IMAGE OF FUNCTORIALITY FOR GENERAL SPIN GROUPS

MAHDI ASGARI AND FREYDOON SHAHIDI

Abstract. We give a complete description of the image of the endoscopic functorial transfer of generic automorphic representations from the quasi-split general spin groups to general linear groups over arbitrary number fields. This result is not covered by the recent work of Arthur on endoscopic classification of automorphic representations of classical groups. The image is expected to be the same for the whole tempered spectrum, whether generic or not, once the transfer for all tempered representations is proved. We give a number of applications including estimates toward the Ramanujan conjecture for the groups involved and the characterization of automorphic representations of GL(6) which are exterior square transfers from GL(4), among others. More applications to reducibility questions for the local induced representations of p-adic groups will also follow.

1. Introduction

The purpose of this article is to completely determine the image of the transfer of globally generic, automorphic representations from the quasi-split general spin groups to the general linear groups. We prove the image is an isobaric, automorphic representation of a general linear group. Moreover, we prove that the isobaric summands of this representation have the property that their twisted symmetric square or twisted exterior square L-function has a pole at $s = 1$, depending on whether the transfer is from an even or an odd general spin group. We are also able to determine whether the image is a transfer from a split group, or a quasi-split, non-split group.

Arthur's recent book on endoscopic classification of representations of classical groups [Ar] establishes similar results for automorphic representations of orthogonal and symplectic groups, generic or not, but does not cover the case of the general spin groups we consider here.

To describe our results in more detail, let k be a number field and let $\mathbb{A} = \mathbb{A}_k$ denote its ring of adèles. Let $\mathbf{G}$ be the split group $\mathrm{GSpin}(2n+1)$, $\mathrm{GSpin}(2n)$, or one of its quasi-split non-split forms $\mathrm{GSpin}^*(2n)$ associated with a quadratic extension K/k (cf. Section 2). There is a natural embedding

$$\iota : {}^L\mathbf{G} \longrightarrow \mathrm{GL}(2n, \mathbb{C}) \times \Gamma_k \tag{1.1}$$

of the L-group of $\mathbf{G}$, as a group over k, into that of $\mathrm{GL}(2n)$ described in Section 3. Let π be a globally generic, (unitary) cuspidal, automorphic representation of $G = \mathbf{G}(\mathbb{A})$. For almost all places v of k the local representation π_v is parameterized by a homomorphism

$$\phi_v : W_v \longrightarrow {}^L\mathbf{G}_v, \tag{1.2}$$

where W_v is the local Weil group of k_v and ${}^L\mathbf{G}_v$ is the L-group of $\mathbf{G}$ as a group over k_v. Langlands Functoriality then predicts that there is an automorphic representation Π of $\mathrm{GL}(2n, \mathbb{A})$ such that for almost all v, the local representation Π_v is parameterized by $\iota \circ \phi_v$.

Our main result is to prove that the transferred representation Π of $\mathrm{GL}(2n)$, from either an even or an odd general spin group, is indeed an isobaric, automorphic representation (cf. Theorem 4.20 and its corollary). We then give a complete description of the image of these transfers in terms of the twisted symmetric square or exterior square L-functions (cf. Theorem 4.26). We refer to [L1] for the notion of isobaric representations.

To prove that the image is isobaric, one needs to know some of the analytic properties of the Rankin-Selberg type L-functions $L(s, \pi \times \tau)$, where τ is a cuspidal representation of $\mathrm{GL}(m, \mathbb{A})$ and π is a generic representation of $\mathbf{G}(\mathbb{A})$. In particular, one needs to know that the L-functions for $\mathbf{G} \times \mathrm{GL}(m)$ for m up to $2n$ are holomorphic for $\Re(s) > 1$ and to know under what conditions these L-function have poles at $s = 1$. These facts are established in Section 4. Here, we use a result on possible poles of the Rankin-Selberg L-functions for $\mathbf{G} \times \mathrm{GL}(m)$ (Proposition 4.9) which we establish in [ACS1] as part of a more extensive project. Consequently, we are able to prove that the transferred representation Π is unique; it is an isobaric sum of pairwise inequivalent, unitary, cuspidal, automorphic representations

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_t, \tag{1.3}$$

and each Π_i satisfies the condition that its twisted symmetric square or twisted exterior square L-function has a pole at $s = 1$, depending on whether we are transferring from even or odd general spin groups (cf. Theorem 4.26). In particular, we prove that $\Pi \cong \widetilde{\Pi} \otimes \omega$ (and not just nearly equivalent as we showed in [AS1]). Here, $\widetilde{\Pi}$ denotes the contragredient of Π.

The automorphic representations Π of $\mathrm{GL}(2n, \mathbb{A})$ which are transfers from representations π of general spin groups satisfy

$$\Pi \cong \widetilde{\Pi} \otimes \omega, \tag{1.4}$$

as predicted by the theory of twisted endoscopy [KoSh]. In fact, these representations comprise precisely the image of the transfer. While we prove half of this statement, we note that the other half of this, i.e., the fact that any representation of $\mathrm{GL}(2n, \mathbb{A})$ satisfying (1.4) is a transfer from a representation of a general spin group has now also been proved thanks to the work of J. Hundley and E. Sayag [HS1, HS2], extending the descent theory results of Ginzburg, Rallis, and Soudry [GRS] from the case of classical groups ($\omega = 1$) to general spin groups .

If a representation Π of $\mathrm{GL}(2n, \mathbb{A})$ satisfies (1.4), then

$$L^T\left(s, \Pi \times (\Pi \otimes \omega^{-1})\right) = L^T(s, \Pi, \mathrm{Sym}^2 \otimes \omega^{-1}) L^T(s, \Pi, \wedge^2 \otimes \omega^{-1}), \tag{1.5}$$

where T is a sufficiently large finite set of places of k and L^T denotes the product over $v \notin T$ of the local L-functions. The L-function on the left hand side of (1.5) has a pole at $s = 1$, which implies that one, and only one, of the two L-functions on the right hand side of (1.5) has a pole at $s = 1$. If the twisted exterior square L-function has a pole at $s = 1$, then Π is a transfer from an odd general spin group and if the twisted symmetric square L-function has a pole at $s = 1$, then Π is a transfer from an even general spin group (which may be split or quasi-split non-split).

To tell the split and quasi-split non-split cases apart, note that from (1.4) we have

$$\omega_\Pi^2 = \omega^{2n}. \tag{1.6}$$

In other words, $\mu = \omega_\Pi \omega^{-n}$ is a quadratic idèle class character. If μ is the trivial character, then Π is the transfer of a generic representation of the even split general spin group and if μ is a non-trivial quadratic character, then Π is a transfer from a generic representation of a quasi-split group associated with the quadratic extension of k determined by μ through class field theory.

Our results here along with those of Hundley and Sayag [HS1, HS2] give a complete description of the image of the transfer for the generic representations of the general spin groups. It remains to study the transfers of non-generic, cuspidal, automorphic representations of the general spin groups, which our current method cannot handle. However, the image of the generic spectrum is conjecturally the full image of the tempered spectrum, generic or not, of the general spin groups since each tempered L-packet is expected to contain a generic member. We refer to [Sh6] for more details on this conjecture. We point out that Arthur's upcoming book [Ar] would answer this question in the case of classical groups. However, his book does not cover the case of general spin groups.

While we characterize the automorphic representation Π in the image, we should remark that the existence of Π in the case of the split groups $\mathrm{GSpin}(2n)$ and $\mathrm{GSpin}(2n+1)$ was established in [AS1]. However, we were not able to prove the quasi-split case then. One of our results, therefore, is also to establish the transfer of globally generic, automorphic representations from the quasi-split non-split even general spin group, $\mathrm{GSpin}^*(2n, \mathbb{A}_k)$, to $\mathrm{GL}(2n, \mathbb{A}_k)$ (cf. Theorem 3.5). The quasi-split case had to wait because the local technical tools of "stability of γ-factors" (cf. Proposition 3.15) and a result on local L-functions and normalized intertwining operators (Proposition 3.13) were not available in the quasi-split non-split case. The local result is now available in our cases thanks to the thesis of Wook Kim [WKim], and more generally the work of Heiermann and Opdam [HO], and the stability of γ-factors is available in appropriate generality thanks to a recent work of Cogdell, Piatetski-Shapiro and Shahidi [CPSS1].

