Intersection homology.

Let X be a n-dim, cpt, oriented mfld.

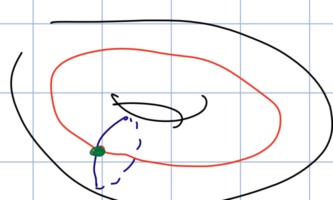
$V, W \subseteq X$

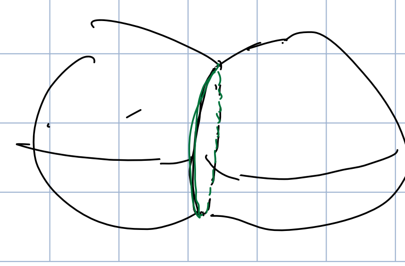
i-cycle, j-cycle

Poincaré + Lefschetz (1890's - 1920's)

1. If V, W are general, then $V \cap W$ is $(i+j-n)$-cycle.

2. The homology class $[V \cap W]$ is determined by $[V], [W]$.

Ex)  $X = T_2$ (torus)



Consequences:

① There is an intersection product

$$H_i(X) \times H_j(X) \xrightarrow{\cap} H_{i+j-n}(X)$$

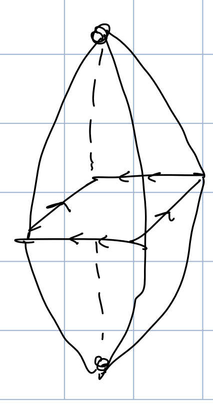
② Poincaré duality:

$$H_i(X) \times H_{n-i}(X) \to H_0(X) \simeq \mathbb{Z}$$

is nondegenerate

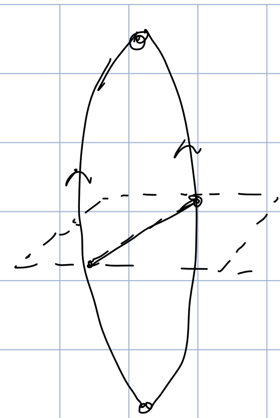
If X is singular, these properties can fail.

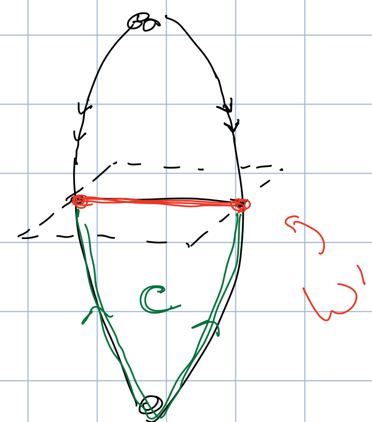
Ex) X = suspended torus

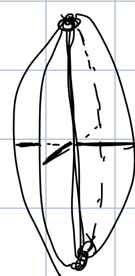


$= X \times [0,1]/\sim$

3-dim pseudomanifold.

$V =$ 

$W =$ 

① $V \cap W =$  not a cycle.

② $V \cap W' = pt \rightsquigarrow [V \cap W'] = [pt]$

but $W' = \partial C$, so $[W'] = 0$

Thus $[V \cap W']$ is not determined by

$[V], [W']$.

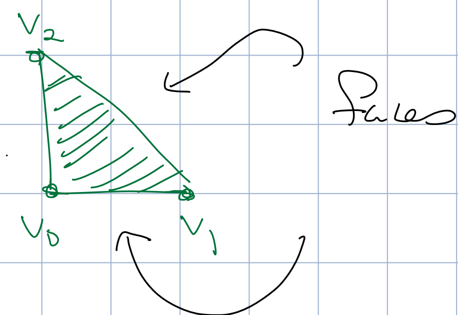
Q: Is there a homology theory of singular spaces with "manifold" properties?

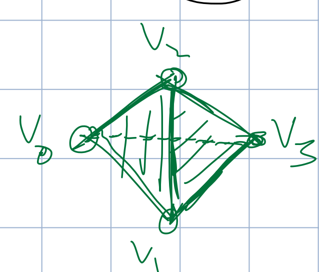
Intersection homology

Review of simplicial homology

<u>Defn</u>: A n-simplex σ of $\mathbb{R}^N$ is the convex hull of points $v_0, \dots, v_n$ s.t. $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ linearly independent vectors

The faces of σ are the $(n-1)$-simplices given by $v_0, \dots, \hat{v}_i, \dots, v_n$ for $i \in \{0, \dots, n\}$

Ex) $\sigma =$ 

Ex) $\sigma =$ 

An orientation of σ is an ordering of vertices up to even permutations

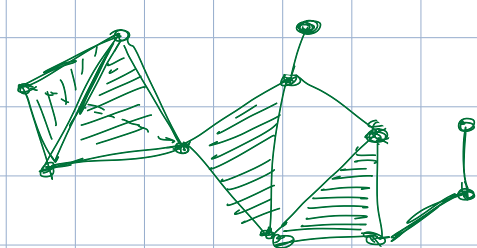
Defn: A <u>simplicial complex</u>, K of $\mathbb{R}^N$ a set of simplices such that:

1. If $\sigma \in K$, τ a face of σ, then $\tau \in K$.

2. If $\sigma, \tau \in K$ and $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a simplex whose vertices are also vertices of both σ and τ.

