LECTURE 17

Homogeneous Equations with Constant Coefficients, Cont’d

Recall that the general solution of a $2^{nd}$ order linear homogeneous differential equation

\begin{equation}
L[y] = y'' + p(x)y' + q(x)y = 0
\end{equation}

is always a linear combination

\begin{equation}
y(x) = c_1 y_1(x) + c_2 y_2(x)
\end{equation}

of two linearly independent solutions $y_1$ and $y_2$, and we’ve seen that if we’re given one solution $y_1(x)$ we can compute a second linearly independent solution using the method of reduction of order. We will now turn to the problem of actually finding a single solution $y_1(x)$ of (17.1).

We let us now return to the special case of a homogeneous second order linear differential equation with constant coefficients; i.e., differential equations of the form

\begin{equation}
y'' + py' + qy = 0
\end{equation}

where $p$ and $q$ are constant.

We saw in Lecture 11, that one can construct solutions of the differential equation (17.3) by looking for solutions of the form

\begin{equation}
y(x) = e^{\lambda x}
\end{equation}

Let us recall that construction. Plugging (17.4) into (17.3) yields

\begin{equation}
0 = \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + q e^{\lambda x} = (\lambda^2 + p\lambda + q) e^{\lambda x}.
\end{equation}

Since the exponential function $e^{\lambda x}$ never vanishes we must have

\begin{equation}
\lambda^2 + p\lambda + q = 0.
\end{equation}

Equation (17.6) is called the **characteristic equation** for (17.3) since for any $\lambda$ satisfying (17.6) we will have a solution $y(x) = e^{\lambda x}$ of (17.3).

Now because (17.6) is a quadratic equation we can employ the Quadratic Formula to find all of its roots:

\begin{equation}
\lambda^2 + p\lambda + q = 0 \quad \Rightarrow \quad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.
\end{equation}

Note that a root $\lambda$ of (17.6) need not be a real number. Indeed, if $p^2 - 4q < 0$, then in order to compute $\lambda$ via the Quadratic Formula we have to take the square root of a negative number and that forces us into the realm of complex numbers. We shall postpone temporarily the case when a root $\lambda$ is complex and first discuss the case when the roots of (17.6) are all real. This requires $p^2 - 4q \geq 0$.

*Case (i):* $p^2 - 4q > 0$
Because \( p^2 - 4q \) is positive, \( \sqrt{p^2 - 4q} \) is a positive real number and

\[
\begin{align*}
\lambda_+ &= \frac{-p + \sqrt{p^2 - 4q}}{2} \\
\lambda_- &= \frac{-p - \sqrt{p^2 - 4q}}{2}
\end{align*}
\]

are distinct real roots of (17.6). Thus,

\[
\begin{align*}
y_1 &= e^{\lambda_+ x} \\
y_2 &= e^{\lambda_- x}
\end{align*}
\]

will both be solutions of (17.3). Noting that

\[
W(y_1, y_2) = y_1 y_2' - y_2 y_1' = \lambda_+ e^{\lambda_+ x} - \lambda_- e^{\lambda_- x}
\]

\[
= (\lambda_+ - \lambda_-) e^{(\lambda_+ + \lambda_-) x}
\]

\[
= \sqrt{p^2 - 4q} e^{-\frac{x}{2}}
\]

is non-zero, we conclude that if \( p^2 - 4q \neq 0 \), then the roots (17.8) furnish two linearly independent solutions of (17.3) and so the general solution is given by

\[
y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x}.
\]

Case (ii): \( p^2 - 4q = 0 \)

If \( p^2 - 4q = 0 \), however, this construction only gives us one distinct solution; because in this case \( \lambda_+ = \lambda_- \). To find a second fundamental solution we must use the method of Reduction of Order.

So suppose \( y_1(x) = e^{-\frac{p}{2} x} \) is the solution corresponding to the root

\[
\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{-p \pm 0}{2} = \frac{-p}{2}
\]

of

\[
\lambda^2 + p\lambda - q = 0 \quad , \quad p^2 - 4q = 0.
\]

Then the Reduction of Order formula gives us a second linearly independent solution

\[
y_2(x) = y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp \left[ \int^s -p(t)dt \right] ds
\]

which gives us a second linearly independent solution. Plugging in \( y_1(x) = e^{-\frac{p}{2} x} \) and \( p(t) = p \), yields

\[
y_2(x) = e^{-\frac{p}{2} x} \int^x \frac{1}{(e^{-\frac{p}{2} s})^2} \exp \left[ \int^s -pdt \right] ds
\]

\[
= e^{-\frac{p}{2} x} \int^x \frac{1}{e^{-p s}} \exp [-ps] ds
\]

\[
= e^{-\frac{p}{2} x} \int^x e^{ps} e^{-ps} ds
\]

\[
= e^{-\frac{p}{2} x} \int^x ds
\]

\[
= xe^{-\frac{p}{2} x}
\]

\[
= xy_1(x)
\]

In summary, for the case when \( p^2 - 4q = 0 \), we only have one root of the characteristic equation, and so we get only one distinct solution \( y_1(x) \) of the original differential equation by solving the characteristic equation for \( \lambda \). To get a second linearly solution we must use the Reduction of Order formula; however, the
result will always be the same: the second linearly independent solution will always be \( x \) times the solution \( y_1(x) = e^{-\frac{b}{2}x} \). Thus, the general solution in this case will be
\[
y(x) = c_1 e^{-\frac{b}{2}x} + c_2 x e^{-\frac{b}{2}x} \quad \text{if } p^2 - 4q = 0.
\]

We now turn to the third and last possibility.

