Consider the differential equation
\begin{equation}
  y'' + p(x)y' + q(x)y = g(x)  \tag{20.1}
\end{equation}

Suppose \( y_1(x) \) and \( y_2(x) \) are two linearly independent solutions of the homogeneous problem corresponding to (20.1); i.e., \( y_1 \) and \( y_2 \) satisfy
\begin{equation}
  y'' + p(x)y' + q(x)y = 0  \tag{20.2}
\end{equation}

and
\begin{equation}
  W[y_1,y_2] \neq 0  \tag{20.3}
\end{equation}

We seek to determine two functions \( u_1(x) \) and \( u_2(x) \) such that
\begin{equation}
  y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)  \tag{20.4}
\end{equation}
is a solution of (20.1). To determine the two functions \( u_1 \) and \( u_2 \) uniquely we need to impose two (independent) conditions. First, we shall require (20.4) to be a solution of (20.1); and second, we shall require
\begin{equation}
  u_1'y_1 + u_2'y_2 = 0  \tag{20.5}
\end{equation}
(This latter condition is imposed not only because we need a second equation, but also make the calculation a lot easier.)

Differentiating (20.4) yields
\begin{equation}
  y'_p = u_1'y_1 + u_1y'_1 + u_2'y_2 + u_2y'_2  \tag{20.6}
\end{equation}
which because of (20.5) becomes
\begin{equation}
  y'_p = u_1'y_1 + u_2y'_2  \tag{20.7}
\end{equation}

Differentiating again yields
\begin{equation}
  y''_p = u_1'y'_1 + u_1y''_1 + u_2'y'_2 + u_2y''_2  \tag{20.8}
\end{equation}

We now plug (20.4), (20.7), and (20.8) into the original differential equation (20.1).
\begin{align}
  g(x) &= (u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p(x)(u_1'y_1' + u_2'y_2') + q(x)(u_1y_1 + u_2y_2)) \\
        &= u_1'y_1' + u_2'y_2' + u_1(y_1'' + p(x)y_1' + q(x)y_1) + u_2(y_2'' + p(x)y_2' + q(x)y_2)  \tag{20.9}
\end{align}

The last two terms vanish since \( y_1 \) and \( y_2 \) are solutions of (20.2). We thus have
\begin{equation}
  u_1'y_1 + u_2'y_2 = 0  \tag{20.10}
\end{equation}

\begin{equation}
  u_1'y_1' + u_2'y_2' = g  \tag{20.11}
\end{equation}

We now can now solve this pair of equations for \( u_1 \) and \( u_2 \). The result is
\begin{align}
  u_1' &= \frac{-y_2g}{W[y_1,y_2]} = \frac{-y_2g}{W[y_1,y_2]}  \\
  u_2' &= \frac{y_1g}{W[y_1,y_2]} = \frac{y_1g}{W[y_1,y_2]}  \tag{20.12}
\end{align}
(Note that division by $W(y_1, y_2)$ causes no problems since $y_1$ and $y_2$ were chosen such that $W(y_1, y_2) \neq 0$.)

Hence

$$u_1(x) = \int^x \frac{y_1(t) y_2(t) dt}{W[y_1, y_2](t)}$$
$$u_2(x) = \int^x \frac{y_1(t) y_2(t) dx'}{W[y_1, y_2](t)}$$

and so

$$y_p(x) = -y_1(x) \int^x \frac{y_1(t) y_2(t) dt}{W[y_1, y_2](t)} + y_2(x) \int^x \frac{y_1(t) y_2(t) dt}{W[y_1, y_2](t)}$$

is a particular solution of (20.1).

**EXAMPLE 20.1.** Find the general solution of

$$y'' - y' - 2y = 2e^{-x}$$

using the method of Variation of Parameters.

Well, the corresponding homogeneous problem is

$$y'' - y' - 2y = 0$$

This is a second order linear equation with constant coefficients whose characteristic equation is

$$\lambda^2 - \lambda - 2 = 0$$

The characteristic equation has two distinct real roots

$$\lambda = -1, 2$$

and so the functions

$$y_1(x) = e^{-x}$$
$$y_2(x) = e^{2x}$$

form a fundamental set of solutions to (20.16).

To find a particular solution to (20.15) we employ the formula (20.14). Now

$$g(x) = 2e^{-x}$$

and

$$W[y_1, y_2](x) = (e^{-x}) (2e^{2x}) - (-e^{-x}) (e^{2x}) = 3e^x$$

so

$$y_p(x) = -y_1(x) \int^x \frac{y_1(t) y_2(t) dt}{W[y_1, y_2](t)} + y_2(x) \int^x \frac{y_1(t) y_2(t) dt}{W[y_1, y_2](t)}$$

$$= -e^{-x} \int^x \frac{e^{2x}(2e^{2x}) dt}{3e^x} + e^{2x} \int^x \frac{e^{-x}(2e^{-x}) dt}{3e^x}$$

$$= -e^{-x} \int^x \frac{2e^{2x} dt}{3} + e^{2x} \int^x \frac{2e^{-3x} dt}{3}$$

$$= -\frac{2}{3}xe^{-x} - \frac{2}{3}e^{-x}$$

The general solution of (20.15) is thus

$$y(x) = y_p(x) + c_1 y_1(x) + c_2(x)$$

$$= -\frac{2}{3}xe^{-x} + (c_1 - \frac{2}{3}) e^{-x} + c_2 e^{2x}$$

$$= -\frac{2}{3}xe^{-x} + C_1 e^{-x} + C_2 e^{2x}$$

where we have absorbed the $-\frac{2}{3}$ in the second line into the arbitrary parameter $C_1$. 