Sheaves + Sheaf cohomology

What is a sheaf?

Defn: A presheaf $\mathcal{F}$ on a top. space X is:

① $\forall$ open set $U \subseteq X$, abelian group $\mathcal{F}(U)$

② $\forall$ $U \subseteq V$, a homomorphism

$$\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U) \quad \text{s.t.}$$

restriction

(a) $\mathcal{F}(\emptyset) = 0$

(b) $\rho_{U,U} = \mathrm{id}.$

(c) If $U \subseteq V \subseteq W$, then $\rho_{W,U} = \rho_{W,V} \circ \rho_{V,U}$

Notation: if $s \in \mathcal{F}(V)$, $U \subseteq V$,

$$s|_U := \rho_{V,U}(s) \in \mathcal{F}(U)$$

A presheaf is a sheaf if

(d) For any $\{V_i\}_{i \in I}$ and $s_i \in F(V_i)$

s.t. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$

$\exists ! \; s \in F(\bigcup_{i \in I} V_i)$ s.t. $s|_{V_i} = s_i \;\; \forall i$.

Examples: Fix X.

(1) $F(U) = \{ \phi : U \to A \mid \phi \text{ continuous} \}$

abelian

holomorphic, C^∞, fns

(2) Fix $\pi : Y \to X$ continuous

$F(U) = \{ \sigma : U \to Y \mid \pi \circ \sigma(x) = x \}$
$= \{ \text{sections of } \pi \text{ over } U \}$.

ex) $Y = TX \rightsquigarrow F(U)$ vector fields
$Y = \Lambda^k TX \rightsquigarrow F(U)$ k-forms.

(3) $F(U) = \{ \phi : U \to A \mid \phi \text{ constant} \}$

is a pre sheaf, but not a sheaf.

$F(U) \cong A$

ex) $X = U_1 \sqcup U_2$

$s_1 : U_1 \mapsto k_1 \qquad s_2 : U_2 \mapsto k_2$

If $k_1 \neq k_2$, then s_1, s_2 can't be "glued" to make $s \in F(X)$.

Constant pre-sheaf : $F = A_X^{pre}$

(3a) $F(U) = \{ \phi : X \to A \mid \phi \text{ continuous} \}$

discrete topology

Constant fns connected components of X.

If U is connected, $F(U) \cong A$.

$F = A_X$ constant sheaf. (locally)

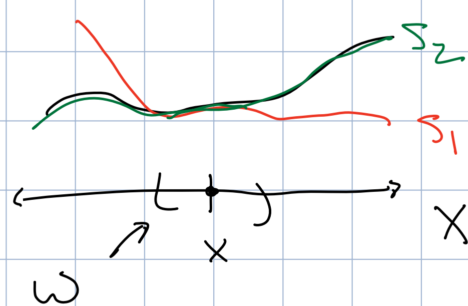
sheaf associated to A_X^{pre}

Defn.

For $x \in X$, the stalk $F_x := \varinjlim_{x \in U} F(U)$

i.e. $F_x = \coprod_{\substack{U \subseteq X \\ x \in U}} F(U)$ $\quad s_1 \sim s_2$ if $s_1|_W = s_2|_W$

for some $W \subseteq U_1 \cap U_2$.



Prop: F presheaf $\rightsquigarrow F^+$ sheaf

$$Y = \coprod_{x \in X} F_x \xrightarrow{\pi} X$$

$\pi(s) = x$ if $s \in F_x$.

$F^+(U) := \{$ sections of π over $U \}$

Čech cohomology of sheaves.

Fix a sheaf F on X.

Goal: form a "cochain complex" using F.

$\mathcal{U} = \{U_i\}_{i \in I}$ open cover of X.

$$I^{(p)} := \{ K \subseteq I \mid |K| = p+1 \}$$

For $K = \{i_0, \dots, i_p\} \in I^{(p)}$

$$\rightsquigarrow U_K = U_{i_0} \cap \cdots \cap U_{i_p}$$

Defn: $C^p(\mathcal{U}, F) = \prod_{K \in I^{(p)}} F(U_K)$

Remark: $\alpha \in C^p(\mathcal{U}, F)$ is determined by

$$\alpha_K \in F(U_K) \quad \forall K \in I^{(p)}$$

For each $K \in I^{(p)}$ fix an orientation

Coboundary maps:

$$\partial^p : C^p(\mathcal{U}, F) \to C^{p+1}(\mathcal{U}, F)$$

Let $k = \{i_0, \dots, i_{p+1}\} \in I^{(p+1)}$

$\alpha \in C^p(\mathcal{U}, F)$

$\hookrightarrow \quad \alpha_{k'} \in F(U_{k'}) \quad \forall\, k' \in I^{(p)}$

$$(\partial^p \alpha)_k = \sum_{j=0}^{p+1} \pm\, \alpha_{k - \{i_j\}} \Big|_{U_k}$$

signs $(i_j, k - \{i_j\})$ vs k

Claim: $\partial^p \circ \partial^{p-1} = 0$ so cochain complex!

Defn: $\check{H}^p(\mathcal{U}, F) = \ker \partial^p / \operatorname{im} \partial^{p-1}$

depends on open cover.

