Nonlinear Covers I

Objects of Study: central extensions $\tilde{G}$ of a linear alg gp G over a field k.
$\tilde{G}$ may be nonlinear, i.e. it does not embed into GL_N.

Motivation and History

- Segal-Shale-Weil rep'n (or oscillator rep'n)

  Weil (1964), Kubota (1967), Rao (1993): k a local field. Then there is a unique 2-fold cover of $Sp(2n, k)$, $n \geq 1$ (the metaplectic group $Mp(2n,k)$)

  Weil and others did a comprehensive study of this gp and its Weil/oscillator rep'n. Weil's goal was to formulate the theory of theta functions in rep'n theoretic terms and repeat the previous results of Siegel (e.g. Siegel mass formula, Siegel-Weil formula) in this framework.

  $\rightsquigarrow$ Roger Howe's theory of dual pairs/theta correspondence.

- half-integral weight modular forms

  Shimura (1973 Annals), Gelbart (1976, SLN 530), Gelbart-Jacquet (1978)

  theory of Hecke operators for half-integral weight modular forms $\rightsquigarrow$ Shimura correspondence: $\left\{ \begin{array}{c} \frac{1}{2}\text{-integral} \\ \text{wt mod forms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{integral wt} \\ \text{mod forms} \end{array} \right\}$

  $\rightsquigarrow$ Gelbart-Jacquet's construction of Sym^2.

  $\rightsquigarrow$ Flicker-Kazhdan (1980) & Kazhdan-Patterson covering (1984)

  trace formula approach to Shimura corresp. (n-fold cover GL_2)

  (n-fold covers of GL_r via pull back of n-fold cover of SL_{r+1})

  $\rightsquigarrow$ Waldspurger (1980): aut discrete spectrum of $Mp(2, \mathbb{A})$ using theta corr. & Bump-Friedberg-Hoffstein ..

- The Congruence Subgroup Problem

Bass–Lazard–Serre (1964); Bass–Milnor–Serre (1967)

Thm For $n \geq 3$, every sbgp of finite index in $SL(n, \mathbb{Z})$ is a congruence sbgp.

Thm Same for both $SL(n, \mathbb{Z})$ $n \geq 3$, $Sp(2n, \mathbb{Z})$, $n \geq 2$.

Note: not true for $SL(2, \mathbb{Z})$.

What is a congruence sbgp? $\mathfrak{a}$ proper nonzero ideal in $\mathbb{Z}$.

The sbgp $SL(2, \mathfrak{a}) := \{ g \in SL(2, \mathbb{Z}) : g \equiv I_2 \pmod{\mathfrak{a}} \}$ is a normal sbgp w/ finite index in $SL(2, \mathbb{Z})$

A congruence sbgp of $SL(2, \mathbb{Z})$ is a sbgp containing some $SL(2, \mathfrak{a})$.

Question: (towards the end of 19th century) Are there other examples of normal sbgps of finite index?

Fricke–Klein: $\exists$ a surjective homomorphism $\varphi : SL(2, \mathbb{Z}) \to A_5$ and $\Gamma = \ker(\varphi)$ can not contain a sbgp of the form $SL(2, \mathfrak{a})$.

So for $SL(2, \mathbb{Z})$ the answer is YES.

But: the congruence sbgp problem proves that $SL(2)$ is an exception!

Setup:

abstract gps

G a group, A an abelian gp $\circlearrowleft G$ (i.e. A is a G-module)

An extension of G by A is a short exact sequence

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

such that the action of G on A by inner automorphisms of E agrees with the G-module action.

Often: A is trivial G-module. Then we call E a central extension of G by the abelian gp A.

top. gps

G loc. compact, Hausdorff, second countable topological gp (abbrev. as loc. cpt gp). A loc cpt abelian gp

Def. A central extension of G by A, is a short exact sequence

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

such that

1. E is a loc. cpt gp
2. i is continuous and $i(A)$ is a closed sbgp of the center of G
3. p is continuous and induces a top. iso $E/_{i(A)} \cong G$.

[3 may be replaced by the condition that p is cont. and open.]

Our interest: mostly when A is finite.

Study of central extns
- → categorical view
- → cohomological view
- → as a moduli functor
- → construct them w/ generators & relations

Also
- → over $\mathbb{R}, \mathbb{C}$
- → over p-adic
- → over adeles.

Some Examples

① (central extns of certain finite abelian gps)

a) central extensions of $G = \mathbb{Z}/2\mathbb{Z}$ by $A = \mathbb{Z}/2\mathbb{Z}$. there are 2

one: trivial - $\mathbb{Z}/2 \times \mathbb{Z}/2$

~~other~~: nontrivial $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

b) (Heisenberg gp) $n \in \mathbb{Z}_{>0}$, p prime, $\mathbb{F}_q$, $q = p^n$.

there is a nontrivial central extension $E = H(\mathbb{F}_q^{2n})$ of $G = \mathbb{F}_q^{2n}$ by $A = \mathbb{F}_q$, called the Heisenberg group:

$$0 \longrightarrow \mathbb{F}_q \longrightarrow H(\mathbb{F}_q^{2n}) \longrightarrow \mathbb{F}_q^{2n} \longrightarrow 0$$

the simplest realization of $H(\mathbb{F}_q^{2n})$ is:

the gp of $(n+2)\times(n+2)$ upper triangular unipotent matrices with only non-diagonal nonzero entries from $\mathbb{F}_q$ in the 1st row and the last column.

e.g. $n=1$

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

$n \geq 1$

$$\begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_n & z \\ & 1 & & & & y_1 \\ & & \ddots & 0 & & y_2 \\ & 0 & & \ddots & & \vdots \\ & & & & 1 & y_n \\ & & & & & 1 \end{bmatrix}$$

n

Center $= \begin{bmatrix} 1 & & & z \\ & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{bmatrix} \cong \mathbb{F}_q.$

② (real groups).

(a) $SU(n)$ is simply-connected and has no central extensions.

(b) $SO(n)$ has a 2-fold simply-connected cover:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(n) \longrightarrow SO(n) \longrightarrow 1$$

(c) $$\pi_1(SL(n,\mathbb{R})) = \begin{cases} \mathbb{Z} & \text{if } n=2 \\ \mathbb{Z}/2\mathbb{Z} & n>2 \end{cases}$$

(d) $$\pi_1(\mathrm{Spin}(p,q)) = \begin{cases} 1 & \text{if } \min(p,q) \leq 1 \\ \mathbb{Z} & \text{if } \min(p,q) = 2 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \min(p,q) > 2. \end{cases}$$

- Next time I first give some references and a summary of the historical development of these central extensions for linear alg. gps over local/global fields and then restrict our discussion to a limited situation of this for the remainder of the reading seminar.

- In particular, I'll explain the (algebraic) $\pi_1(G)$ and $H^2(G,A)$.