1. (15 pts) Use a change of variable to solve the following first order equation. (Hint: note that it is homogeneous of degree zero).

\[ y' = \frac{xy + y^2}{x^3} \]

\[ \frac{dy}{dx} = \left( \frac{y}{x} \right) + \left( \frac{y}{x} \right)^2 \]

Under the substitution \( u = y/x \), or equivalently \( y = xu \), we have

\[ \frac{dy}{dx} = u + x \frac{du}{dx} \]

Therefore, our original differential equation is equivalent to

\[ u + x \frac{du}{dx} = \frac{dy}{dx} = \left( \frac{y}{x} \right) + \left( \frac{y}{x} \right)^2 = u + u^2 \]

 Cancelling the terms \( u \) that appear on the extreme sides of this equation yields

\[ x \frac{du}{dx} = u^2 \]

or

\[ \frac{du}{u^2} = \frac{dx}{x} \]

Integrating both sides yields

\[ -\frac{1}{u} = \ln |x| + C \]

Now we recall \( u = y/x \) to get

\[ -\frac{x}{y} = \ln |x| + C \]

or

\[ y(x) = -\frac{x}{\ln |x| + C} \]

2. Given that \( y_1(x) = x \) and \( y_2(x) = x^3 \) are solutions to \( x^2y'' - 3xy' + 3y = 0 \)

(a) (5 pts) Show that the functions \( y_1(x) \) and \( y_2(x) \) are linearly independent.

\[ W[y_1, y_2](x) = (x)(3x^2) - (1)(x^3) = 2x^3 \neq 0 \]

so \( y_1(x) \) and \( y_2(x) \) must be linearly independent. □
(b) (5 pts) Write down the general solution.

\[ y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2 x^3 \]

(c) (5 pts) Find the solution satisfying the initial conditions \( y(1) = 1, y'(1) = 1 \).

\[ 1 = y(1) = c_1 (1) + c_2 (1^2) = c_1 + c_2 \]
\[ 1 = y'(1) = c_1 + c_2 (3x^2) \big|_{x=1} = c_1 + 3c_2 \]

Subtracting the second equation from the first yields
\[ 0 = 0 - 2c_2 \implies c_2 = 0 \]

But then the first equation implies \( c_1 = 1 \). Thus,
\[ c_1 = 1 \]
\[ c_2 = 0 \]

and the solution satisfying the given initial conditions is
\[ y(x) = x \]

3. (10 pts) Given that \( y_1(x) = x^2 \) is one solution of \( x^2 y'' - 4xy' + 6y = 0 \), use Reduction of Order to determine the general solution.

- Putting the differential equation in standard form we see that the term \( p(x) \) in the Reduction of Order formula is \( p(x) = -\frac{4}{x^2} \). Thus, a second linearly independent solution is

\[ y_2(x) = y_1(x) \int \frac{1}{(y_1(s))^2} \exp \left[ -\int \frac{p(t)dt}{t} \right] ds \]
\[ = x^2 \int_1^x \frac{1}{s^4} \exp \left[ +4 \int \frac{dt}{t} \right] ds \]
\[ = x^2 \int_1^x s^{-4} \exp (4 \ln |s|) ds \]
\[ = x^2 \int_1^x s^{-4} ds \]
\[ = x^2 \int_1^x ds \]
\[ = x^3 \]

Now that we have two linearly independent solutions, we can write down the general solution:
\[ y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x^2 + c_2 x^3 \]

4. Determine the general solution of the following differential equations.

(a) (5 pts) \( y'' - 3y' - 3y = 0 \)
- The characteristic equation for this homogeneous linear equation with constant coefficients is
  \[ \lambda^2 - 3\lambda - 3 = 0. \]

  The roots of this equation are determined by the Quadratic Formula
  \[ \lambda = \frac{3 \pm \sqrt{9 - (4)(-3)}}{2} = \frac{3 \pm \sqrt{33}}{2} \]

  So
  \[ y_1(x) = e^{\frac{3 + \sqrt{33}}{2}x} \]
  \[ y_2(x) = e^{\frac{3 - \sqrt{33}}{2}x} \]

  and the general solution is
  \[ y(x) = c_1 e^{\frac{3 + \sqrt{33}}{2}x} + c_2 e^{\frac{3 - \sqrt{33}}{2}x} \]

(b) (5 pts) \( y'' + 10y' + 25y = 0 \)

- The characteristic equation for this homogeneous linear equation with constant coefficients is
  \[ 0 = \lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 \]

  and so we have a single root \( \lambda = -5 \). The two linearly independent solutions are thus
  \[ y_1(x) = e^{-5x} \]
  \[ y_2(x) = xe^{-5x} \]

  and the general solution is
  \[ y(x) = c_1 e^{-5x} + c_2 xe^{-5x} \]

(c) (5 pts) \( y'' - 4y' + 13y = 0 \)

- The characteristic equation for this homogeneous linear equation with constant coefficients is
  \[ \lambda^2 - 4\lambda + 13 = 0. \]

  Applying the Quadratic Formula we obtain
  \[ \lambda = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i \]

  We thus have a pair of complex roots. The corresponding linearly independent (real-valued) solutions are
  \[ y_1(x) = e^{2x} \cos(3x) \]
  \[ y_2(x) = e^{2x} \sin(3x) \]

  and so the general solution is
  \[ y(x) = c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x) \]

