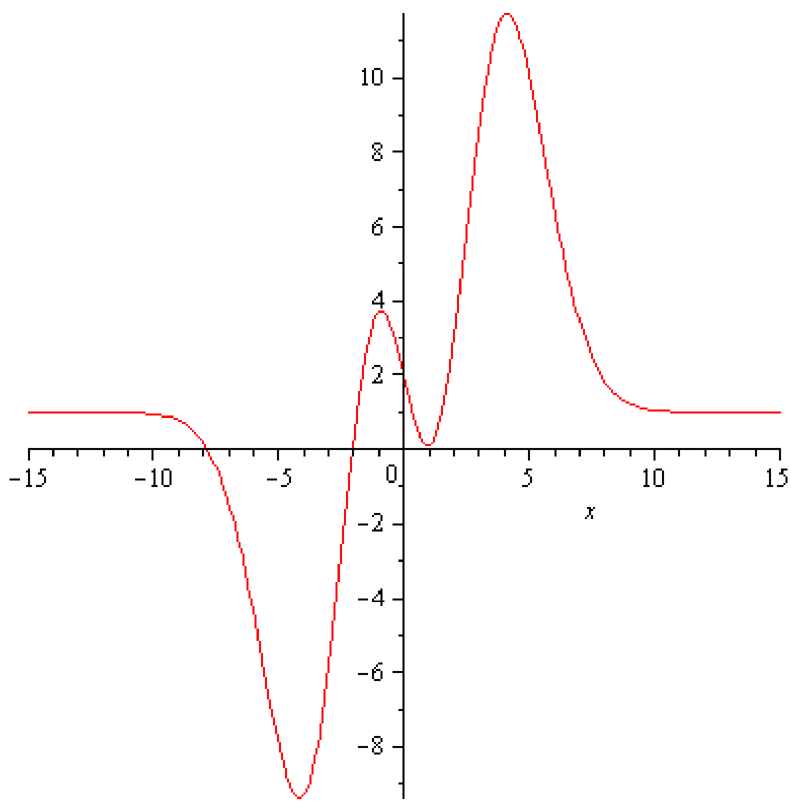
LECTURE 20

Maxima and Minima

Consider a function f with the following graph



Just looking at the graph we see that function attains a maximum value of about 12 around $x = 4$ and a minimum value of about 9 when $x = -4$. Today we'll try to answer two questions simultaneously:

- Can we find a more accurate way of determining the maxima and minima of a function (that is, something better than just "eyeballing" the graph of f)?
- Can we find a way of determining the maxima and minima of a function without having to graph it first?

Well, the answer to both these questions is sort of obvious once you figure out a suitable graphical characterization maxima and minima. Look at the the shape of the graphs near the maxima and minimas. Maxima are shaped like little hill tops, and minima are shaped like little valleys. We shall refine this characterization in just a moment. But before doing so we'll induce a little mathematical terminology.

DEFINITION 20.1. *By a* ***neighborhood*** *of a point $a \in \mathbb{R}$, we mean a open interval containing a. i.e. some open interval of the form*

$$I = \{a - \delta < x < a + \delta\}$$

DEFINITION 20.2. *A function f has a* ***local maximum*** *at $x = a$ if*

$$f(x) \leq f(a) \qquad \text{for all } x \text{ in a neighborhood of } a$$

and a function f a ***local minimum*** *at $x = a$ if*

$$f(x) \geq f(a) \qquad \text{for all } x \text{ in a neighborhood of } a$$

We say that $x = a$ is a ***local extrema*** *if it is either a local maxima or a local minima.*

A function f has a ***maximum*** *at $x = a$ if*

$$f(x) \leq f(a) \qquad \text{for all } x \text{ in the domain of } f$$

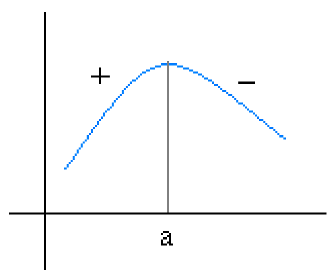
and a ***minimum*** *at $x = a$ if*

$$f(x) \geq f(a) \qquad \text{for all } x \text{ in the domain of } f$$

In the figure above, f has two local maxima and two local minima but just a single maximum and a single minimum.

Next we'll try to quantify our "hilltop or valley" characterization of local extrema.

So what makes a "hilltop" in the graph.



Well, to the left of the hilltop, the graph is going uphill, at the very top of the hill the graph is flat, and to the right of the hilltop the graph is going downhill. This behavior we can also characterize in terms of the derivative of f. For a function is going uphill if its slope (actually, the slope of its tangent line) is positive:

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{positive number}}{\text{positive number}} = \text{some positive number}$$

But this then means

$$f'(x) > 0 \qquad \text{for all } x \text{ just to the left of } a$$

because the derivative $f'(x)$ is just the slope of the tangent line to graph of f at $(x, f(x))$.

