LECTURE 19

Hyperbolic Functions

The purpose of this lecture is to introduce you to some basic functions that are frequently used in engineering, physics, and a variety of other applications.

The good news is that they really pretty simple functions, especially when expressed in terms of the exponential function:

The first two are the hyperbolic sine and hyberbolic cosine functions:

$$\begin{aligned}
\sinh(x) &:= \frac{e^{x}-e^{-x}}{2} \\
\cosh(x) &:= \frac{e^{x}+e^{-x}}{2}
\end{aligned}$$

The other hyperbolic functions are obtained from these two in a manner that mimics exactly how the other trigonometric functions can be obtained from the sine and cosine functions:

notation	name	"trig-like formula"	formula in tems of exponentials
$\tanh(x)$	hyperbolic tangent	$\frac{\sinh(x)}{\cosh(x)}$	$\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$
$\coth(x)$	hyperbolic cotangent	$\frac{\cosh(x)}{\sinh(x)}$	$\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$
$\mathrm{sech}(x)$	hyperbolic secant	$\frac{1}{\cosh(x)}$	$\frac{2}{e^{x}+e^{-x}}$
$\mathrm{csch}(x)$	hyberbolic cosecant	$\frac{1}{\sinh(x)}$	$\frac{2}{e^{x}-e^{-x}}$

Not only do these functions have definitions similar to trignometric functions, their derivatives are also quite similar - in fact, even a little bit simpler because you don't have to worry so much about which formulas use minus signs. Let's compute a few derivatives.

First of all

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{e^{x}-e^{-x}}{2}\right) = \frac{1}{2}\frac{d}{dx}e^{x} - \frac{1}{2}\frac{d}{dx}e^{-x}$$

Using

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$$

we obtain

$$\frac{d}{dx}\sinh(x) = \frac{1}{2}e^{x} - \frac{1}{2}\left(-e^{-x}\right) = \frac{e^{x}+e^{-x}}{2} = \cosh(x)$$

Similarly,

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\left(\frac{e^{x}+e^{-x}}{2}\right) = \frac{1}{2}\frac{d}{dx}e^{x} + \frac{1}{2}\frac{d}{dx}e^{-x} = \frac{1}{2}e^{x} + \frac{1}{2}\left(-e^{-x}\right) = \frac{e^{x}-e^{-x}}{2} = \sinh(x)$$

Before computing more derivatives let me show another way in which the hyberbolic functions act like the trigonometric functions: that is, that they also satisfy a number of simple identities. For example,

$$\cosh^{2}(x) - \sinh^{2}(x) = 1$$

To see this, let's write the left hand side in terms of exponential functions:

$$\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) \\
&= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}\right) \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

To compute the derivative of $\tanh(x)$, it is easiest to use its "trig-like" formula, the preceding results, and the Quotient Rule.

$$\begin{aligned}
\frac{d}{dx}\tanh(x) &= \frac{d}{dx}\frac{\sinh(x)}{\cosh(x)} = \frac{\left(\frac{d}{dx}\sinh(x)\right)\cosh(x) - \sinh(x)\left(\frac{d}{dx}\cosh(x)\right)}{(\cosh(x))^2} \\
&= \frac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{\cosh^2(x)} \\
&= \frac{1}{\cosh^2(x)} \\
&: = \operatorname{sech}^2(x)
\end{aligned}$$

Below we give a table of the derivatives of all the hyperbolic functiosn

$$\begin{aligned}
\frac{d}{dx}\sinh(x) &= \cosh(x) \\
\frac{d}{dx}\cosh(x) &= \sinh(x) \\
\frac{d}{dx}\tanh(x) &= \operatorname{sech}^2(x) \\
\frac{d}{dx}\coth(x) &= -\operatorname{csch}^2(x) \\
\frac{d}{dx}\operatorname{sech}(x) &= -\operatorname{sech}(x)\tanh(x) \\
\frac{d}{dx}\operatorname{csch}(x) &= -\operatorname{csch}(x)\coth(x)
\end{aligned}$$

We can also define inverse hypebolic functions by

$$\begin{aligned}
\sinh^{-1}(\sinh(x)) &= x = \sinh\left(\sinh^{-1}(x)\right) \\
\cosh^{-1}(\cosh(x)) &= x = \cosh\left(\cosh^{-1}(x)\right) \\
\tanh^{-1}(\tanh(x)) &= x = \tanh\left(\tanh^{-1}(x)\right)
\end{aligned}$$

These turn out to have fairly simply formulas that we can state directly in terms of x :

$$\begin{aligned}
\sinh^{-1}(x) &= \ln\left(x+\sqrt{x^2+1}\right) \\
\cosh^{-1}(x) &= \ln\left(x+\sqrt{x^2-1}\right) \\
\tanh^{-1}(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad , \qquad x^2<1 \\
\coth^{-1}(x) &= \frac{1}{2}\ln\left(\frac{1-x}{1+x}\right) \qquad , \qquad x^2>1 \\
\operatorname{sech}^{-1}(x) &= \log\left(\frac{1+sgn(x)\sqrt{1+x^2}}{x}\right) \\
\operatorname{csch}^{-1}(x) &= \log\left(\frac{1+sgn(x)\sqrt{1-x^2}}{x}\right)
\end{aligned}$$

The derivatives of the inverse hyperbolic functions are given by the following table

$$\begin{aligned}
\frac{d}{dx}\sinh^{-1}(x) &= \frac{1}{\sqrt{1+x^2}} \\
\frac{d}{dx}\cosh^{-1}(x) &= \frac{1}{\sqrt{x^2-1}} \\
\frac{d}{dx}\tanh^{-1}(x) &= \frac{1}{1-x^2} \\
\frac{d}{dx}\coth^{-1}(x) &= \frac{1}{1-x^2} \\
\frac{d}{dx}\operatorname{sech}^{-1}(x) &= -\frac{1}{|x|\sqrt{x^2+1}} \\
\frac{d}{dx}\operatorname{csch}^{-1}(x) &= -\frac{1}{|x|\sqrt{1-x^2}}
\end{aligned}$$