3. If $x \in \sigma \in K$, then $\exists$ a nbd. U of x such that $U \cap \tau \neq \emptyset$ for only finitely many simplices $\tau \in N$.

Ex)



For each $\sigma \in K$, fix an orientation and define

$$K^{(i)} := \{ \sigma \in K \mid \sigma \text{ i-simplex} \}.$$

An <u>i-chain</u> is a formal l.c.

$$\xi = \sum_{\sigma \in K^{(i)}} c_\sigma \cdot \sigma, \quad c_\sigma \in \mathbb{Z}$$

(finite sum)

Let $C_i(K)$ denote the space of i-chains.

The boundary map:

$$\partial_i : C_i(K) \to C_{i-1}(K)$$

is given by

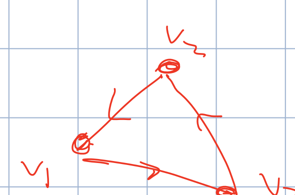
orientation of σ

$$\partial_i(\sigma) = \sum_{j=0}^{i} (-1)^j \langle v_0, \ldots, \hat{v}_j, \ldots, v_i \rangle$$

faces of σ

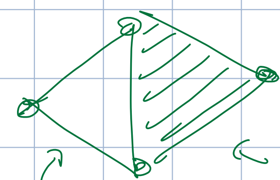
An i-chain ξ is called an <u>i-cycle</u> if

$\sigma \in \ker \partial_i$



Fact: $\partial_i \circ \partial_{i+1} = 0$.

So $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$



Defn: The <u>i^{th} homology group</u> of K is

$$H_i(K) := \ker \partial_i \,/\, \operatorname{im} \partial_{i+1}$$

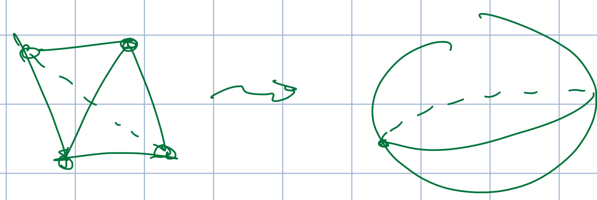
Triangulable spaces

The <u>support</u> of a simplicial complex K is

$$|K| = \bigcup_{\sigma \in N} \sigma \subseteq \mathbb{R}^N$$

A <u>triangulation</u> of a topological space X is a homeomorphism

$$T : |K| \to X.$$

Ex) T: 

$|K|$ $\quad\quad$ S^2

If $T : |K| \to X$ is a triangulation, then

define
$$C_i^T(X) := C_i^T(K)$$

and
$$H_i^T(X) := H_i(K),$$

the i^{th} homology group of X w.r.t. T.

If T, T' are triangulations of X s.t.

$$\forall \sigma \in K, \exists \sigma' \in K' \text{ s.t. } T(\sigma) \subseteq T'(\sigma'),$$

then T is said to be a refinement of T'.

Let $C_i(X) = \varinjlim_{T} C_i^T(X)$

(Direct limit under refinement of triangulations)

denote the space of p.l. i-chains on X.

Defn: the i^{th}-simplicial homology group

$$H_i(X) := \frac{\ker \partial_i : C_i(X) \to C_{i-1}(X)}{\operatorname{im} \partial_{i+1} : C_{i+1}(X) \to C_i(X)}$$

Thm: If X is triangulable, then

$$H_i(X) \simeq H_i^T(X) \cong H_i^{sing}(X)$$

Piecewise linear (PL) stratified pseudomanifolds

A compact space X is a n-dim **pseudomanifold** if there is a closed subspace Σ s.t.

① $X - \Sigma$ is a n-dim manifold

② $\dim(\Sigma) \leq n-2$

Stratification of X is a sequence

$$X = X_n \supseteq X_{n-2} \supseteq X_{n-3} \supseteq \cdots \supseteq X_0 \supseteq X_{-1} = \emptyset$$

s.t. ① $X_{n-2} = \Sigma$

② $X_{n-k} - X_{n-k-1}$ is either a

ⓐ $(n-k)$-dim manifold

ⓑ empty.

③ If $x \in X_{n-k} - X_{n-k-1}$, then

$\exists$ a nbhd N of x s.t.

$$N \cong \mathbb{R}^{n-k} \times cL$$

where L is a stratified $(k-1)$-dim pseudomanifold

and cL is the open cone over L.