As in the split case, the method of proving the existence of an automorphic representation Π is to use converse theorems. This requires knowledge of the analytic properties of the L-functions for $\mathrm{GL}(m) \times \mathrm{GL}(2n)$ for $m \leq 2n-1$. The two local tools allow us to relate the L-functions for $\mathbf{G} \times \mathrm{GL}$ from the Langlands-Shahidi method to those required in the converse theorems in the following way. Due to the lack of the local Langlands correspondence in general, there is no natural choice for the local components of our candidate representation Π at the finite number of exceptional places of k where some of our data may be ramified. This means that we have to pick these local representations essentially arbitrarily. However, we show that the local γ-factors appearing will become independent of the representation, depending only on the central character, if we twist by a highly ramified character. Globally we can afford to twist our original representation by an idèle class character which is highly ramified at a finite number of places. With this technique we succeed in applying an appropriate version of the converse theorems. The conclusion is to have an automorphic representation Π of $\mathrm{GL}(2n, \mathbb{A})$ which is locally the transfer of π associated with ι outside a finite number of finite places. Moreover, if $\omega = \omega_\pi$ is the central character of π, then $\omega_\Pi = \omega^n \mu$, where μ is a quadratic idèle class character, only non-trivial in the quasi-split non-split case. In that case it determines the defining quadratic extension. This settles the existence of Π as an automorphic representation in all cases, whose complete description we give, as we explained above.

We summarize the results on Π being isobaric and its description in terms of twisted symmetric or exterior square L-functions in Theorem 4.26.

These results allow us to give a number of applications. As a first application, we are able to describe the local component of the transferred representation at the ramified places. In particular, we show that these local components are generic (cf. Proposition 5.1).

Another application is to prove estimates toward the Ramanujan conjecture for the generic spectrum of the general spin groups. We do this by using the best estimates currently known for the general linear groups [LRS]. In particular, our estimates show that if we know the Ramanujan conjecture for $\mathrm{GL}(m)$ for m up to $2n$, then the Ramanujan conjecture for the generic spectrum of $\mathrm{GSpin}(2n+1)$ and $\mathrm{GSpin}(2n)$ follows. We note that the Ramanujan conjecture is expected *not* to hold for certain non-generic representations of the general spin groups.

Yet another application of our main results is to give more information about H. Kim's exterior square transfer from GL(4) to GL(6) with the help of some recent work of J. Hundley and E. Sayag. We prove that a cuspidal representation Π of GL(6) is in the image of Kim's transfer if and only if the (partial) twisted symmetric square L-function of Π has a pole at $s = 1$ (cf. Proposition 5.10).

We can apply our results in this paper, along with those of Cogdell, Kim, Krishnamurthy, Piatetski-Shapiro, and Shahidi for the classical and unitary groups, to give some uniform results on reducibility of local induced representations of non-exceptional p-adic groups. We will address this question along with other local applications of generic functoriality in a forthcoming paper [ACS2].

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Contents

2. The Preliminaries

Let k be a number field and let $\mathbb{A} = \mathbb{A}_k$ be the ring of adèles of k. Let $n \geq 1$ be an integer. We consider the general spin groups. The group GSpin($2n+1$) is a split, connected, reductive group of type B_n defined over k whose dual group is $\mathrm{GSp}(2n, \mathbb{C})$. Similarly, the split, connected, reductive group $\mathrm{GSpin}(2n)$ over k is of type D_n and its dual is isomorphic to $\mathrm{GSO}(2n, \mathbb{C})$, the connected component of the group $\mathrm{GO}(2n, \mathbb{C})$. There are also quasi-split non-split groups $\mathrm{GSpin}^*(2n)$ in the even case. They are of type ${}^2D_n$ and correspond to quadratic extensions of k. A more precise description is given below. We also refer to [CPSS2, §7] for a review of the generalities about these quasi-split groups.

We fix a Borel subgroup $\mathbf{B}$ and a Cartan subgroup $\mathbf{T} \subset \mathbf{B}$. The associated based root datum to $(\mathbf{B}, \mathbf{T})$ will be denoted by $(X, \Delta, X^\vee, \Delta^\vee)$, which we further explicate below. Our choice of the notation for the root data below is consistent with the Bourbaki notation [Bou].

2.1. **Structure of** GSpin **Groups.** We describe the odd and even GSpin groups by introducing a based root datum for each as in [Spr, §7.4.1]. More detailed descriptions can also be found in [AS1, §2] and [HS2, §4]. We use these data as our tools to work with the groups in question due to the lack of a convenient matrix representation.

2.1.1. *The root datum of* $\mathrm{GSpin}(2n+1)$. The root datum of $\mathrm{GSpin}(2n+1)$ is given by $(X, R, X^\vee, R^\vee)$, where X and $X^\vee$ are $\mathbb{Z}$-modules generated by generators $e_0, e_1, \cdots, e_n$ and $e_0^*, e_1^*, \cdots, e_n^*$, respectively. The pairing

$$\langle\ ,\ \rangle : X \times X^\vee \longrightarrow \mathbb{Z} \tag{2.1}$$

is the standard pairing. Moreover, the roots and coroots are given by

$$\begin{aligned} R = R_{2n+1} &= \{\pm(e_i \pm e_j) \,:\, 1 \le i < j \le n\} \cup \{\pm e_i \,|\, 1 \le i \le n\} && (2.2)\\ R^\vee = R^\vee_{2n+1} &= \left\{\pm(e_i^* - e_j^*) \,:\, 1 \le i < j \le n\right\} \cup && (2.3)\\ &\quad \left\{\pm(e_i^* + e_j^* - e_0^*) \,:\, 1 \le i < j \le n\right\} \cup \{\pm(2e_i^* - e_0^*) \,|\, 1 \le i \le n\} \end{aligned}$$

along with the bijection $R \longrightarrow R^\vee$ given by

$$\begin{aligned} (\pm(e_i - e_j))^\vee &= \pm(e_i^* - e_j^*) && (2.4)\\ (\pm(e_i + e_j))^\vee &= \pm(e_i^* + e_j^* - e_0^*) && (2.5)\\ (\pm e_i)^\vee &= \pm(2e_i^* - e_0^*). && (2.6) \end{aligned}$$

It is easy to verify that the conditions (RD 1) and (RD 2) of [Spr, §7.4.1] hold. Moreover, we fix the following choice of simple roots and coroots:

$$\begin{aligned} \Delta &= \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_n\}, && (2.7)\\ \Delta^\vee &= \left\{e_1^* - e_2^*, e_2^* - e_3^*, \cdots, e_{n-1}^* - e_n^*, 2e_n^* - e_0^*\right\}. && (2.8) \end{aligned}$$

This datum determines the group $\mathrm{GSpin}(2n+1)$ uniquely, equipped with a Borel subgroup containing a maximal torus.

2.1.2. *The root datum of* $\mathrm{GSpin}(2n)$. Next, we give a similar description for the even case. The root datum of $\mathrm{GSpin}(2n)$ is given by $(X, R, X^\vee, R^\vee)$ where X and $X^\vee$ and the pairing is as above and the roots and coroots are given by

$$\begin{aligned} R = R_{2n} &= \{\pm(e_i \pm e_j) \,:\, 1 \le i < j \le n\} && (2.9)\\ R^\vee = R^\vee_{2n} &= \left\{\pm(e_i^* - e_j^*) \,:\, 1 \le i < j \le n\right\} \cup && (2.10)\\ &\quad \left\{\pm(e_i^* + e_j^* - e_0^*) \,:\, 1 \le i < j \le n\right\} \end{aligned}$$

along with the bijection $R \longrightarrow R^\vee$ given by

$$\begin{aligned} (\pm(e_i - e_j))^\vee &= \pm(e_i^* - e_j^*) && (2.11)\\ (\pm(e_i + e_j))^\vee &= \pm(e_i^* + e_j^* - e_0^*). && (2.12) \end{aligned}$$

It is easy again to verify that the conditions (RD 1) and (RD 2) of [Spr, §7.4.1] hold. Similar to the odd case we fix the following choice of simple roots and coroots:

$$\Delta = \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_{n-1} + e_n\}, \tag{2.13}$$

$$\Delta^\vee = \left\{e_1^* - e_2^*, e_2^* - e_3^*, \cdots, e_{n-1}^* - e_n^*, e_{n-1}^* + e_n^* - e_0^*\right\}. \tag{2.14}$$

This based root datum determines the split group $\mathrm{GSpin}(2n)$ uniquely, equipped with a Borel subgroup containing a maximal torus.

2.1.3. *The quasi-split forms of* $\mathrm{GSpin}(2n)$. In the even case, quasi-split non-split forms also exist. We fix a splitting $(\mathbf{B}, \mathbf{T}, \{x_\alpha\}_{\alpha\in\Delta})$, where $\{x_\alpha\}$ is a collection of root vectors, one for each simple root of $\mathbf{T}$ in $\mathbf{B}$. The quasi-split forms of $\mathrm{GSpin}(2n)$ over k are in one-one correspondence with homomorphisms from $\mathrm{Gal}(\bar{k}/k)$ to the group of automorphisms of the character lattice preserving Δ. This group has two elements: the trivial and the one switching $e_{n-1} - e_n$ and $e_{n-1} + e_n$ while keeping all other simple roots fixed (cf. [HS2, §4.3], and [CPSS2, §7.1] for the quasi-split forms of $\mathrm{SO}(2n)$).

