LECTURE 22

The Mean Value Theorem

We've just seen how derivatives can be used to find and classify the local maxima and minima of a function. In fact, with just a little more work we can actually obtain a sketch of what the graph of a function has to look like without actually plotting the graph point by point.

Before doing so, however, I need to state the fundamental theorems upon which this sort of analysis is based.

We'll start with Rolle's theorem.

THEOREM 22.1 (Rolle's Theorem). *Let f be a function that satisfies the following three conditions*

(i) $f(x)$ *is continuous on the closed interval* $[a,b]$
(ii) $f(x)$ *is differentiable on the open interval* (a,b)
(iii) $f(a) = f(b)$

Then there is a point $c \in (a,b)$ such that $f'(c) = 0$.

Proof. There are three possibilities to consider.

- $f(x) = K$, a constant.
  In this case, $f'(x) = 0$ identically, and so for any number $c \in [a,b]$ we will have $f'(c) = 0$
- $f(x) > f(a)$ at some point $x \in (a,b)$.
  By the Extreme Value Theorem, we know that f attains a maximum value somewhere in $[a,b]$. However, since $f(a) < f(x)$, and $f(b) < f(x)$ (by condition (iii)) this maximum will not occur at the endpoints of the interval. So it must be a local maximum inside the interval; say this maximum occurs at $x = c$. But then since f has a local maximum at $x = c$ we must have $f'(c) = 0$.
- $f(x) < f(a)$ at some point $x \in (a,b)$.
  The Extreme Value Theorem, tells us that f attains a minimum value somewhere in $[a,b]$, and since there exists $x \in (a,b)$ such that $f(x) < f(a) = f(b)$, this minimal value doesn't occur at the endpoints a or b. Thus, there must be a local minimum inside (a,b), say at the point $x = c$. But then since $x = c$ is a local minimum of f, we have $f'(c) = 0$.

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EXAMPLE 22.2. Prove that $x^3 + x - 1$ has exactly one real root.

- First we show that $x^3 + x - 1$ has at least one real root. To see this, we note $f(x) = x^3 + x - 1$ is continuous and
$$\begin{aligned} f(0) &= -1 \\ f(1) &= 1 \end{aligned}$$
  By the Intermediate Value Theorem, we know that since 0 lies between $f(0)$ and $f(1)$, somewhere in the inteval $(0,1)$ there's a number c such that $f(c) = 0$. That number c will of course also satisfy $c^3 + c - 1 = 0$.

Now suppose we had two numbers c_1 and c_2 such that $f(c_i) = 0$. Then Rolle's Theorem tells us that there is a point d between c_1 an c_2 such that $f'(d) = 0$. However,

$$f'(x) = 3x^2 + 1 \geq 1 \qquad \text{for all } x$$

We conclude that we cannot have two such roots c_1 and c_2 without contradicting Rolle's Theorem. So the statement is proved.

THEOREM 22.3 (The Mean Value Theorem). *Let f be a function that satisfies*

- *f is continuous on the closed interval $[a, b]$*
- *f is differentiable on the open interval (a, b)*

Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof.

Consider the function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

(This happens to be the function that measures that measures the vertical distance between the graph of $f(x)$ and the secant line through $(a, f(a))$ and $(b, f(b))$.) Since f is continuous and differentiable and since

$$g(x) = -\frac{f(b) - f(a)}{b - a}(x - a)$$

is continuous and differentiable, so will be $h(x) = f(x) + g(x)$. We also have

$$\begin{aligned} h(a) &= f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0 \\ h(b) &= f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = 0 \end{aligned}$$

So we can apply Rolle's Theorem to conclude that somewhere between $x = a$ and $x = b$ there is a point $x = c$ where $h'(c) = 0$. But then

$$0 = h'(c) = \left(f'(x) - \frac{f(b) - f(a)}{b - a} \right)\bigg|_{x=c} = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Hence there is a point $c \in (a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

COROLLARY 22.4. *Suppose $f'(x) = 0$ for all x in an interval $[a, b]$, then $f(x)$ is a constant function on (a, b).*

Proof. Let x_1, x_2 be any two points inside the interval $[a, b]$ chosen so that $a < x_1 < x_2 < b$. By the Mean Value Theorem we know that there exists $c \in (x_1, x_2)$ so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(x) = 0$ everywhere in $[a, b]$. Hence,

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \Longrightarrow \quad f(x_2) = f(x_1)$$

for all points $x_1, x_2 \in (a, b)$. We conclude that f is constant on (a, b).

EXAMPLE 22.5. Prove the identity

$$\tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}$$

Set

$$f(x) = \tan^{-1}(x) + \cot^{-1}(x)$$

Using the identities

$$\begin{aligned} \frac{d}{dx}\tan^{-1}(x) &= \frac{1}{1+x^2} \\ \frac{d}{dx}\cot^{-1}(x) &= -\frac{1}{1+x^2} \end{aligned}$$

we find

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

Therefore, by the Corollary above, $f(x)$ must be a constant.

$$\tan^{-1}(x) + \cot^{-1}(x) = C$$

To figure out the constant we can put $x = 1$. Now when

$$1 = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \quad \Longrightarrow \quad \theta = 45^\circ \sim \pi/4 \text{ radians}$$

so $\tan^{-1}(1) = \pi/4$. Similarly,

$$1 = \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} \quad \Longrightarrow \quad \theta = 45^\circ \sim \pi/4 \text{ radians}$$

Hence,

$$C = \tan^{-1}(1) + \cot^{-1}(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Thus, the constant C is $\pi/2$. The stated identity now follows.