Taylor Series Methods

In this lecture I shall describe one last general method that is available to use to find approximate solutions of a first order differential equation.

Recall that the \( n \)th order Taylor expansion of a (smooth) function \( f(x) \) about the point \( x = x_0 \) is the degree \( n \) polynomial defined by

\[
T_n(x) = \sum_{i=0}^{n} \frac{1}{i!} f^{(i)}(x_0) (x - x_0)^i
\]

(5.1)

\[
= f(x_0) + f'(x_0) (x - x_0) + \frac{1}{2} f''(x_0) (x - x_0)^2 + \frac{1}{6} f'''(x_0) (x - x_0)^3 + \cdots
\]

and that such expansions are extremely useful in that they can (for sufficiently small \( |x - x_0| \)) be used as approximate expressions for the original function \( f \). Indeed, Taylor's theorem says

\[
f(x) = T_n(x) + O \left( |x - x_0|^{n+1} \right)
\]

and that moreover

\[
f(x) = \lim_{n \to \infty} T_n(x)
\]

(so long as \( f(x) \) is smooth).

Therefore, one way to get an approximate solution of a differential equations would be to figure out what its Taylor series looks like and try to find the remaining terms. That turns out to be a relatively easy thing to do.

Suppose \( y(x) \) is a solution of

\[
y' = F(x, y)
\]

(5.2)

satisfying the initial condition

\[
y(x_0) = y_0.
\]

(5.3)

Since \( x = x_0 \) implies \( y = y_0 \), and because the differential equation tells us what \( y'(x) \) must be given \( x \) and \( y \), we can infer that

\[
y'(x_0) = y_0.
\]

Thus, we already know the first two terms of the Taylor expansion of \( y(x) \):

\[
y(x) = y(x_0) + y'(x_0) (x - x_0) + \cdots
\]

(5.1)

\[
= y_0 + F(x_0, y_0) (x - x_0) + \cdots
\]
What about the higher order terms? To get the second order term we can differentiate the original differential equation with respect to \( x \) to get

\[
y''(x) = \frac{d}{dx} F(x, y(x))
\]

\[
= \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx}
\]

\[
= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y'(x)
\]

\[
= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} F(x, y(x))
\]

So

\[
y''(x_0) = \left. \frac{\partial F}{\partial x} \right|_{x=x_0} + \left. \frac{\partial F}{\partial y} F(x, y) \right|_{x=x_0, y=y_0}
\]

which after carrying out the partial differentiations and plugging in for \( x \) and \( y \) is just a number. And so we now have the second order term of the Taylor expansion of our solution \( y(x) \) about \( x = x_0 \). To get the third order term, we can differentiate the differential equation again to obtain

\[
y'''(x) = \frac{d^2}{dx^2} \left( \frac{dy}{dx} \right) = \frac{d^2}{dx^2} F(x, y(x))
\]

So

\[
y'''(x_0) = \left. \frac{d^2}{dx^2} F(x, y(x)) \right|_{x=x_0, y=y_0}
\]

Let’s now look at a specific example.

**Example 5.1.** Find the first four terms of the Taylor expansion of the solution of

\[
y' = x + y^2
\]

\[
y(0) = 1
\]

about \( x = x_0 = 0 \).

Suppose \( y(x) \) is a solution of the differential equation (5.5). Its Taylor series about \( x = 0 \) is then

\[
y(x) = y(0) + y'(0)(x - 0) + \frac{1}{2!}y''(0)(x - 0)^2 + \frac{1}{3!}y'''(0)(x - 0)^3 + \cdots
\]

\[
y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \cdots
\]

Now the initial condition (5.6) gives us a value for the first term; namely, \( y(0) = 1 \). The differential equation gives us a value for the factor \( y'(0) \) in the second term; for Equation (5.9) says

\[
y'(x) = x + (y(x))^2
\]

so in particular

\[
y'(0) = 0 + (y(0))^2 = 0 + 1^2 = 1.
\]

To get the factor \( y''(0) \) in the third term we differentiate Equation (5.9) with respect to \( x \):

\[
y''(x) = \frac{d}{dx} \left( x + (y(x))^2 \right)
\]

\[
= 1 + 2y(x)y'(x)
\]

\[
= 1 + 2y(x) \left( x + (y(x))^2 \right)
\]

\[
= 1 + 2y(x) + 2(y(x))^3
\]
(In passing from the second line to the third we have again employed Equation (5.9).) At $x = 0$, we then have

$$y''(0) = 1 + 2 \cdot 0 \cdot y(0) + 2 (y(0))^3$$
$$= 1 + 0 + 2 \cdot 1^3$$
$$= 3$$

To get a number for $y'''(0)$, we differentiate Equation (5.13) to get

$$y'''(x) = 0 + 2 y(x) + 2 x y'(x) + 6 (y(x))^2 y'(x)$$
$$= 0 + 2 y(x) + 2 x \left( x + (y(x))^2 \right) + 6 (y(x))^2 \left( x + (y(x))^2 \right)$$
$$= 2 y(x) + 2 x^2 + 2 x (y(x))^2 + 6 x (y(x))^2 + 6 (y(x))^4$$

Evaluating this equation at $x = 0$ yields

$$y'''(0) = 2 y(0) + 2 \cdot 0^2 + 2 \cdot 0 \cdot (y(0))^2 + 6 \cdot 0 \cdot (y(0))^2 + 6 (y(0))^4$$
$$= 2 \cdot 1 + 0 + 0 + 6 \cdot 1^4$$
$$= 2 + 6$$
$$= 8$$

Finally, we can plug the values we found for $y(0)$, $y'(0)$, $y''(0)$, and $y'''(0)$ into the right hand side of Equation (5.8) to get

$$y(x) = 1 + 1 \cdot x + \frac{1}{2} \cdot 3 \cdot x^2 + \frac{1}{6} \cdot 8 \cdot x^3 + \cdots$$
$$= 1 + x + \frac{3}{2} x^2 + \frac{4}{3} x^3 + \cdots$$

which we would could then view as an approximate solution of the initial value problem, accurate to order $x^4$. 