By Class Field Theory such homomorphisms correspond to quadratic characters

$$\mu : k^\times \backslash \mathbb{A}_k^\times \longrightarrow \{\pm 1\}. \tag{2.15}$$

When μ is non-trivial we denote the associated quasi-split non-split group with $\mathrm{GSpin}^\mu(2n)$ or simply $\mathrm{GSpin}^*(2n)$ when the particular μ is unimportant. We will also denote the quadratic extension of k associated with μ by K^μ/k or simply K/k.

3. Weak Transfer for the Quasi-split $\mathrm{GSpin}(2n)$

In this section, $n \geq 1$ will be an integer and $\mathbf{G} = \mathrm{GSpin}^*(2n)$ will denote one of the quasi-split non-split forms of $\mathrm{GSpin}(2n)$ as in 2.1.3. We will denote the associated quadratic extension by K/k and $\mathbb{A} = \mathbb{A}_k$ will continue to denote the ring of adèles of k. Also, $\mathbf{G}$ is associated with a non-trivial quadratic character $\mu : k^\times \backslash \mathbb{A}_k^\times \longrightarrow \{\pm 1\}$.

The connected component of the L-group of $\mathbf{G}$ is ${}^L\mathbf{G}^0 = \mathrm{GSO}(2n, \mathbb{C})$ and the L-group can be written as

$${}^L\mathbf{G} = \mathrm{GSO}(2n, \mathbb{C}) \rtimes W_k, \tag{3.1}$$

where the Weil group acts through the quotient

$$W_k/W_K \cong \mathrm{Gal}(K/k). \tag{3.2}$$

The L-group of $\mathrm{GL}(2n)$ is $\mathrm{GL}(2n, \mathbb{C}) \times W_k$, a direct product because $\mathrm{GL}(2n)$ is split. These are the Weil forms of the L-group, or we can equivalently use the Galois forms of the L-groups.

We define a map

$$\iota : \mathrm{GSO}(2n, \mathbb{C}) \rtimes \Gamma_k \longrightarrow \mathrm{GL}(2n, \mathbb{C}) \times \Gamma_k \tag{3.3}$$

$$(g, \gamma) \mapsto \begin{cases} (g, \gamma) & \text{if } \gamma_{|K} = 1, \\ (hgh^{-1}, \gamma) & \text{if } \gamma_{|K} \neq 1, \end{cases}$$

where $\gamma \in \Gamma_k$, $g \in \mathrm{GSO}(2n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{C})$, and

$$h = h^{-1} = \begin{pmatrix} I_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-1} \end{pmatrix}. \tag{3.4}$$

(We refer to [CPSS2, §7.1] for more details.) The map ι is an L-homomorphism. We also have a compatible family of local L-homomorphisms $\iota_v : {}^L\mathbf{G}_v \longrightarrow \mathrm{GL}(2n, \mathbb{C}) \times W_v$. Our purpose in this section is to prove the existence of a weak transfer of globally generic, cuspidal, automorphic representations of $G = \mathbf{G}(\mathbb{A})$ to automorphic representations of $\mathrm{GL}(2n, \mathbb{A})$ associated with ι.

Theorem 3.5. *Assume that $n \geq 1$ is an integer. Let K/k be a quadratic extension of number fields and let $\mathbf{G} = \mathrm{GSpin}^*(2n)$ be as above. Let ψ be a non-trivial continuous additive character of $k\backslash\mathbb{A}_k$. The choice of ψ and the splitting above defines a non-degenerate additive character of $\mathbf{U}(k)\backslash\mathbf{U}(\mathbb{A})$, again denoted by ψ.*

Let $\pi = \otimes_v \pi_v$ be an irreducible, globally ψ-generic, cuspidal, automorphic representation of $G = \mathbf{G}(\mathbb{A}_k)$. Write $\psi = \otimes_v \psi_v$. Let S be a non-empty finite set of non-archimedean places v of k such that for every non-archimedean $v \notin S$ both π_v and ψ_v, as well as K_w/k_v for $w|v$, are unramified. Then there exists an automorphic representation $\Pi = \otimes_v \Pi_v$ of $\mathrm{GL}(2n, \mathbb{A}_k)$ such that for all $v \notin S$ the homomorphism parameterizing the local representation Π_v is given by

$$\Phi_v = \iota_v \circ \phi_v : W_{k_v} \to \mathrm{GL}(2n, \mathbb{C}),$$

where W_{k_v} denotes the local Weil group of k_v and $\phi_v : W_{k_v} \longrightarrow {}^L\mathbf{G}$ is the homomorphism parameterizing π_v. Moreover, if ω_Π and ω_π denote the central characters of Π and π, respectively, then $\omega_\Pi = \omega_\pi^n \mu$, where μ is the non-trivial quadratic idèle class character corresponding to K/k, the quadratic extension defining $\mathbf{G}$. Furthermore, Π and $\widetilde{\Pi} \otimes \omega_\pi$ are nearly equivalent.

Remark 3.6. We proved an analogous result for the split groups $\mathrm{GSpin}(2n+1)$ and $\mathrm{GSpin}(2n)$ in [AS1].

To prove the theorem we will use a suitable version of the converse theorems of Cogdell and Piatetski-Shapiro [CPS1, CPS2]. The exact version we need can be found in [CPSS2, §2] which we quickly review below. Next we introduce an irreducible, admissible representation Π of $\mathrm{GL}(2n, \mathbb{A})$ as a candidate for the transfer of π. We then prove that Π satisfies the required conditions of the converse theorem and hence is automorphic. Along the way we also verify the remaining properties of Π stated in Theorem 3.5.

3.1. **The Converse Theorem.** Let k be a number field and fix a non-empty finite set S of non-archimedean places of k. For each integer m let

$$\mathcal{A}_0(m) = \{\tau | \tau \text{ is a cuspidal representation of } \mathrm{GL}(m, \mathbb{A}_k)\} \tag{3.7}$$

and

$$\mathcal{A}_0^S(m) = \{\tau \in \mathcal{A}_0(m) | \tau_v \text{ is unramified for all } v \in S\}. \tag{3.8}$$

Also, for a positive integer N let

$$\mathcal{T}(N-1) = \coprod_{m=1}^{N-1} \mathcal{A}_0(m) \quad \text{and} \quad \mathcal{T}^S(N-1) = \coprod_{m=1}^{N-1} \mathcal{A}_0^S(m) \tag{3.9}$$

and for η a continuous character of $k^\times\backslash\mathbb{A}_k^\times$ let

$$\mathcal{T}(S;\eta) = \mathcal{T}^S(N-1) \otimes \eta = \left\{\tau = \tau' \otimes \eta | \tau' \in \mathcal{T}^S(N-1)\right\}. \tag{3.10}$$

For our purposes we will apply the following theorem with $N = 2n$.

Theorem 3.11. *(Converse theorem of Cogdell and Piatetski-Shapiro) Let $\Pi = \otimes \Pi_v$ be an irreducible, admissible representation of $\mathrm{GL}(N, \mathbb{A}_k)$ whose central character ω_Π is invariant under $k^\times$ and whose L-function*

$$L(s, \Pi) = \prod_v L(s, \Pi_v)$$

is absolutely convergent in some right half plane. Let S be a finite set of non-archimedean places of k and let η be a continuous character of $k^\times\backslash\mathbb{A}^\times$. Suppose that for every $\tau \in \mathcal{T}(S;\eta)$ the L-function $L(s,\tau\times\Pi)$ is nice, i.e., it satisfies the following three conditions:

(1) *$L(s,\tau\times\Pi)$ and $L(s,\widetilde{\tau}\times\widetilde{\Pi})$ extend to entire functions of $s\in\mathbb{C}$.*
(2) *$L(s,\tau\times\Pi)$ and $L(s,\widetilde{\tau}\times\widetilde{\Pi})$ are bounded in vertical strips.*
(3) *The functional equation $L(s,\tau\times\Pi)=\epsilon(s,\tau\times\Pi)L(s,\widetilde{\tau}\times\widetilde{\Pi})$ holds.*

Then there exists an automorphic representation Π' of $\mathrm{GL}(N,\mathbb{A}_k)$ such that $\Pi_v\cong\Pi'_v$ for all $v\notin S$. □

The twisted L- and ϵ-factors in the statement are those in [CPS1]. In particular, they are Artin factors and known to be the same as the ones coming from the Langlands-Shahidi method at all places.

3.2. **L-functions for $\mathrm{GL}(m)\times\mathrm{GSpin}^*(2n)$.** Let π be an irreducible, admissible, globally generic representation of $\mathrm{GSpin}^*(2n,\mathbb{A}_k)$ and let τ be a cuspidal representation of $\mathrm{GL}(m,\mathbb{A}_k)$ with $m\ge 1$. The group $\mathrm{GSpin}^*(2(m+n))$ has a standard maximal Levi $\mathrm{GL}(m)\times\mathrm{GSpin}^*(2n)$ and we have the completed L-functions

$$L(s,\tau\times\pi)=\prod_v L(s,\tau_v\times\pi_v)=\prod_v L(s,\tau_v\otimes\pi_v,\iota'_v\otimes\iota_v)=L(s,\tau\otimes\pi,\iota'\otimes\iota), \tag{3.12}$$

with similar ϵ- and γ-factors, defined via the Langlands-Shahidi method in [Sh3]. Here, ι is the representation of the L-group of $\mathrm{GSpin}^*(2n)$ we described before and ι' is the projection map onto the first factor in the L-group ${}^L\mathrm{GL}(m)=\mathrm{GL}(m,\mathbb{C})\times W_k$.

