) \right) \bigg|_{0}^{N} + \frac{s}{a} \int_{0}^{\infty} e^{-sx} \cos(ax) \, dx
\]

we find

\[
(28.6) \quad \mathcal{L}[f](s) = \frac{a}{1 + \frac{s^2}{a^2}} = \frac{a}{a^2 + s^2} .
\]

(If \( s \leq 0 \), the integral on the first line does not converge, so \( \mathcal{L}[f](s) \) is only defined for \( s > 0 \).)

Example 28.3. If \( f(x) = e^{bx} \), then

\[
(28.7) \quad \mathcal{L}[f] = \int_{0}^{\infty} e^{bt} e^{-sx} \, dt = \int_{0}^{\infty} e^{(b-s)t} \, dt = \frac{1}{b-s} e^{(b-s)t} \bigg|_{0}^{\infty} = \frac{1}{b-s} \quad (\text{if } s > b)
\]

(If \( s \leq b \) then the integral does not converge.)

The following theorem explains under what conditions we can expect the Laplace transform of a function \( f(x) \) to exist.
28. The Laplace Transform

Theorem 28.4. Suppose that \( f(x) \) is a piecewise continuous function for \( 0 \leq t \leq A \) and there exist constants \( K,a,M \) such that
\[
|f(t)| \leq Ke^{at}, \quad \forall t > M > 0.
\]
Then the Laplace transform \( \mathcal{L}[f] \) defined by
\[
\mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st} \, dt
\]
exists for all \( s > a \).

The condition (28.8) is a rather moderate “growth” condition on the function \( f(x) \); it says that for large enough \( t \), \( |f(t)| \) grows no faster than an exponential function of the form \( Ke^{at} \). This condition is easily satisfied by any polynomial function of \( x \).

Theorem 28.5. Properties of the Laplace Transform

(i) Suppose \( f_1(x) \) and \( f_2(x) \) are two functions satisfying the hypotheses of Theorem 6.2. Then if \( g(x) = c_1f_1(x) + c_2f_2(x) \), \( \mathcal{L}[g] \) exists and
\[
\mathcal{L}[g](s) = c_1\mathcal{L}[f_1](s) + c_2\mathcal{L}[f_2](s)
\]
(ii) Suppose that \( f \) is continuous and that both \( f \) and its derivative \( f' \) satisfy the hypotheses of Theorem 6.2. Then \( \mathcal{L}[f'](s) \) exists for \( s > a \) and moreover
\[
\mathcal{L}[f'](s) = s\mathcal{L}[f] - f(0)
\]
(iii) Suppose that \( f \) and its derivatives \( f', \ldots, f^{(n-1)} \) are continuous and satisfy the hypotheses of Theorem 6.2. Then \( \mathcal{L}[f^{(n)}](s) \) exists for \( s > a \) and
\[
\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)
\]

Proof of (i).

This follows from the linearity property integration:
\[
\mathcal{L}[c_1f_1 + c_2f_2](s) = \int_0^\infty (c_1f_1(x) + c_2f_2(x))e^{-sx} \, dx
\]
\[
= c_1\int_0^\infty f_1(x)e^{-sx} \, dx + c_2\int_0^\infty f_2(x)e^{-sx} \, dx
\]
\[
= c_1\mathcal{L}[f_1](s) + c_2\mathcal{L}[f_2](s)
\]

Proof of (ii).

Integrating by parts one finds
\[
\mathcal{L}[f'](s) = \int_0^\infty e^{-st}f'(t) \, dt
\]
\[
= e^{-st}f(t)|_0^\infty - \int_0^\infty (-se^{-st})f(t) \, dt
\]
\[
= 0 - f(0) + s\int_0^\infty e^{-st}f(t) \, dt
\]
\[
= s\mathcal{L}[f] - f(0)
\]

Similarly, (iii) is proved by integrating by parts repeatedly.