Case (iii): \( p^2 - 4q < 0 \)

In this case
\[
\sqrt{p^2 - 4q}
\]
will be undefined unless we introduce complex numbers. But when we set
\[
\sqrt{-1} = i
\]
we have
\[
\sqrt{p^2 - 4q} = \sqrt{(-1)(4q - p^2)} = \sqrt{-1} \sqrt{4q - p^2} = i \sqrt{4q - p^2}.
\]
The square root on the right hand side is well-defined since \( 4q - p^2 \) is a positive number. Thus,
\[
\lambda_\pm = \frac{-p \pm i \sqrt{4q - p^2}}{2} = \alpha \pm i \beta
\]
where
\[
\alpha = -\frac{b}{2}, \quad \beta = \frac{\sqrt{4q - p^2}}{2},
\]
will be a complex solution of (17.6) and
\[
y(x) = c_1 e^{\alpha x + i \beta x} + c_2 e^{\alpha x - i \beta x}
\]
would be a solution of (17.3) if we could make sense out the notion of an exponential function with a complex argument.

Thus, we must address the problem of ascribing some meaning to
\[
e^{\alpha x + i \beta x}
\]
as a function of \( x \). To ascribe some sense to this expression we considered the Taylor series expansion of \( e^x \)
\[
e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots
\]
Now although we do not yet understand what \( e^{\alpha x + i \beta x} \) means, we can nevertheless substitute \( \alpha x + i \beta x \) for \( x \) on the right hand side of (17.19), and get a well defined series with values in the complex numbers. One can show that this series converges for all \( \alpha, \beta \) and \( x \). We thus take
\[
e^{\alpha x + i \beta x} = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{i!} (\alpha x + i \beta x)^i
\]
which agrees with (17.19) when \( \beta = 0 \).

One can also show that
\[
e^{\alpha x + i \beta x} = e^{\alpha x} e^{i \beta x}.
\]
where

\begin{equation}
\alpha = \frac{-p}{2}, \quad \beta = \frac{\sqrt{4q - p^2}}{2}. \tag{17.23}
\end{equation}

A general solution of (17.3) would then be

\begin{equation}
y(x) = c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}. \tag{17.24}
\end{equation}

However, this is rarely the form in which one wants a solution of (17.3). One would prefer solutions that are real-valued functions of \(x\) rather than complex-valued functions of \(x\). But these can be had as well, since if \(z = x + iy\) is a complex number, then

\begin{align}
\text{Re}(z) &= \frac{1}{2} (z + \bar{z}) = x \\
\text{Im}(z) &= \frac{1}{2i} (z - \bar{z}) = y
\end{align} \tag{17.25}

are both real numbers. Applying the Superposition Principle, it is easy to see that if

\begin{equation}
y(x) = e^{\alpha x} e^{i\beta x} \tag{17.26}
\end{equation}

and

\begin{equation}
\bar{y}(x) = e^{\alpha x} e^{-i\beta x} \tag{17.27}
\end{equation}

are two complex-valued solutions of (17.3), then

\begin{equation}
y_r(x) = \frac{1}{2} (y(x) + \bar{y}(x)) = e^{\alpha x} \left( \frac{e^{i\beta x} + e^{-i\beta x}}{2} \right) \tag{17.28}
\end{equation}

and

\begin{equation}
y_i(x) = \frac{1}{2i} (y(x) - \bar{y}(x)) = e^{\alpha x} \left( \frac{e^{i\beta x} - e^{-i\beta x}}{2i} \right) \tag{17.29}
\end{equation}

are both real-valued solutions of (17.3).

Let us now compute the series expansion of

\begin{equation}
\frac{e^{ix} + e^{-ix}}{2} \tag{17.30}
\end{equation}

and

\begin{equation}
\frac{e^{ix} - e^{-ix}}{2i} \tag{17.31}
\end{equation}

\begin{equation}
\frac{1}{2} (e^{ix} + e^{-ix}) = \frac{1}{2} \left( 1 + (-ix) + \frac{1}{3!}(-ix)^2 + \frac{1}{5!}(-ix)^3 + \cdots \right)
\end{equation}

\begin{equation}
= \frac{1}{2} (1 + (ix) + \frac{1}{3!}(ix)^2 + \frac{1}{5!}(ix)^3 + \cdots)
\end{equation}

\begin{equation}
= \frac{1}{2} (1 - \frac{1}{2}ix^2 + \frac{1}{4!}ix^4 + \cdots)
\end{equation}

The expression on the right hand side is readily identified as the Taylor series expansion of \(\cos(x)\). We thus conclude

\begin{equation}
\cos(x) = \frac{e^{ix} + e^{-ix}}{2}. \tag{17.33}
\end{equation}

Similarly, one can show that

\begin{equation}
\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}. \tag{17.34}
\end{equation}
On the other hand, if one adds (17.33) to \(i\) times (17.34) one gets

\[
\cos(x) + i\sin(x) = \frac{e^{ix} + e^{-ix}}{2} + i \frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{ix} + e^{-ix} + e^{ix} - e^{-ix}}{2} = e^{ix}
\]

or

\[
e^{ix} = \cos(x) + i\sin(x)
\]

Thus, the real part of \(e^{ix}\) is \(\cos(x)\), while the pure imaginary part of \(e^{ix}\) is \(\sin(x)\).