5. (10 pts) Explain in words how one could use Reduction of Order and the Method of Variation of Parameters to construct the general solution of \( x^2y'' - 2y = 3x^3 - 1 \), given that \( y_1(x) = x^2 \) is a solution of \( x^2y'' - 2y = 0 \). (It is not necessary to carry out any of the calculations.)
• First we’d use the Reduction of Order formula

\[ y_2(x) = y_1(x) \int^{x} \frac{1}{(y_1(s))^2} \exp \left[ - \int^{s} p(t) \, dt \right] \, ds \]

to compute a second linearly independent solution \( y_2(x) \) of \( x^2 y'' - 2y = 0 \). Next we’d use the Variation of Parameters formula

\[ y_p(x) = -y_1(x) \int^{x} \frac{y_2(s)g(s)}{W[y_1, y_2](s)} \, ds + y_2(x) \int^{x} \frac{y_1(s)g(s)}{W[y_1, y_2](s)} \, ds \]

with \( g(s) = (3x^2 - 1)/x^2 \) to construct a particular solution of \( x^2 y'' - 2y = 3x^2 - 1 \). We’d then have all the ingredients necessary to write down the general solution:

\[ y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) \]

6. Given that \( y_1(x) = e^x \) and \( y_2(x) = e^{2x} \) are solutions of \( y'' - 3y' + 2y = 0 \).

(a) (10 pts) Use the Method of Variation of Parameters to find a particular solution of

\[ y'' - 3y' + 2y = e^x \]

• Before applying the Variation of Parameters formula we note that

\[ W[y_1, y_2](x) = (e^x) (2e^{2x}) - (e^x) (e^{2x}) = e^{3x} \]

and

\[ y(x) = e^x \]

We can now compute a particular solution to the non-homogeneous equation

\[ y_p(x) = -y_1(x) \int^{x} \frac{y_2(s)g(s)}{W[y_1, y_2](s)} \, ds + y_2(x) \int^{x} \frac{y_1(s)g(s)}{W[y_1, y_2](s)} \, ds \]

\[ = -e^x \int^{x} \frac{e^{2x}(e^s)}{e^{3x}} \, ds + e^{2x} \int^{x} \frac{e^s(e^s)}{e^{3x}} \, ds \]

\[ = -e^x \int^{x} ds + e^{2x} \int e^{-s} ds \]

\[ = -xe^x + e^{2x} (-e^{-x}) \]

\[ = -xe^x - e^x \]

\[ \approx -xe^x \]

(The second term can be ignored since it is a solution to the corresponding homogeneous equation.)

(b) (10 pts) Find the solution satisfying \( y(0) = 0, y'(0) = 2 \).

• The general solution to the non-homogeneous problem in part (a) is

\[ y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) = -xe^x + c_1 e^x + c_2 e^{2x} \]

Applying the initial conditions yields

\[ 0 = y(0) = -(0)e^0 + c_1 e^0 + c_2 e^0 = c_1 + c_2 \]

\[ 2 = y'(0) = -e^x - xe^x + c_1 e^x + 2c_2 e^{2x} \bigg|_{x=0} = -1 + c_1 + 2c_2 \]

We thus have

\[ c_1 + c_2 = 0 \]

\[ c_1 + 2c_2 = 3 \]
Using the first equation we can substitute \( c_1 = -c_2 \) into the second to obtain
\[
c_2 = 3
\]
from which we can conclude also that \( c_1 = 3 \). Hence the solution to the initial value problem is
\[
y(x) = -xe^x - 3e^x + 3e^{2x}
\]

7. Find the general solution of the following Euler-type differential equations.

(a) (5 pts) \( x^2y'' + 4xy' + y = 0 \)

- Substituting \( y(x) = x^r \) into the differential equation yields
  \[
  (r(r - 1) + 4r + 1)x^r = 0
  \]
so we must have
\[
0 = r(r - 1) + 4r + 1 = r^2 + 3r + 1
\]
Applying the Quadratic Formula yields
\[
r = \frac{-3 \pm \sqrt{9 - 4(1)(1)}}{2} = \frac{-3 \pm \sqrt{5}}{2}
\]
Hence we have the following two linearly independent solutions
\[
y_1(x) = x^{\frac{-3 + \sqrt{5}}{2}}
\]
\[
y_2(x) = x^{\frac{-3 - \sqrt{5}}{2}}
\]
and the general solution is
\[
y(x) = c_1 x^{\frac{-3 + \sqrt{5}}{2}} + c_2 x^{\frac{-3 - \sqrt{5}}{2}}
\]

(b) (5 pts) \( x^2y'' - 5xy' + 9y = 0 \)

- Substituting \( y(x) = x^r \) into the differential equation yields
  \[
  (r(r - 1) - 5r + 9)x^r = 0
  \]
so we must have
\[
0 = r(r - 1) - 5r + 9 = r^2 - 6r + 9 = (r - 3)^2
\]
We thus have a single root \( r = 3 \) and thus two linearly independent solutions will be
\[
y_1(x) = x^{-3}
\]
\[
y_2(x) = x^{-3}\ln|x|
\]
The general solution is thus
\[
y(x) = c_1 x^{-3} + c_2 x^{-3}\ln|x|
\]

(c) (5 pts) \( x^2y'' - 5xy' + 13y = 0 \)

- Substituting \( y(x) = x^r \) into the differential equation yields
  \[
  (r(r - 1) - 5r + 13)x^r = 0
  \]
so we must have
\[
0 = r(r - 1) - 5r + 13 = r^2 - 6r + 13
\]
Applying the Quadratic Formula we see that

\[ r = \frac{6 \pm \sqrt{36 - (4)(13)}}{2} = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i \]

We thus have the following (real-valued) linearly independent solutions

\[ y_1(x) = x^3 \cos (2 \ln |x|) \]
\[ y_2(x) = x^3 \sin (2 \ln |x|) \]

and so the general solution is

\[ y(x) = c_1 x^3 \cos (2 \ln |x|) + c_2 x^3 \sin (2 \ln |x|) \]