Let $\mathcal{V}$ be a refinement of $\mathcal{U}$

$\rightsquigarrow C^p(\mathcal{U}, F) \to C^p(\mathcal{V}, F)$

Defn: $C^p(X, F) = \varinjlim C^p(\mathcal{U}, F)$

Čech cohomology:

$$\check{H}^p(X, F) := \ker d^p / \operatorname{im} d^{p-1}$$

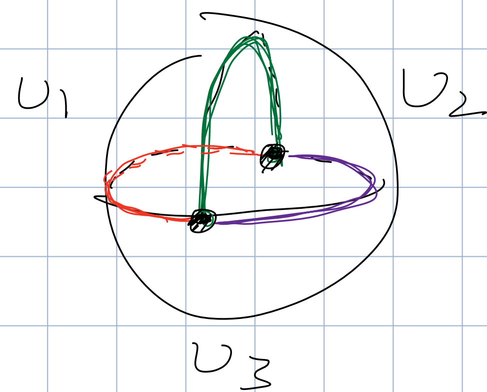
Thm: If X is triangulable, $\exists\ \mathcal{U}$ open cover s.t.

$$\check{H}^p(X, F) = \check{H}^p(\mathcal{U}, F).$$

Thm: If $F = \mathbb{Q}_X$ constant sheaf

$$\check{H}^p(X, \mathbb{Q}_X) \simeq \underset{\substack{\uparrow \\ \text{singular/simplicial cohomology}}}{H^p}(X, \mathbb{Q}).$$

Ex) $X = S^2$



$$\begin{aligned}
C^0(\mathcal{U}, \mathbb{Q}_X) &= F(U_1) \times F(U_2) \times F(U_3) \\
&= \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}
\end{aligned}$$

$$\Big\downarrow \partial^0$$

$$\begin{aligned}
C^1(\mathcal{U}, \mathbb{Q}_X) &= F(U_1 \cap U_2) \times F(U_2 \cap U_3) \\
&\quad \times F(U_1 \cap U_3) \\
&= \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}
\end{aligned}$$

$$\Big\downarrow \partial^1$$

$$\begin{aligned}
C^2(\mathcal{U}, \mathbb{Q}_X) &= F(U_1 \cap U_2 \cap U_3) \\
&= \mathbb{Q} \oplus \mathbb{Q}
\end{aligned}$$

① $(k_1, k_2, k_3) \xrightarrow{\partial^0} (k_1 - k_2,\ k_1 - k_3,\ k_2 - k_3)$

$$\check{H}^0(\mathcal{U}, \mathbb{Q}_X) = \ker \partial^1 = \mathbb{Q}$$

② $(x_{12}, x_{13}, x_{23}) \xrightarrow{\partial^1} (x_{12} - x_{13} + x_{23})^2$

$$\check{H}^1(\mathcal{U}, \mathbb{Q}_X) = \ker \partial^1 / \mathbb{Q}_0 =$$

$$\cancel{\frac{(x_{12} - x_{13} + x_{23} = 0)}{\begin{pmatrix} x_{12} = k_1 - k_2 \\ x_{13} = k_1 - k_3 \\ x_{23} = k_2 - k_3 \end{pmatrix}}} = 0$$

③ $$\check{H}^2(\mathcal{U}, \mathbb{Q}_X) = (z_1, z_2) / (z_1 = z_2)$$

$$= \mathbb{Q}$$

$$\leadsto \check{H}^i(X, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 2 \\ 0 & i = 1 \end{cases}$$

Question: Is there a sheaf F on X s.t.

$$\check{H}^p(X, F) = IH_p^{\bar{m}}(X) \,?$$

Algebra of sheaves.

Sheaves: F, G sheaves on X.

1. homomorphisms: $F \xrightarrow{\varphi} G$

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi(U)} & G(U) \\ \downarrow & & \downarrow \\ F(V) & \xrightarrow{\varphi(V)} & G(V) \end{array}$$

commutes with restriction

$\rightarrow \quad \varphi_x : F_x \rightarrow G_x$ hom

2. $\ker \varphi(U) := \ker(\varphi(U))$ is a sheaf

$\operatorname{im} \varphi(U) := \operatorname{im}(\varphi(U))^{+}$ is a ~~pre~~sheaf

(3) $F \subseteq G$ <u>subsheaf</u> if $F(U) \subseteq G(U)$
is a sheaf.

G/F <u>quotient sheaf</u> is $\left(G/F(U) = G(U)/F(U) \right)^{+}$
is ~~pre~~sheaf

(4) Let $\phi : X \to Y$ continuous map.

$F \rightsquigarrow X$, $G \rightsquigarrow Y$ sheaves.

(a) $\phi_* F$ on Y is a sheaf

$$\phi_* F(U) = F(\phi^{-1}(U))$$

(b) $\phi^* G$ on X is a ~~pre~~sheaf

$$\phi^* G(U) = \left(\lim_{\substack{\phi(U) \subseteq V \\ V \text{ open in } Y}} G(V) \right)^{+?}$$

(Co)homology of complexes.

$\mathcal{A}$ abelian category (vector spaces, modules, ...)

Chain complex: $K_\bullet = (K_n, \partial_n)_{n\in\mathbb{Z}}$

$$\cdots \to K_{i+1} \xrightarrow{\partial_{i+1}} K_i \xrightarrow{\partial_i} K_{i+1} \xrightarrow{\partial_i} \cdots$$

s.t. $\partial_{i+1} \circ \partial_i = 0$. $(\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i)$

$\to H_i(K_\bullet) = \ker \partial_i / \operatorname{im} \partial_{i+1}$

Similarly cochain complex $\rightsquigarrow H^i(K_\bullet)$

Q: Can we do this with sheaves?

$\rightsquigarrow$
1. homomorphisms
2. kernels, images
3. subs, quotients.
4. more tools if needed...