Proposition 3.13. *Let S be a non-empty finite set of finite places of k and let η be a character of $k^\times\backslash\mathbb{A}_k^\times$ such that, for some $v\in S$, η_v^2 is ramified. Then for all $\tau\in\mathcal{T}(S;\eta)$ the L-function $L(s,\tau\times\pi)$ is nice, i.e., it satisfies the following three conditions:*

(1) *$L(s,\tau\times\pi)$ and $L(s,\widetilde{\tau}\times\widetilde{\pi})$ extend to entire functions of $s\in\mathbb{C}$.*
(2) *$L(s,\tau\times\pi)$ and $L(s,\widetilde{\tau}\times\widetilde{\pi})$ are bounded in vertical strips.*
(3) *The functional equation $L(s,\tau\times\pi)=\epsilon(s,\tau\times\pi)L(s,\widetilde{\tau}\times\widetilde{\pi})$ holds.*

Proof. Twisting by η is necessary for conditions (1) and (2). Both (2) and (3) hold in wide generality.

Condition (2) follows from [GS, Cor. 4.5] and is valid for all $\tau\in\mathcal{T}(N-1)$, provided that one removes neighborhoods of the finite number of possible poles of the L-function. Condition (3) is a consequence of [Sh3, Thm. 7.7] and is valid for all $\tau\in\mathcal{T}(N-1)$.

Condition (1) follows from a more general result, [KS1, Prop. 2.1]. Note that this result rests on Assumption 1.1 of [KS1], sometimes called Assumption A [K1], on certain normalized intertwining operators being holomorphic and non-zero. Fortunately the assumption has been verified in our cases. The assumption requires two ingredients: the so-called "standard modules conjecture" and the "tempered L-functions conjecture". Both of these have been verified in our cases in Wook Kim's thesis [WKim]. For results proving various cases of this assumption we refer to [Sh3, CSh, MuSh, Mu, A, K3, Hei, KK]. Recently V. Heiermann and E. Opdam have proved the assumption in full generality in [HO]. □

The key now is to relate the L-functions $L(s,\tau\times\pi)$, defined via the Langlands-Shahidi method, to the L-functions $L(s,\Pi\times\tau)$ in the converse theorem. We note that, when we introduce our candidate for Π, for archimedean places and those non-archimedean places at which all data are unramified we know the equality of the local L-functions. However, we do not know this to be the case for ramified places. We get around this problem through stability of γ-factors, which basically makes the choice of local components of Π at the ramified places irrelevant as long as we can twist by highly ramified characters.

3.3. **Stability of γ-factors.** In this subsection let F denote a non-archimedean local field of characteristic zero. Let $G = \mathrm{GSpin}^*(2n, F)$, where the quasi-split non-split group is associated with a quadratic extension E/F.

Fix a non-trivial additive character ψ of F. Let π be an irreducible, admissible, ψ-generic representation of G and let η denote a continuous character of $\mathrm{GL}(1, F)$. Let $\gamma(s, \eta \times \pi, \psi)$ be the associated γ-factor defined via the Langlands-Shahidi method [Sh3, Theorem 3.5]. We have

$$\gamma(s, \eta \times \pi, \psi) = \frac{\epsilon(s, \eta \times \pi, \psi) L(1-s, \eta^{-1} \times \widetilde{\pi})}{L(s, \eta \times \pi)}. \tag{3.14}$$

Proposition 3.15. *Let π_1 and π_2 be two irreducible, admissible, ψ-generic representations of G having the same central characters. Then for a suitably highly ramified character η of $\mathrm{GL}(1, F)$ we have*

$$\gamma(s, \eta \times \pi_1, \psi_v) = \gamma(s, \eta \times \pi_2, \psi_v).$$

Proof. This is a special case of a more general theorem which is the main result of [CPSS1]. We note that in our case one has to apply that theorem to the self-associate maximal Levi subgroup $\mathrm{GL}(1) \times \mathrm{GSpin}^*(2n)$ in $\mathrm{GSpin}^*(2n+2)$ which does satisfy the assumptions of that theorem. ☐

3.4. **The Candidate Transfer.** We construct now a candidate global transfer $\Pi = \otimes_v \Pi_v$ as a restricted tensor product of its local components Π_v, which are irreducible, admissible representations of $\mathrm{GL}(2n, k_v)$. There are three cases to consider: (i) archimedean v, (ii) non-archimedean unramified v, (iii) non-archimedean ramified v.

3.4.1. *The archimedean transfer.* If v is an archimedean place of k, then by the local Langlands correspondence [L2, Bor] the representation π_v is parameterized by an admissible homomorphism ϕ_v and we choose Π_v to be the irreducible, admissible representation of $\mathrm{GL}(2n, k_v)$ parameterized by Φ_v as in the statement of Theorem 3.5. We then have

$$L(s, \pi_v) = L(s, \iota_v \circ \phi_v) = L(s, \Pi_v) \tag{3.16}$$

and

$$\epsilon(s, \pi_v, \psi_v) = \epsilon(s, \iota_v \circ \phi_v, \psi_v) = \epsilon(s, \Pi_v, \psi_v), \tag{3.17}$$

where the middle factors are the local Artin-Weil L- and ϵ-factors attached to representations of the Weil group as in [T]. The other L- and ϵ-factors are defined via the Langlands-Shahidi method which, in the archimedean case, are known to be the same as the Artin factors defined through the arithmetic Langlands classification [Sh1].

If τ_v is an irreducible, admissible representation of $\mathrm{GL}(m, k_v)$, then it is parameterized by an admissible homomorphism $\phi'_v : W_{k_v} \longrightarrow \mathrm{GL}(m, \mathbb{C})$ and the tensor product homomorphism $(\iota_v \circ \phi_v) \otimes \phi'_v : W_{k_v} \longrightarrow \mathrm{GL}(2mn, \mathbb{C})$ is another admissible homomorphism and we again have

$$L(s, \pi_v \times \tau_v) = L(s, (\iota_v \circ \phi_v) \otimes \phi'_v) = L(s, \Pi_v \times \tau_v) \tag{3.18}$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, (\iota_v \circ \phi_v) \otimes \phi'_v, \psi_v) = \epsilon(s, \Pi_v \times \tau_v, \psi_v). \tag{3.19}$$

Hence, we get the following matching of the twisted local L- and ϵ-factors.

Proposition 3.20. *Let v be an archimedean place of k and let π_v be an irreducible, admissible, generic representation of* $\mathrm{GSpin}^*(2n, k_v)$*,* Π_v *its local functorial transfer to* $\mathrm{GL}(2n, k_v)$*, and τ_v an irreducible, admissible, generic representation of* $\mathrm{GL}(m, k_v)$*. Then*

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v), \quad L(s, \widetilde{\pi}_v \times \widetilde{\tau}_v) = L(s, \widetilde{\Pi}_v \times \widetilde{\tau}_v)$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, \Pi_v \times \tau_v, \psi_v).$$

□

3.4.2. *The non-archimedean unramified transfer.* If v is a non-archimedean place of k such that π_v, as well as all K_w/k_v for $w|v$, are unramified, then by the arithmetic Langlands classification or the Satake classification [Bor, Sat], the representation π_v is parameterized by an unramified admissible homomorphism $\phi_v : W_{k_v} \longrightarrow {}^L\mathbf{G}_v$. Again we take Φ_v as in the statement of the theorem. It defines an irreducible, admissible, unramified representation Π_v of $\mathrm{GL}(2n, k_v)$ [HT, H1].

Given that π_v is unramified, its parameter ϕ_v factors through an unramified homomorphism into the maximal torus ${}^L\mathbf{T}_v \hookrightarrow {}^L\mathbf{G}_v$. Then Φ_v has its image in a torus of $\mathrm{GL}(2n, \mathbb{C})$ and Π_v is the corresponding unramified representation. We have

$$L(s, \pi_v) = L(s, \Pi_v) \tag{3.21}$$

and

$$\epsilon(s, \pi_v, \psi_v) = \epsilon(s, \Pi_v, \psi_v) \tag{3.22}$$

and the factors on either side of the above equations can be expressed as products of one-dimensional abelian Artin factors by multiplicativity of the local factors.

Let τ_v be an irreducible, admissible, generic, unramified representation of $\mathrm{GL}(m, k_v)$. Again appealing to the general multiplicativity of local factors [JPSS, Sh3, Sh4] we have

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v) \tag{3.23}$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, \Pi_v \times \tau_v, \psi_v). \tag{3.24}$$

Hence, we again get the following matching of the twisted local L- and ϵ-factors.

