LECTURE 16

Exponential Growth and Decay

Recall that the natural exponential function

$$\exp(x) = e^x \qquad , \qquad e := 2.718....$$

is the one elementary function whose derivative is equal to itself

$$\frac{d}{dx}e^x = e^x \tag{1}$$

In fact, one can almost say that $\exp(x)$ is the unique function that is equal to its own derivative, except that any constant times e^x will have the same property; for if $f(x) = ce^x$

$$\frac{df}{dx} = \frac{d}{dx}ce^x = c\frac{d}{dx}e^x = ce^x = f(x)$$

The other *general* exponential functions

$$f(x) = a^x$$

do not obey a derivative rule quite this simple. As we saw last time, these functions obey the rule

$$\frac{d}{dx}a^x = \log(a)\, a^x \tag{2}$$

While this might *appear* more complicated than (1), its characterization is really pretty simple. We first note that for any given a, $\log(a)$ is just a constant. One can thus say that the general exponential function $f(x) = a^x$ is a function such that its derivative is just a constant multiple of itself.

In many applications, in fact, it is not the exponential base a that is most immediate, but rather the proportionality constant $\log(a)$ that relates an exponential function to its derivative.

Example 16.1. The growth rate of a population of bacteria is proportional to the size of a population. This is because at any given time t the number of bacteria cells dividing and multiplying is going to a certain fixed percentage of the total population. Thus,

$$\text{instantaneous rate of change of bacteria population} = \frac{dP}{dt} = \lambda P(t)$$

where λ is the constant percentage of the bacteria population P that is in the process of dividing and multiplying.

Example 16.2. Observation: The rate at which a geiger counter clicks is proportional to the size of a sample of a uranium.

Since the radioactivity that geiger counter is measuring is simply the by-products of uranium atoms undergoing nuclear decay, we can infer that the rate at which a uranium sample is decaying is proportional to its size

$$\frac{dQ}{dt} = -\lambda Q$$

(I put the minus sign in because it clear that the quantity Q of uranium should be decreasing with time, and so should have a negative derivative).

Given the preceding remarks about functions whose derivatives are proportional to themselves we should expect the function $P(t)$ describing the size of a bacteria as a function of time t, or a function $Q(t)$ describing the amount of radioactive material present at time t to be some kind of general exponential function; that is a function of the form

$$f(t) = a^t$$

or even

$$f(t) = ca^t$$

because in either case we still have

$$\frac{d}{dt} f(t) \propto f(t)$$

However, in practice, it is much more convenient to write such a function in the form

$$f(t) = Ae^{\lambda t}$$

The reason for this is the fact that the parameters A and λ have each have a very direct physical interpretation. We first note that at $t = -$

$$f(t) = Ae^0 = A$$

and so the constant A corresponds to the amount present at time $t = 0$. Next, we note that

$$f'(t) = \frac{d}{dt} Ae^{\lambda t} = A\frac{d}{dt}\left(e^{\lambda t}\right) = A\lambda e^{\lambda t} = \lambda f(t)$$

and so λ is the proportionality constant relating $f(t)$ to its rate of change.

Definition 16.3. *A (general) exponential function is a function of the form*

$$f(x) = Ce^{\lambda x} \tag{3}$$

The constant C is called the initial value of $f(x)$ and the constant λ is called the growth or decay rate of $f(x)$. When λ is positive, $\frac{df}{dx} = \lambda f$ is positive and so $f(x)$ is an increasing function of x and we therefore refer to λ as the growth rate of $f(x)$. When λ is negative, $\frac{df}{dx}$ is negative, and so $f(x)$ is a decreasing function of x, and so we call λ the decay rate of $f(x)$.

To see how functions of the form

$$f(x) = Ce^{\lambda x}$$

are related to the functions of the form

$$g(x) = ca^x$$

we note that when $A := \log(a)$, we have

$$a = e^{\log(a)} = e^A$$

and

$$g(x) = ca^x = c\left(e^A\right)^x = ce^{Ax} \tag{4}$$

Comparing (3) with (4) we see we can make $f(x)$ coincide with $g(x)$ be setting $c = C$ and $\lambda = A = \log(a)$. Thus the exponential functions we define by (3) are exactly the same sort of functions we wrote before as ca^x. It's just that we have a more direct physical interpretation of the parameters C and λ than we have for the parameters c and a.

Example 16.4. Assuming the rate of growth of the world's population is proportional to its size, estimate the size of the world's population in the year 2020 using the fact that in 1950 the world's population was 2.560 billion, and 1960 the population was 3.040 billion.

Our assumption implies that the world's population should be governed by an exponential function. So we set

$$P(t) = Ce^{\lambda t}$$

Let $t = 0$ correspond to 1950. Then

$$2.560 \times 10^9 = P(0) = Ce^0 = C$$

This tells us how to fix C. The problem statement tells us also that 10 years later the population was 3.040 billion

$$3.040 \times 10^9 = Ce^{10\lambda} = \left(2.560 \times 10^9\right) e^{10\lambda}$$

Dividing both sides by 2.560×10^9 we get

$$e^{10\lambda} = 1.1875$$

Taking the logarithm of both sides and solving for λ yields

$$10\lambda = \log(1.1875) \quad \Longrightarrow \quad \lambda = \frac{1}{10}\log(1.1875) = 0.017185$$

We now can write down an explicit formula for $P(t)$,

$$P(t) = Ce^{10\lambda} = \left(2.560 \times 10^9\right) e^{(0.017185)t} \tag{5}$$

The problem asks to determine the population in the year 2020 which is 70 years after 1950. So all we have to do now is plug in $t = 10$ into the formula (5)

$$\text{population in 2020} = P(70) = \left(2.560 \times 10^9\right) e^{(0.017185)70} = 8.525 \times 10^9$$