LECTURE 31

Definite and Indefinite Integrals

The Fundamental Theorem of Calculus came in two parts:

THEOREM 31.1. *Let f be a continuous function on the interval $[a, b]$. Then*

(i) *If $g(x) \equiv \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.*
(ii) *If $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.*

The expression

$$\int_a^b f(x)\,dx$$

is called **definite integral** of $f(x)$ from a to b. It is just a number (for any fixed choice of a and b). The second part of the Fundamental Theorem says that that this number can be computed by finding an antiderivative F of f and then computing the difference between its value at $x = b$ and its value at $x = a$.

In general, given an expression of the form

$$\int_a^y f(x)\,dx$$

we refer to

- $\int$ as the *integration sign*,
- y as the *upper endpoint of integration*,
- a as the *lower endpoint of integration*,
- f as the *integrand of the integral*, and
- x as the *variable of integration.*.

The expression

$$\int^x f(t)\,dt$$

is called an **indefinite integral** of the function $f(x)$. It is just the anti-derivative of $f(x)$ (up to a constant that gets ignored). Sometimes we write this as simply

$$\int f(x)\,dx$$

but continue to regard the result as a function of x.

What we formally called a Table of Antiderivatives, will now be called a Table of Indefinite Integrals

$f(x)$	$\int f(x)\,dx$
x^n	$\frac{1}{n+1}x^{n+1}$
$\frac{1}{x}$	$\ln\|x\|$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\sec^2(x)$	$\tan(x)$
$\csc^2(x)$	$-\cot(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\csc(x)\cot(x)$	$-\csc(x)$
$e^{\lambda x}$	$\frac{1}{\lambda}e^{\lambda x}$

1. The Net Change Theorem

Recall that the Part (ii) of the Fundamental Theorem of Calculus says

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any anti-derivative of f. If we replace the integrand f by its derivative, then we get the statement

$$(*) \qquad \int_a^b f'(x)\,dx = f(b) - f(a)$$

because $f(x)$ is alway an anti-derivative of its derivative. Now let us reinteprete this last equation: When we think of $f'(x) = \frac{df}{dx}$ as prescribing the rate at which $f(x)$ changes with respect to x, then equation (*) says the following

Theorem 31.2. *The integral of a rate of change of a function f between $x = a$ and $x = b$ yields precisely the difference between the value of $f(x)$ at $x = b$ and $x = a$.*

This interpretation of the second part of the Fundamental Theorem is very useful for applications.

Example 31.3. An object moves along a line in such a way that its velocity at time t is given by

$$v(t) = t^2 - t + 1.$$

How far does it move between $t = 1$ and $t = 4$.

- Let $x(t)$ be the function that prescribes the position of the object at time t. We need to compute

  $$x(4) - x(1)$$

  from the fact the its velocity is given by

  $$v(t) = \frac{dx}{dt} = t^2 - t + 1$$

  We can do this using Theorem 31.2, which says for the situation at hand that,

  $$\int_1^4 \frac{dx}{dt}dt = x(4) - x(1)$$

  And so we just need to compute

  $$\begin{aligned} \int_1^4 \frac{dx}{dt}dt &= \int_1^4 \left(t^2 - t + 1\right)dt = \left.\left(\frac{1}{3}t^3 - \frac{1}{2}t^2 + t\right)\right|_1^4 \\ &= \frac{64}{3} - \frac{16}{2} + 4 - \frac{1}{3} + \frac{1}{2} - 1 \\ &= \frac{33}{2} \end{aligned}$$

So the total distance travelled between $t = 1$ and $t = 4$ is $\frac{33}{2}$.

2. Varying Endpoints of Integration

In Part (i) of the Fundamental theorem

$$(**) \qquad \text{if } g(x) = \int_a^x f(t)\,dt \quad \Longrightarrow \quad g'(x) = f(x)$$

the upper end point of integration x is regarded as a variable parameter, and the integral of f is used to define a new function $g(x)$. Now once we have a function of x we can use it to build more complicated functions. For example we could define a function

$$h(x) = g\left(x^2\right)$$

What is this new function, well all we need to do is replace the x in (**) by x^2

$$h(x) = g\left(x^2\right) = \int_a^{x^2} f(t)\,dt$$

Ok. Now what does the Fundamental Theorem say about the derivative of $h(x)$? To answer this question correctly we have to first utilize the Chain Rule:

$$\frac{dh}{dx} = \left.\frac{dg}{du}\right|_{u=x^2}\left(\frac{d}{dx}x^2\right) = \left.\frac{dg}{du}\right|_{u=x^2}(2x)$$

Now $g'(u)$ is given by the Fundamental Theorem

$$g(u) = \int_a^u f(t)\,dt \quad \Longrightarrow \quad g'(u) = f(u)$$

Thus

$$\left.\frac{dg}{du}\right|_{u=x^2} = \left.f(u)\right|_{u=x^2} = f\left(x^2\right)$$

and so

$$\frac{d}{dx}\int_a^{x^2} f(t)\,dt = f\left(x^2\right)(2x)$$

In general,

THEOREM 31.4. *Suppose $h(x)$ is defined by*

$$h(x) = \int_a^{p(x)} f(t)\,dt$$

then its derivative is

$$f(p(x))\,p'(x)$$

In other words,

$$\frac{d}{dx}\int_a^{p(x)} f(t)\,dt = f(p(x))\,p'(x)$$

Proof. We can regard $h(x)$ as a composed function

$$h(x) = g(p(x))$$

where

$$g(u) = \int_a^u f(t)\,dt$$

The Chain Rule then says

$$\frac{d}{dx}h(x) = \frac{d}{dx}g(p(x)) = \left(\left.\frac{dg}{du}\right|_{u=p(x)}\right)\left(\frac{dp}{dx}\right)$$

Now the Fundamental Theorem of Calculus, Part (i), tells us that

$$\frac{dg}{du} = \frac{d}{du}\int_a^u f(t)\,dt = f(u)$$

and so we have

$$\frac{d}{dx}h(x) = \left(\left.\frac{dg}{du}\right|_{u=p(x)}\right)\left(\frac{dp}{dx}\right) = \left(f(u)|_{u=p(x)}\right)\left(\frac{dp}{dx}\right) = f(p(x))\,p'(x)$$

and the stated result follows.