Last time:

1. Simplicial complexes K

2. Triangulated space X

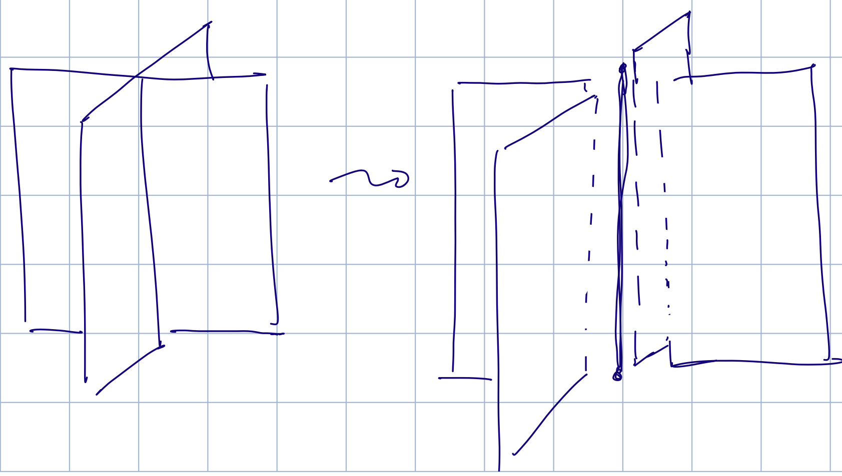
3. Simplicial homology $H_*(X)$

Goal: Find a homology th. for singular spaces with "manifold" properties.

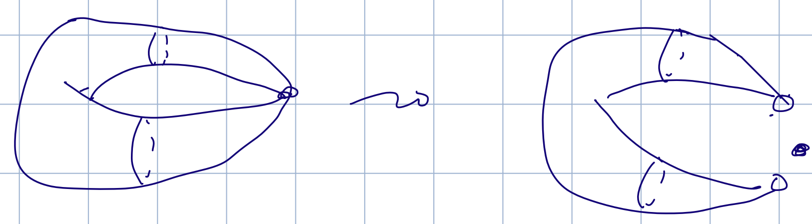
Stratified spaces

Idea: Given X singular, "break" X into smooth pieces

Ex)



Ex)



Whitney stratification

X quasi-proj. variety of pure dim n.

A Whitney stratification is a filtration

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 \quad \text{s.t.}$$

① X_j is a closed subvariety of X

② $X_j - X_{j-1}$ is a smooth quasi-proj variety of pure dim j. or empty.

Let S_α denote connected components of $X_j - X_{j-1}$

strata

ⓐ If $a_i \in S_\alpha$ with $\lim_{i \to \infty} a_i = b \in S_\beta$.

then $T_b S_\beta \subseteq \lim_{i \to \infty} T_{a_i} S_\alpha$.

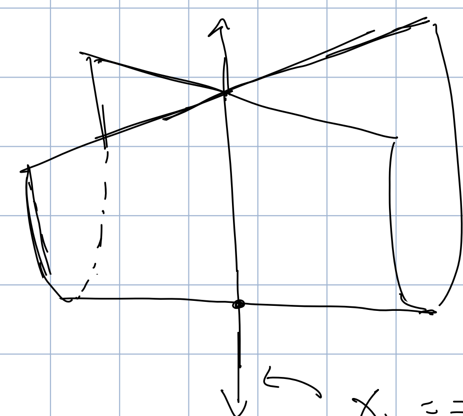
(b) If $a_i \in S_\alpha$, $b_i \in S_\beta$, $c \in S_\beta$ s.t.

$\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = c$, then

$$\lim_{i\to\infty} \overrightarrow{a_i b_i} \subseteq \lim_{i\to\infty} T_{a_i} S_\alpha .$$

lines joining a_i, b_i

Ex)



Whitney's umbrella ($zy^2 - x^2 = 0$)

X_2

$X_1 = z$-axis, $X_0 = \emptyset$.

$X_2 - X_1$, X_1 smooth. but at $(0,0,0)$ fails (a) + (b).

Pick $X = X_2 \supseteq X_1 \supseteq X_0$

X_1: z-axis, X_0: origin

Thm (Whitney): Any quasi-proj variety of pure dim n has a Whitney strat.

Topological stratifications

A space X is a stratified pseudomanifold if there is a filtration by closed subspaces:

$$X = X_n \supseteq \underset{\underset{\Sigma}{\shortparallel}}{X_{n-2}} \supseteq X_{n-3} \supseteq \cdots \supseteq X_0 \quad \text{s.t.}$$

(1) $X - \Sigma$ is a n-dim manifold with $\dim(\Sigma) \leq n-2$.