Proposition 3.25. *Let v be a non-archimedean place of k and let π_v be an irreducible, admissible, generic, unramified representation of* $\mathrm{GSpin}^*(2n, k_v)$*,* Π_v *its local functorial transfer to* $\mathrm{GL}(2n, k_v)$*, and τ_v an irreducible, admissible, generic representation of* $\mathrm{GL}(m, k_v)$*. Then*

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v), \quad L(s, \widetilde{\pi}_v \times \widetilde{\tau}_v) = L(s, \widetilde{\Pi}_v \times \widetilde{\tau}_v)$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, \Pi_v \times \tau_v, \psi_v).$$

□

Proposition 3.25 is what is needed for the application of the converse theorem, establishing that the global representation Π is automorphic. However, to prove further properties of Π, we also make the local transfer in this case explicit. The analysis is similar to that of the quasi-split $\mathrm{SO}(2n)$ carried out in [CPSS2, §7.2], which we refer to for more detail.

Let $\mathbf{G} = \mathrm{GSpin}^*(2n)$, associated with the quadratic extension K/k of number fields. Let μ be the associated quadratic idèle class character of k as in (2.15). Assume that v is a non-archimedean place of k where all the data is unramified. There are two cases: either v

splits as two places in K or v is inert, i.e., there is a single place in K above the place v in k.

First, consider the case that v is inert in K and let w be the single place of K above v. The maximal torus $\mathbf{T}_v$ in $\mathbf{G}_v$ is isomorphic to $\mathrm{GL}(1)^{n-1} \times \mathbf{T}_0$, where $\mathbf{T}_0 \cong \mathrm{GSpin}^*(2)$ is a non-split torus in $\mathbf{G}_v$. We have $\mathbf{T}_v(k_v) \cong (k_v^\times)^{n-1} \times K_w^\times$ and the center of $\mathbf{G}_v(k_v)$ sits as a copy of $k_v^\times$ in $K_w^\times$. The unramified representation π_v is then determined by $n-1$ unramified characters $\chi_1, \ldots, \chi_{n-1}$ of $k_v^\times$ along with a character $\tilde{\chi}_n$ of $K_w^\times$ satisfying

$$\tilde{\chi}_n|_{k_v^\times} = \omega_{\pi_v}. \tag{3.26}$$

Note that the χ_i's are characters of $k_v^\times$ even though no v appears; they are not necessarily local components of global idèle class characters.

The representation Π_v of $\mathrm{GL}(2n, k_v)$ is induced from

$$\left(\chi_1, \ldots, \chi_{n-1}, \pi\left(\mathrm{Ind}_{W_{K_w}}^{W_{k_v}} \tilde{\chi}_n\right), \chi_{n-1}^{-1}\omega_{\pi_v}, \ldots, \chi_1^{-1}\omega_{\pi_v}\right), \tag{3.27}$$

where $\pi\left(\mathrm{Ind}_{W_{K_w}}^{W_{k_v}} \tilde{\chi}_n\right)$ is the Weil representation of $\mathrm{GL}(2, k_v)$ defined by $\mathrm{Ind}_{W_{K_w}}^{W_{k_v}} \tilde{\chi}_n$. (For a quick review of its definition and properties, see [GL, Appendix B], for example.)

Being a Weil representation, the central character of $\pi\left(\mathrm{Ind}_{W_{K_w}}^{W_{k_v}} \tilde{\chi}_n\right)$ is equal to

$$\mu_v \cdot \tilde{\chi}_n|_{k_v^\times} = \mu_v \cdot \omega_{\pi_v}, \tag{3.28}$$

where μ_v, the component of the global μ at v, is the generating character of $k_v^\times / N_{K_w/k_v}(K_w^\times)$, i.e., the non-trivial character of $k_v^\times$ which is trivial on norm elements, $N_{K_w/k_v}(K_w^\times)$. Hence, the central character of Π_v is equal to $\omega_{\pi_v}^n \mu_v$.

Moreover, by properties of the Weil representation, the contragredient of $\pi\left(\mathrm{Ind}_{W_{K_w}}^{W_{k_v}} \tilde{\chi}_n\right)$ is the Weil representation defined by $\mathrm{Ind}_{W_{K_w}}^{W_{k_v}} \tilde{\chi}_n^{-1}$ and

$$\begin{aligned}
\widetilde{\pi}\left(\mathrm{Ind}_{W_{K_w}}^{W_{k_v}} \tilde{\chi}_n\right) \otimes \omega_{\pi_v} &\cong \pi\left(\mathrm{Ind}_{W_{K_w}}^{W_{k_v}} \tilde{\chi}_n^{-1}\right) \otimes \omega_{\pi_v} \\
&\cong \pi\left(\mathrm{Ind}_{W_{K_w}}^{W_{k_v}} \tilde{\chi}_n^{-1} \cdot \left(\omega_{\pi_v} \circ N_{K_w/k_v}\right)\right) \\
&\cong \pi\left(\mathrm{Ind}_{W_{K_w}}^{W_{k_v}} \tilde{\chi}_n\right).
\end{aligned} \tag{3.29}$$

The last equivalence holds because

$$\tilde{\chi}_n^{-1} \cdot \left(\omega_{\pi_v} \circ N_{K_w/k_v}\right) = \tilde{\chi}_n{}^{\gamma}, \tag{3.30}$$

or equivalently,

$$\omega_{\pi_v} \circ N_{K_w/k_v} = \tilde{\chi}_n \cdot \tilde{\chi}_n{}^{\gamma}, \tag{3.31}$$

which immediately follows from (3.26). Here, γ denotes the non-trivial element of the Galois group $\mathrm{Gal}(K_w/k_v)$. Hence, $\Pi_v \cong \widetilde{\Pi}_v \otimes \omega_{\pi_v}$.

Furthermore, we note that for the transferred representation Π_v to be a principal series, the character $\tilde{\chi}_n$ must factor through the norm. Write $\tilde{\chi}_n = \chi_n \circ N_{K_w/k_v}$ with χ_n a character of $k_v^\times$. Then, (3.26) implies that $\chi_n^2 = \omega_{\pi_v}$ and the representation Π_v is a constituent of the representation of $\mathrm{GL}(2n, k_v)$ induced from

$$\left(\chi_1,\, \ldots,\, \chi_{n-1},\, \chi_n\mu_v,\, \chi_n^{-1}\omega_{\pi_v},\, \chi_{n-1}^{-1}\omega_{\pi_v},\, \ldots,\, \chi_1^{-1}\omega_{\pi_v}\right). \tag{3.32}$$

Next, consider the case that the place v splits in K. We have

$$\mathbf{G}_v(k_v) \cong \mathrm{GSpin}(2n, k_v). \tag{3.33}$$

The unramified representation π_v is given by n unramified characters $\chi_1, \ldots, \chi_n$ of $k_v^\times$ along with the central character ω_{π_v}.

The transferred representation Π_v of $\mathrm{GL}(2n, k_v)$ is now the same as in the case of the split $\mathrm{GSpin}(2n)$ we treated in [AS1, Eq. (64)]. More precisely, the representation Π_v is a constituent of the representation of $\mathrm{GL}(2n, k_v)$ induced from

$$\left(\chi_1, \ldots, \chi_{n-1}, \chi_n\mu_v, \chi_n^{-1}\omega_{\pi_v}, \chi_{n-1}^{-1}\omega_{\pi_v}, \ldots, \chi_1^{-1}\omega_{\pi_v}\right), \tag{3.34}$$

where μ_v, the component at v of the quadratic idèle class character μ associated with the quadratic extension K/k is now trivial. Hence, we again have $\Pi_v \cong \widetilde{\Pi}_v \otimes \omega_v$ and $\omega_{\Pi_v} = \omega_{\pi_v}^n \mu_v$.

Therefore, we have proved the following.

Proposition 3.35. *Let v be a non-archimedean place of k and let π_v be an irreducible, admissible, generic, unramified representation of* $\mathrm{GSpin}^*(2n, k_v)$ *with Π_v its local functorial transfer to* $\mathrm{GL}(2n, k_v)$ *defined above. Then*

$$\omega_{\Pi_v} = \omega_{\pi_v}^n \mu_v$$

and

$$\Pi_v \cong \widetilde{\Pi}_v \otimes \omega_{\pi_v}.$$

Here, μ_v is a quadratic character of $k_v^\times$ associated with the quadratic extension defining $\mathrm{GSpin}^*(2n)$*, i.e., the non-trivial character of $k_v^\times$ which is trivial on norm elements, $N_{K_w/k_v}(K_w^\times)$.* □

3.4.3. *The non-archimedean ramified transfer.* For v a non-archimedean ramified place of k we take Π_v to be an arbitrary, irreducible, admissible representation of $\mathrm{GL}(2n, k_v)$ whose central character satisfies

$$\omega_{\Pi_v} = \omega_{\pi_v}^n \mu_v, \tag{3.36}$$

where μ_v is the v-component of the quadratic idèle class character of k associated with the quadratic extension K/k.