We now have a means of interpreting the function

\[
e^{\alpha x + i\beta x}
\]

in terms of elementary functions (rather than as a power series); namely,

\[
e^{\alpha x + i\beta x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x)).
\]

Thus,

\[
\begin{align*}
\text{Re}
\left[
\begin{array}{c}
e^{\alpha x + i\beta x} \\
e^{\alpha x + i\beta x}
\end{array}
\right]
& = e^{\alpha x} \cos(\beta x), \\
\text{Im}
\left[
\begin{array}{c}
e^{\alpha x + i\beta x} \\
e^{\alpha x + i\beta x}
\end{array}
\right]
& = e^{\alpha x} \sin(\beta x).
\end{align*}
\]

I now want to show how (17.33) and (17.34) allow us to write down the general solution of a differential equation of the form

\[
y'' + py' + qy = 0, \quad p^2 - 4q < 0
\]

as a linear combination of real-valued functions.

Now when \(p^2 - 4q < 0\), then

\[
\lambda_{\pm} = -p \pm i\sqrt{4q - p^2} = \alpha \pm i\beta
\]

are the (complex) roots of the characteristic equation

\[
\lambda^2 + p\lambda + q = 0
\]

corresponding to (17.40) and

\[
y_{\pm}(x) = e^{\alpha x \pm i\beta x}
\]

are two (complex-valued) solutions of (17.40). But since (17.40) is linear, since \(y_{\pm}\) and \(y_{-}\) are solutions so are

\[
y_1(x) = \frac{1}{2} \left( y_+ (x) + y_- (x) \right) = e^{\alpha x} \left( \frac{e^{i\beta x} + e^{-i\beta x}}{2} \right) = e^{\alpha x} \cos(\beta x)
\]

and

\[
y_2(x) = \frac{1}{2i} \left( y_+ (x) - y_- (x) \right) = e^{\alpha x} \left( \frac{e^{i\beta x} - e^{-i\beta x}}{2i} \right) = e^{\alpha x} \sin(\beta x).
\]

Note that \(y_1\) and \(y_2\) are both real-valued functions.

We conclude that if the characteristic equation corresponding to

\[
y'' + py' + qy = 0
\]
has two complex roots 

\begin{equation}
\lambda = \alpha \pm i\beta \tag{17.47}
\end{equation}

then the general solution is 

\begin{equation}
y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) \tag{17.48}
\end{equation}

**Example 17.1.** The differential equation 

\begin{equation}
y'' - 2y' - 3y \tag{17.49}
\end{equation}

has as its characteristic equation 

\begin{equation}
\lambda^2 - 2\lambda - 3 = 0 \tag{17.50}
\end{equation}

The roots of the characteristic equation are given by 

\begin{equation}
\lambda = \frac{2 \pm \sqrt{4 - 12}}{2} = 3, -1 \tag{17.51}
\end{equation}

These are distinct real roots, so the general solution is 

\begin{equation}
y(x) = c_1 e^{3x} + c_2 e^{-x} \tag{17.52}
\end{equation}

**Example 17.2.** The differential equation 

\begin{equation}
y'' + 4y' + 4y = 0 \tag{17.53}
\end{equation}

has 

\begin{equation}
\lambda^2 + 4\lambda + 4 = 0 \tag{17.54}
\end{equation}

as its characteristic equation. The roots of the characteristic equation are given by 

\begin{equation}
\lambda = \frac{-4 \pm \sqrt{16 - 16}}{2} = -2 \tag{17.55}
\end{equation}

Thus we have a double root and the general solution is 

\begin{equation}
y(x) = c_1 e^{-2x} + c_2 xe^{-2x} \tag{17.56}
\end{equation}

**Example 17.3.** The differential equation 

\begin{equation}
y'' + y' + y = 0 \tag{17.57}
\end{equation}

has 

\begin{equation}
\lambda^2 + \lambda + 1 = 0 \tag{17.58}
\end{equation}

as its characteristic equation. The roots of the characteristic equation are 

\begin{equation}
\lambda = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{3} i}{2} \tag{17.59}
\end{equation}

and so the general solution is 

\begin{equation}
y(x) = c_1 e^{-\frac{1}{2}x} \cos \left( \frac{\sqrt{3}}{2} x \right) + c_2 e^{-\frac{1}{2}x} \sin \left( \frac{\sqrt{3}}{2} x \right) \tag{17.60}
\end{equation}