(2) $X_{n-k} - X_{n-k-1}$ is a $(n-k)$-dim manifold or empty.

(3) If $x \in X_{n-k} - X_{n-k-1}$, then $\exists$ nbhd N of x s.t.

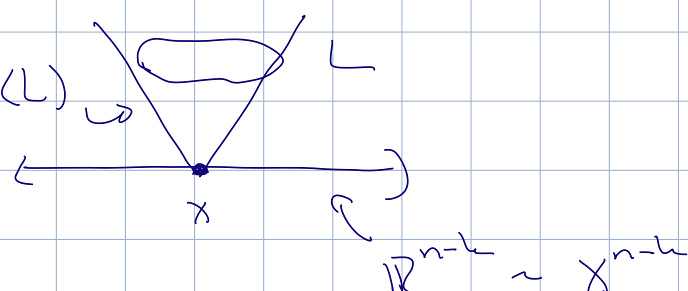
$$N \cong \mathbb{R}^{n-k} \times cL$$

for some $(k-1)$-dim cpt., strat., p. manifold L, and

$c(L)$ is the open cone of L

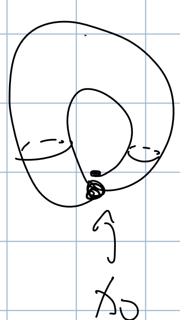
$$= L \times [0,\infty) / (x,0) \sim (y,0)$$

$X - X^{n-k} \sim c(L) \hookrightarrow$



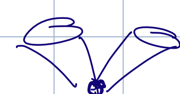
N

Ex)

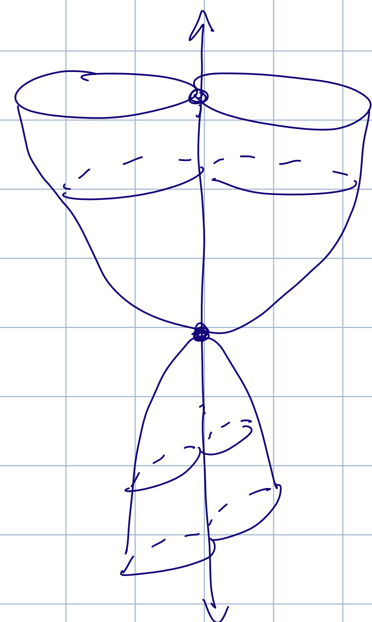


$X = X_2$

$$N \cong \mathbb{R}^0 \times c(\underbrace{S^1 \sqcup S^1}_{=L})$$



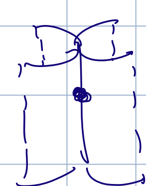
Ex) $X : x^4 + y^4 = xyz$



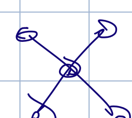
$X_2 = X$

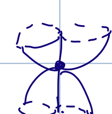
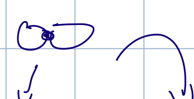
$X_1 = z\text{-axis}$

$X_0 = \text{origin}$

$x \in X_1 - X_0 \rightsquigarrow$ 

$N = \mathbb{R}^1 \times c(L)$

 $L = 4$ pts.

$x = \text{origin}$  

$N = \mathbb{R}^0 \times c(L) \qquad L = (S^1 \vee S^1) \sqcup (S^1 \vee S^1)$

Ex) If X is a manifold, then

$X = X_n \supseteq \phi \supseteq \phi \supseteq \cdots \supseteq \phi$ is a topological stratification

Thm (Borel): Any Whitney stratification of a complex quasi-proj variety of dim n is a stratified pseudomanifold of dim $2n$.

$$\overset{\text{Whitney}}{\curvearrowright} Y_n \supseteq Y_{n-1} \supseteq \cdots \supseteq Y_0$$

$$\overbrace{X_{2n} = X_{2n-1}} \supseteq \overbrace{X_{2n-2} = X_{2n-3}} \supseteq \cdots \supseteq \overbrace{X_1 = X_0}$$

(topological)

Thm (Łojasiewicz, Goresky)

$\exists$ a triangulation of X compatible with any Whitney stratification.