We can no longer expect equality of L- and ϵ-factors as in the previous cases. However, we do still get equality if we twist by a highly ramified character thanks to stability of γ-factors.

Proposition 3.37. *Let v be a non-archimedean place of k such that the irreducible, admissible, generic representation π_v of* $\mathrm{GSpin}^*(2n, k_v)$ *is ramified. Let Π_v be an irreducible, admissible representation of* $\mathrm{GL}(2n, k_v)$ *as above. If $\tau_v = \tau_v' \otimes \eta_v$ is an irreducible, admissible, generic representation of* $\mathrm{GL}(m, k_v)$ *with τ_v' unramified and η_v a sufficiently ramified character of* $\mathrm{GL}(1, k_v)$*, then*

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v), \quad L(s, \widetilde{\pi}_v \times \widetilde{\tau}_v) = L(s, \widetilde{\Pi}_v \times \widetilde{\tau}_v)$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, \Pi_v \times \tau_v, \psi_v).$$

Proof. The representation τ_v' can be written as a full induced principal series. Hence,

$$\tau_v = \mathrm{Ind}\left(\nu^{b_1} \otimes \cdots \otimes \nu^{b_m}\right) \otimes \eta_v = \mathrm{Ind}\left(\eta_v\nu^{b_1} \otimes \cdots \otimes \eta_v\nu^{b_m}\right), \tag{3.38}$$

where $\nu(\cdot) = |\cdot|_v$. By multiplicativity of the L- and ϵ-factors we have

$$L(s, \pi_v \times \tau_v) = \prod_{i=1}^{m} L(s + b_i, \pi_v \times \eta_v) \tag{3.39}$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{m} L(s + b_i, \pi_v \times \eta_v, \psi_v). \tag{3.40}$$

Similarly,

$$L(s, \Pi_v \times \tau_v) = \prod_{i=1}^{m} L(s + b_i, \Pi_v \times \eta_v) \tag{3.41}$$

and

$$\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{m} L(s + b_i, \Pi_v \times \eta_v, \psi_v). \tag{3.42}$$

This reduces the proof to the case of $m = 1$.

Next, note that because η_v is sufficiently ramified (depending on π_v) the L-functions stabilize to one and we have

$$L(s, \pi_v \times \eta_v) \equiv 1 \tag{3.43}$$

and

$$\epsilon(s, \pi_v \times \eta_v, \psi_v) = \gamma(s, \pi_v \times \eta_v, \psi_v). \tag{3.44}$$

On the other hand, by stability of gamma factors, Proposition 3.15, we may replace π_v with another representation with the same central character. Hence, for n arbitrary characters $\chi_1, \chi_2, \ldots, \chi_n$, $\chi_0 = \omega_{\pi_v}$ and the quadratic character μ_v we have

$$\begin{aligned} \gamma(s, \pi_v \times \eta_v, \psi_v) \quad &= \quad \left(\prod_{i=1}^{n-1} \gamma(s, \eta_v \chi_i, \psi_v) \gamma(s, \eta_v \chi_i^{-1} \chi_0, \psi_v) \right) \\ &\cdot \quad \gamma(s, \eta_v \chi_n \mu, \psi_v) \gamma(s, \eta_v \chi_n^{-1} \chi_0, \psi_v). \end{aligned}$$

We refer to [AS1, §6] for more details in the split case. The calculations in the quasi-split case are similar, the only difference being the appearance of the quadratic character μ_v.

We have similar relations also for the GL case. More precisely, because $\omega_{\Pi_v} = \chi_0^n \mu$, by [JS3, Proposition 2.2] we have

$$L(s, \Pi_v \times \eta_v) \equiv 1 \tag{3.45}$$

and

$$\begin{aligned} \epsilon(s, \Pi_v \times \eta_v, \psi_v) \quad &= \quad \left(\prod_{i=1}^{n-1} \gamma(s, \eta_v \chi_i, \psi_v) \gamma(s, \eta_v \chi_i^{-1} \chi_0, \psi_v) \right) \\ &\cdot \quad \gamma(s, \eta_v \chi_n \mu, \psi_v) \gamma(s, \eta_v \chi_n^{-1} \chi_0, \psi_v). \end{aligned} \tag{3.46}$$

Note that this is a special case of the multiplicativity of the local factors. This gives the equalities for the case of $m = 1$ and hence completes the proof. □

3.5. **Proof of Theorem 3.5.** Let ω denote the central character ω_π of π and let S be as in the statement of Theorem 3.5. We let $\Pi = \otimes_v \Pi_v$ with Π_v the candidates we constructed in 3.4.1–3.4.3. Also, let $\mu = \otimes_v \mu_v$ be a quadratic idèle class character associated, by class field theory, with the quadratic extension K/k, where K is the quadratic extension of k over which $\mathrm{GSpin}^*(2n)$ is split.

Choose an idèle class character η of k which is sufficiently ramified at places $v \in S$ so that the requirements of Propositions 3.13 and 3.15 are satisfied. We apply Theorem 3.11 to the representation Π and $\mathcal{T}(S; \eta)$ with S and η as above.

By construction the central character ω_Π of Π is equal to $\omega^n\mu$. Therefore, it is invariant under $k^\times$. Moreover, by (3.16) and (3.21) we have

$$L^S(s,\Pi) = \prod_{v\notin S} L(s,\Pi_v) = \prod_{v\notin S} L(s,\pi_v) = L^S(s,\pi). \tag{3.47}$$

This implies that $L(s,\Pi) = \prod_v L(s,\Pi_v)$ is absolutely convergent in some right half plane. Furthermore, by Propositions 3.20, 3.25, and 3.37, we may verify the remaining properties of being nice for the L-functions $L(s,\Pi\times\tau)$ and the corresponding ϵ-factors, for $\tau\in\mathcal{T}(S;\eta)$, by establishing the similar properties for the L-functions $L(s,\pi\times\tau)$ and their corresponding ϵ-factors. Now the converse theorem can be applied thanks to Proposition 3.13 to conclude that there exists an automorphic representation of $\mathrm{GL}(2n,\mathbb{A}_k)$ whose local components at $v\notin S$ agree with those of Π. This automorphic representation is what we are calling Π in the statement of the theorem.

Finally, by Proposition 3.35, outside the finite set $S\cup\{v : v|\infty\}$, the central character of the automorphic representation Π agrees with the idèle class character $\omega^n\mu$, which implies that it is equal to $\omega^n\mu$. The same proposition also gives that $\Pi_v \cong \widetilde{\Pi}_v \otimes \omega_{\pi_v}$ for $v\notin S\cup\{v : v|\infty\}$. This completes the proof. □

4. The Transferred Representation

In this section k will continue to denote an arbitrary number field and $\mathbb{A}=\mathbb{A}_k$ will denote its ring of adèles.

Fix $n\geq 1$ and let $\mathbf{G}=\mathbf{G}(n)$ denote $\mathrm{GSpin}(2n+1)$, the split $\mathrm{GSpin}(2n)$, or one of the non-split quasi-split groups $\mathrm{GSpin}^*(2n)$ associated with a non-trivial quadratic extension K of the number field k. We will refer to the case of $\mathbf{G}(n)=\mathrm{GSpin}(2n+1)$ as the odd case and the remaining cases as the even case.

4.1. **Low Rank Cases.** Assume that $n=1$. In these low rank cases, the transfers are well-known. We quickly review them below before assuming $n\geq 2$ in what follows.

(i) *Odd General Spin Groups.* The group $\mathrm{GSpin}(3)$ is isomorphic to $\mathrm{GL}(2)$ and the transfer from $\mathrm{GSpin}(3)$ to $\mathrm{GL}(2)$ that we are dealing with is simply the identity. Obviously, the L-functions are preserved in this case: $L(s,\pi)=L(s,\Pi)$.

(ii) *Even General Spin Groups.* In the split case, the group $\mathrm{GSpin}(2)$ is isomorphic to a two-dimensional split torus. A cuspidal automorphic representation π of $\mathrm{GSpin}(2,\mathbb{A}_k)$ is given by two unitary idèle class characters χ and ω. The dual of $\mathrm{GSpin}(2)$ embeds in the diagonal of $\mathrm{GL}(2,\mathbb{C})$ with local unramified Satake parameters appearing as

$$\begin{pmatrix}\chi_v(\varpi_v) & 0\\ 0 & \omega_v(\varpi_v)\chi_v^{-1}(\varpi_v)\end{pmatrix}, \tag{4.1}$$

where ϖ_v denotes a uniformizer at v. The transfer Π of $\pi=\pi_{\omega,\chi}$ to $\mathrm{GL}(2,\mathbb{A}_k)$ is the appropriate constituent of

$$\mathrm{Ind}\left(\chi\otimes\omega\chi^{-1}\right), \tag{4.2}$$

namely, the isobaric sum $\Pi=\chi\boxplus\omega\chi^{-1}$, which takes the local Langlands quotient at each place when there is reducibility.