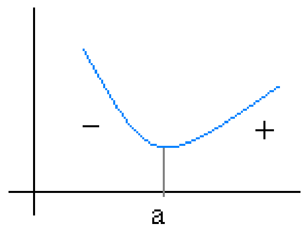
At a local maximum, i.e., at the top of a hill, the graph looks essentially flat. This means the slope of the tangent line should be zero, or

$$f'(a) = 0$$

Finally, to the right of $x = a$, the graph should be going downhill, and this means that the slope of the tangent lines immediately to the right of $x = a$ should be negative; hence

$$f'(x) < 0 \qquad \text{for all } x \text{ just to the right of } a$$

A similar analysis of a local minimum



Shows that near a local minimum we should have

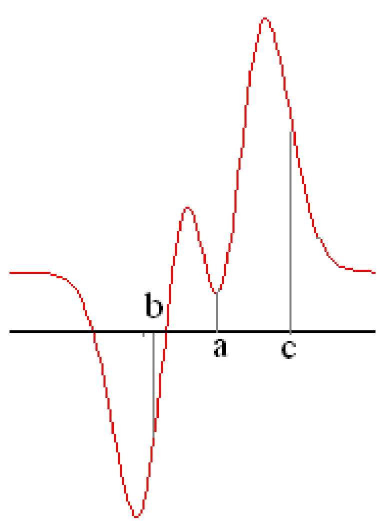
$$\begin{array}{lcll} f'(x) & < & 0 & \text{for all } x \text{ immediately to the left of a local minimum } x = a \\ f'(a) & = & 0 & \text{at a local minimum } x = a \\ f'(x) & > & 0 & \text{for all } x \text{ immediately to the right of a local minimum } x = a \end{array}$$

This *suggests* the following procedure for identifying local maximas and local minimas.

- Compute $f'(x) = \frac{df}{dx}$, and identify the points where $f'(x) = 0$. These points will be candidates for local maxima and local minima.
- Check the value of $f'(x)$ just to left and just to the right of each of the candidate points a. Then

$$\begin{array}{lclclll} f'(a-\delta) & > & 0 & , & f'(a) = 0 \text{ and } f'(a+\delta) < 0 & \Longrightarrow & a \text{ is a local maximum} \\ f'(a-\delta) & < & 0 & , & f'(a) = 0 \text{ and } f'(a+\delta) > 0 & \Longrightarrow & a \text{ is a local minimum} \end{array}$$

The only problem with this procedure is that if we choose δ too large (and perhaps even $\delta = 10^{-6}$ is too large) we could mischaracterize a local maximum as a local minimum:



Notice in the figure above that $x = a$ is a local minimum, yet if we sample $f'(x)$ at the points $x = b$ (a point "just to the left of"' a) and $x = c$ (a point "just to the right of" a), we'll have

$$f'(b) > 0 \quad \text{and} \quad f'(c) < 0$$

and incorrectly characterize $x = a$ as a local maximum.

Here's how we fix matters up. Recall that whenever we can always approximate a differentiable function f by a linear function

$$L(x) = f(a) + f'(a)(x-a)$$

and, moreover, the closer we are to $x = a$ the better the approximation. So let's find a linear approximation to $f'(x) = \frac{df}{dx}$. This would be

$$f'(a) + f''(a)(x-a)$$

(because the derivative of $f'(x)$ is $f''(x) = \frac{d}{dx}\frac{df}{dx}$, the double derivative of f).

Now suppose $f(a) = 0$, and $f''(a) > 0$. Then to the left of $x = a$

$$f'(x) \approx f'(a) + f''(a)(x-a) = 0 + (\text{positive number})(\text{negative number}) < 0$$

and to the right of $x = a$, we have

$$f'(x) \approx f'(a) + f''(a)(x-a) = 0 + (\text{positive number})(\text{postive number}) > 0$$

and so we'd have

$$\left.\begin{array}{r} f'(x) < 0 \text{ for all } x \text{ immediately to the left of } a \\ f'(a) = 0 \\ f'(x) > 0 \text{ for all } x \text{ immediately to the right of } a \end{array}\right\} \implies f \text{ has a local minimum at } x = a$$

On the other hand, if $f'(a) = 0$, and $f''(a) < 0$. Then to the left of $x = a$

$$f'(x) \approx f'(a) + f''(a)(x-a) = 0 + (\text{negative number})(\text{negative number}) > 0$$

and to the right of $x = a$, we have

$$f'(x) \approx f'(a) + f''(a)(x-a) = 0 + (\text{negative number})(\text{positive number}) < 0$$

and so we'd have

$$\left.\begin{array}{r} f'(x) > 0 \text{ for all } x \text{ immediately to the left of } a \\ f'(a) = 0 \\ f'(x) < 0 \text{ for all } x \text{ immediately to the right of } a \end{array}\right\} \implies f \text{ has a local maximum at } x = a$$

We thus have

THEOREM 20.3 (Second Derivative Test). *Suppose $f'(a) = 0$. Then*

$$\begin{array}{lclcl} f''(a) & < & 0 & \implies & x = a \textit{ is a local maximum of } f \\ f''(a) & > & 0 & \implies & x = a \textit{ is a local minimum of } f \end{array}$$