(piecewise linear stratification)

Intersection chains + perversities

Let $X = X_n \supseteq \cdots \supseteq X_0$ be a PL-stratified pseudomanifold (dim $= n$) with triangulation

$$T : |K| \to X$$

i-chain: $\xi = \sum_{\sigma \in K^{(i)}} c_\sigma \cdot \sigma \in C^T_i(X)$

and

support:
$$|\xi| = \bigcup_{c_\sigma \neq 0} T(\sigma) \subseteq X.$$

<u>Defn</u>: A <u>perversity</u> is a map

$$p : \{2, \ldots, n\} \to \mathbb{Z}$$

Ex) $(0, \ldots, 0)$ zero perversity

$(0, 1, 2, \ldots, n-2)$ top perversity

We say $\xi \in C_i^{T}(X)$ is (p,i)-allowable if

$$\dim_{\mathbb{R}}\left(|\xi| \cap X_{n-k}\right) \leq i-k+p(k)$$

codim: $n-i$, k

for all $k \geq 1$

$\rightarrow$ codim: $n-i+k$ $\rightarrow$ dim: $i-k$

<u>Defn</u>: Let $IC_i^{p,T}(X)$ denote the subspace of i-chains in $\xi \in C_i^{T}(X)$ s.t.

① ξ is (p,i)-allowable

② $\partial\xi$ is $(p,i-1)$-allowable.

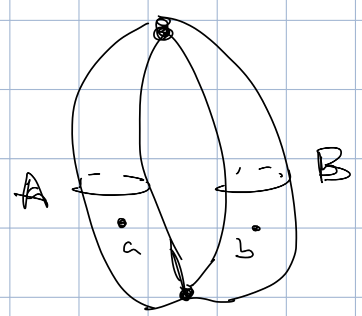
Chain complex:

$$\cdots \longrightarrow IC_i^{p}(X) \xrightarrow{\partial} IC_{i-1}^{p}(X) \rightarrow \cdots$$

<u>Defn</u>: The i^{th} intersection homology group

$$IH_i^{T}(X) = \frac{\ker\left(\partial_i : IC_i \rightarrow IC_{i-1}\right)}{\operatorname{im}\left(\partial_{i+1} : IC_{i+1} \rightarrow IC_i\right)}$$

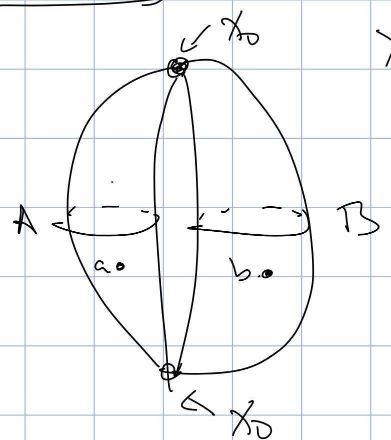
Ex) $X = \text{susp}(S^1 \sqcup S^1)$



H_0 : $[a] = [b]$ $\qquad\qquad \mathbb{Q}$

H_1 : $[\text{susp}\, a - \text{susp}\, b]$ $\qquad \mathbb{Q}$

H_2 : $[\text{susp}\, A], [\text{susp}\, B]$ $\qquad \mathbb{Q} \oplus \mathbb{Q}$



$X = X_2$, X_0 = 2 pts

$\bar{p} = (0,0)$

- 0-cycles, 1 cycles avoid X_0.
- 2-cycles okay if boundary avoids X_0.

$IH_0^{\bar{p}}$: $[a], [b]$ $\qquad \mathbb{Q} \oplus \mathbb{Q}$

$IH_1^{\bar{p}} = 0$ $\qquad 0$

$IH_2^{\bar{p}}$: $[\text{susp}\, A], [\text{susp}\, B]$ $\qquad \mathbb{Q} \oplus \mathbb{Q}$.

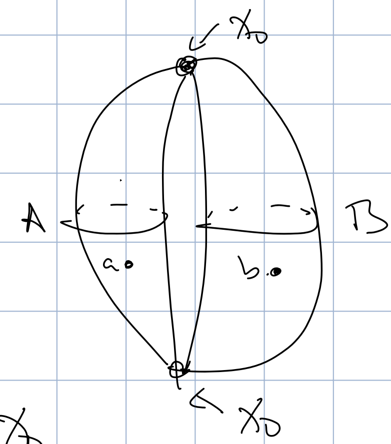
susp a − susp b not allowed

$A = \partial(\text{cone } A)$

↖ allowed.

↙

$B = \partial(\text{cone } B)$



$p = (0,1)$
- 0-cycles avoid x_0
- 1-cycles okay if ∂C_1 avoids x_0
- 2-cycles okay.

IH_0^p : $[a], [b]$ $\qquad \mathbb{Q} \oplus \mathbb{Q}$

IH_1^p : $[\text{susp } a - \text{susp } b]$ $\qquad \mathbb{Q}$

IH_2^p : $[\text{susp } A], [\text{susp } B]$ $\qquad \mathbb{Q} \oplus \mathbb{Q}$.

$p = (0,-1)$ All cycles + boundaries avoid x_0