Next, consider the quasi-split non-split case with $n=1$. Now, the group $\mathbf{G}$ is $\mathrm{GSpin}^*(2)$, associated with a quadratic extension K of k and it contains a copy of $\mathrm{GL}(1)$ over k as its "center". (Note that $\mathbf{G}$ is simply $\mathrm{Res}_{K/k}\,GL(1)$ and abelian.) The L-group ${}^L\mathbf{G}$ is isomorphic to $\mathrm{GSO}(2,\mathbb{C})\rtimes\Gamma_k$ with the embedding of it into $\mathrm{GL}(2,\mathbb{C})\times\Gamma_k$ as in (3.3) again.

A cuspidal automorphic representation π of $\mathbf{G}(\mathbb{A}_k)$ is given as an idèle class character $\widetilde{\chi}$ of the quadratic extension K of k. Its restriction to the "center" is now an idèle class character of k which we denote by ω. The transfer of π to $\mathrm{GL}(2, \mathbb{A}_k)$ is now given by

$$\Pi = \bigotimes_v \pi \left(\mathrm{Ind}_{W_{K_v}}^{W_{k_v}} \widetilde{\chi}_v \right), \tag{4.3}$$

where $\mathrm{Ind}_{W_{K_v}}^{W_{k_v}} \widetilde{\chi}_v = \widetilde{\chi}_v \oplus \widetilde{\chi}_v$ when v splits in K (i.e., $K_v = k_v \oplus k_v$), and it is the Weil representation we described in Section 3.4.2 when v is inert in K. (See [GL, Appendix C].) By our earlier local description, it is again clear that $\Pi \cong \widetilde{\Pi} \otimes \omega$ and $\omega_\Pi = \omega \cdot \mu$, where $\mu = \mu_{K/k}$ is the quadratic idèle class character of k associated with the extension K/k.

We may assume below that $n \geq 2$.

4.2. **The Global Transfer.** Let π be a irreducible, globally generic, unitary, cuspidal, automorphic representation of $\mathbf{G}(\mathbb{A}_k)$. Let Π be a transfer of π to $\mathrm{GL}(2n, \mathbb{A}_k)$ as in [AS1, Theorem 1.1] and Theorem 3.5. By the classification of automorphic representations of general linear groups [JS1, JS2] we know that Π is a constituent of some automorphic representation

$$\Sigma = \mathrm{Ind}\,(|\det|^{r_1}\sigma_1 \otimes \cdots \otimes |\det|^{r_t}\sigma_t) \tag{4.4}$$

with σ_i a unitary, cuspidal, automorphic representation of $\mathrm{GL}(n_i, \mathbb{A}_k)$, $r_i \in \mathbb{R}$, and

$$n_1 + n_2 + \cdots + n_t = 2n. \tag{4.5}$$

Let $\omega = \omega_\pi$ denote the central character of π. Then ω is a unitary idèle class character of k and we have shown that Π is nearly equivalent to $\widetilde{\Pi} \otimes \omega$.

Our first goal in this section is to prove the fact that all the exponents $r_i = 0$ in (4.4). We start by introducing a necessary Eisenstein series and proving a lemma (Lemma 4.11 below) about twisted exterior and symmetric (partial) L-functions.

4.2.1. *Eisenstein Series.* Let m and n denote positive integers and assume $n \geq 2$. Let π be a globally generic, unitary, cuspidal, automorphic representation of $\mathbf{G}(\mathbb{A}_k)$ as before and let τ be a unitary, cuspidal, automorphic representation of $\mathrm{GL}(m, \mathbb{A}_k)$. Denote their central characters by $\omega = \omega_\pi$ and ω_τ, respectively. We construct a certain Eisenstein series for a representation induced from τ as follows.

Let $\mathbf{H} = \mathbf{H}(m)$ be the split group $\mathrm{GSpin}(2m)$ if $\mathbf{G} = \mathrm{GSpin}(2n+1)$ and $\mathrm{GSpin}(2m+1)$ if $\mathbf{G} = \mathrm{GSpin}(2n)$. In other words, $\mathbf{H}$ is of the opposite type to $\mathbf{G}$. Consider the Siegel parabolic $\mathbf{P}_m = \mathbf{M}_m\mathbf{N}_m$ in $\mathbf{H}(m)$ with $\mathbf{M}_m \cong \mathrm{GL}(m) \times \mathrm{GL}(1)$ and its representation

$$\tau_{s';\eta} = \tau|\det|^{s'} \otimes \eta, \tag{4.6}$$

where η is an appropriately chosen idèle class character. (The choice of η and the isomorphism between $\mathbf{M}_m$ and $\mathrm{GL}(m) \times \mathrm{GL}(1)$ are made in [ACS1] so that the Zeta integral we consider there is well-defined and the desired L-functions appear in the constant term of the Eisenstein series; see below.) Here, $s' = s - 1/2$, with $s \in \mathbb{C}$. (The shift in s is for convenience here so that the parameter s appears in the Rankin-Selberg L-functions below.)

Extend $\tau_{s',\eta}$ trivially across $\mathbf{N}_m(\mathbb{A})$ to obtain a representation of $\mathbf{P}_m(\mathbb{A})$. Consider the normalized induced representation

$$\rho_{s';\eta} = \mathrm{Ind}_{\mathbf{P}_m(\mathbb{A})}^{\mathbf{H}(\mathbb{A})} \left(\tau_{s';\eta} \otimes 1 \right). \tag{4.7}$$

For $f = f_{\tau,s';\eta}$ in this induced representation, construct the Eisenstein series

$$E(h, f_{\tau,s';\eta}) = \sum_{\gamma \in \mathbf{P}_m(k)\backslash \mathbf{H}(k)} f(\gamma h), \quad h \in \mathbf{H}(\mathbb{A}). \tag{4.8}$$

This Eisenstein series is absolutely convergent for $\Re(s) \gg 0$, has a meromorphic continuation to all of $\mathbb{C}$ with a finite number of poles, the continuation is of moderate growth, and it converges uniformly on compact subsets away from the poles. Moreover, it has a functional equation relating its values at s' and $1 - s'$.

The Eisenstein series, along with a cusp form in the space of π, appears in a certain Zeta integral we consider in [ACS1]. In particular, we establish the following statement in [ACS1], which we now borrow from that paper.

Proposition 4.9. *With τ and π as before, if $L^S(s, \pi \times \tau)$ has a pole at s_0 with $\Re(s_0) \geq 1$, then, for a choice of $f_{\tau,s';\eta}$, the Eisenstein series $E(h, f_{\tau,s';\eta})$ has a pole at $s = s_0$*

The proposition is established by proving that the the Zeta integral in which it appears is Eulerian and, up to a factor that can be made holomorphic and non-zero for an appropriate choice of $f_{\tau,s';\eta}$, is equal to the quotient of the Rankin-Selberg L-function $L^S(s, \pi \times \tau)$ and either $L^S(2s, \tau, \wedge^2 \otimes \omega^{-1})$ (when $\mathbf{G}$ is odd) or $L^S(2s, \tau, \mathrm{Sym}^2 \otimes \omega^{-1})$ (when $\mathbf{G}$ is even). We carry out the details in [ACS1].

4.2.2. *Twisted Symmetric and Exterior Square L-functions.* We need a result on holomorphy of twisted L-function in the half plane $\Re(s) > 1$. This result, and much more, is the subject of two works, one by Dustin Belt in his thesis at Purdue University, and the other by Suichiro Takeda.

Proposition 4.10. *(*[Blt] *and* [Tk]*) Assume that m is an arbitrary positive integer. Let χ be an arbitrary idèle class character and let τ be a unitary, cuspidal, automorphic representation of* $\mathrm{GL}(m, \mathbb{A})$*. Let S be a finite set of places of k containing all the archimedean places and all the non-archimedean places at which τ ramifies. Then the partial twisted L-functions $L^S(s, \tau, \wedge^2 \otimes \chi)$ and $L^S(s, \tau, \mathrm{Sym}^2 \otimes \chi)$ are holomorphic in $\Re(s) > 1$.*

We remark that Jacquet and Shalika proved that $L^S(s, \tau, \wedge^2 \otimes \chi)$ has a meromorphic continuation to a half plane $\Re(s) > 1 - a$ with $a > 0$ depending on the representation [JS4, §8, Theorem 1]. Proposition 4.10 in the case of $\wedge^2 \otimes \omega$ can also be dug out of their work. However, D. Belt's results show this for all s, with possible poles at $s = 0, 1$.

As far as we know, an analogue of Jacquet-Shalika's result for twisted symmetric square was not available. For $m = 3$ it follows from the results of W. Banks [Bnk] following the untwisted ($\chi = 1$) results of Bump and Ginzburg [BG]. S. Takeda's results build on this line of work. In the case of $\chi = 1$, N. Grbac has recently established a full theory of these L-functions in [Gr].

Lemma 4.11. *Assume that m is an arbitrary positive integer. Let τ be an irreducible, unitary, cuspidal, automorphic representation of* $\mathrm{GL}(m, \mathbb{A})$*. Let ω be an idèle class character and let $s \in \mathbb{C}$. Let S be a finite set of places of k including all the archimedean ones such all data is unramified outside S.*

(a) *Both $L^S(s, \tau, \wedge^2 \otimes \omega^{-1})$ and $L^S(s, \tau, \mathrm{Sym}^2 \otimes \omega^{-1})$ are holomorphic and non-vanishing for $\Re(s) > 1$.*
(b) *If either of the above L-functions has a pole at $s = 1$, then $\tau \cong \widetilde{\tau} \otimes \omega$.*

Proof. We have

$$L^S(s, \tau \otimes (\tau \otimes \omega^{-1})) = L^S(s, \tau, \wedge^2 \otimes \omega^{-1}) L^S(s, \tau, \mathrm{Sym}^2 \otimes \omega^{-1}). \tag{4.12}$$

The left hand side is holomorphic and non-vanishing for $\Re(s) > 1$ by [JS2, Proposition (3.6)]. Moreover, by Proposition 4.10 both of the L-functions on the right hand side are holomorphic for $\Re(s) > 1$. Therefore, both are non-vanishing there, as well. This is part (a).

On the other hand, by [Sh5, Theorem 1.1] both L-functions on the right hand side are non-vanishing on $\Re(s) = 1$. If one has a pole at $s = 1$, then the left hand side must have a pole at $s = 1$. Again by [JS2, Proposition (3.6)] the two representations τ and $\tau \otimes \omega^{-1}$ must be contragredient of each other, i.e., $\tau \cong \widetilde{\tau} \otimes \omega$. This is part (b). □

Proposition 4.13. *Assume that m is an arbitrary positive integer. Let τ be an irreducible, unitary, cuspidal representation of* $\mathrm{GL}(m, \mathbb{A})$*. Fix $s_0 \in \mathbb{C}$ with $\Re(s_0) \geq 1$ and assume that the Eisenstein series $E(h, f_{\tau,s';\eta})$ introduced in (4.8) has a pole at $s = s_0$. Then, $s_0 = 1$ and $L^S(s, \tau, \wedge^2 \otimes \omega^{-1})$ has a simple pole at $s = 1$ in the odd case while $L^S(s, \tau, \mathrm{Sym}^2 \otimes \omega^{-1})$ has a simple pole at $s = 1$ in the even case.*

Proof. We know from the general theory of Euler products of Langlands and the Langlands-Shahidi method that the poles of $E(h, f_{\tau,s';\eta})$ come from its constant term along $\mathbf{P}_m$.

For a decomposable section $f = f_{\tau,s';\eta} = \otimes_v f^{(v)}$ the constant term of $E(h, f_{\tau,s';\eta})$ along $\mathbf{P}_m$ has the form

$$f(I) + \prod_{v \in T} M\left(f^{(v)}\right) \frac{L^T(2s-1, \tau, \wedge^2 \otimes \omega^{-1})}{L^T(2s, \tau, \wedge^2 \otimes \omega^{-1})} \tag{4.14}$$

in the odd case, and

$$f(I) + \prod_{v \in T} M\left(f^{(v)}\right) \frac{L^T(2s-1, \tau, \mathrm{Sym}^2 \otimes \omega^{-1})}{L^T(2s, \tau, \mathrm{Sym}^2 \otimes \omega^{-1})} \tag{4.15}$$

in the even case, where T is a finite set of places of k containing S.

We should recall that in the construction the Eisenstein series we used $s - \frac{1}{2}$ in (4.6) instead of the usual s. This is responsible for the appearance of $2s-1$ and $2s$ instead of the usual $2s$ and $2s+1$ in the constant term.

The terms $M\left(f^{(v)}\right)$, the local intertwining operators at I, are holomorphic for $\Re(s) \geq 1$ for all v [Sh2, Sh3]. Therefore, if $E(h, f_{\tau,s';\eta})$ has a pole at $s = s_0$, then

$$\frac{L^T(2s-1, \tau, \wedge^2 \otimes \omega^{-1})}{L^T(2s, \tau, \wedge^2 \otimes \omega^{-1})} \tag{4.16}$$

has a pole at $s = s_0$ in the odd case, or

$$\frac{L^T(2s-1, \tau, \mathrm{Sym}^2 \otimes \omega^{-1})}{L^T(2s, \tau, \mathrm{Sym}^2 \otimes \omega^{-1})} \tag{4.17}$$

has a pole at $s = s_0$ in the even case.

Assume that $E(h, f_{\tau,s';\eta})$ does have a pole at $s = s_0$ with $\Re(s_0) \geq 1$. Then $\Re(2s_0) \geq 2$ and by [KS2, Prop. 7.3] the denominator in both (4.16) and (4.17) is non-vanishing for $\Re(s) \geq 1$. Therefore, the numerator has a pole at $s = s_0$. Because $\Re(2s_0 - 1) \geq 1$, Lemma 4.11 implies that $s_0 = 1$ and the proof is complete. □

We use Proposition 4.9 in the proof of the following theorem.

Theorem 4.18. *Assume that $n \geq 2$ is an integer. Let π be an irreducible, unitary, globally generic, cuspidal, automorphic representation of $\mathbf{G}(n, \mathbb{A})$. Let τ be an irreducible, unitary, cuspidal, automorphic representation of* $\mathrm{GL}(m, \mathbb{A})$*. Assume that S is a sufficiently large finite set of places including all the archimedean places of k.*

(a) *The L-function $L^S(s,\pi\times\tau)$ is holomorphic for $\Re(s)>1$.*
(b) *If $L^S(s,\pi\times\tau)$ has a pole at s_0 with $\Re(s_0)=1$, then $s_0=1$ and $L^S(s,\tau,\wedge^2\otimes\omega_\pi^{-1})$ has a pole at $s=1$ in the odd case and $L^S(s,\tau,\mathrm{Sym}^2\otimes\omega_\pi^{-1})$ has a pole at $s=1$ in the even case. In particular, $\tau\cong\widetilde{\tau}\otimes\omega_\pi$. Such a pole would be simple.*

Remark 4.19. The partial L-functions in the above statements may be replaced by the completed L-functions as Hundley and Sayag have pointed out in [HS2, §19.3]. The point is that, using the description of generic unitary representations of the general linear groups and analytic properties of local intertwining operators and L-functions, one can show that the local twisted symmetric or exterior square L-functions above are holomorphic and non-vanishing for $\Re(s)\geq 1$. (They do it for the twisted symmetric square L-function and at $s=1$, but the same argument works for the twisted exterior square L-function and for $\Re(s)\geq 1$.) We thank the referee for this remark.

Proof. Assume that $L^S(s,\pi\times\tau)$ has a pole at $s=s_0$ with $\Re(s_0)\geq 1$. By [KS2, Prop. 7.3] we know that both $L^S(s,\tau,\wedge^2\otimes\omega_\pi^{-1})$ and $L^S(s,\tau,\mathrm{Sym}^2\otimes\omega_\pi^{-1})$ are holomorphic and non-vanishing at $s=2s_0$. By Proposition 4.9, the Eisenstein series $E(h,f_{\tau,s';\eta})$ must have a pole at $s=s_0$. Proposition 4.13 now implies that $s_0=1$ and $L^S(s,\tau,\wedge^2\otimes\omega^{-1})$ in the odd case, or $L^S(s,\tau,\mathrm{Sym}^2\otimes\omega^{-1})$ in the even case, has a simple pole at $s=1$. This proves (a) and (b). $\square$

4.3. **Description of the Image of Transfer.** We can now describe the image in general.

Theorem 4.20. *Let π be an irreducible, globally generic, unitary, cuspidal, automorphic representation of $\mathbf{G}(n,\mathbb{A})$ with central character $\omega=\omega_\pi$ and let Π be a transfer of π to $\mathrm{GL}(2n,\mathbb{A})$. Consider Π as a subquotient of Σ as in (4.4) with $n_1+n_2+\cdots+n_t=2n$.*
(a) *We have $r_1=r_2=\cdots=r_t=0$.*
(b) *The representations σ_i are pairwise inequivalent, and $\sigma_i\cong\widetilde{\sigma}_i\otimes\omega$ for all i. Moreover, for S a sufficiently large finite set of places including all the archimedean ones, the L-function $L^S(s,\sigma_i,\wedge^2\otimes\omega^{-1})$ has a pole at $s=1$ in the odd case, and the L-function $L^S(s,\sigma_i,\mathrm{Sym}^2\otimes\omega^{-1})$ has a pole at $s=1$ in the even case.*

Proof. By [AS1, Prop. 7.4] we know that Σ is induced from a representation of a Levi subgroup of $\mathrm{GL}(2n)$ of type $(a_1,\ldots,a_q,b_1\cdots,b_\ell,a_q,\cdots,a_1)$ which can be written as

$$\begin{gathered}\delta_1|\det(\cdot)|^{z_1}\otimes\cdots\otimes\delta_q|\det(\cdot)|^{z_q}\otimes\\ \sigma_1\otimes\sigma_2\otimes\cdots\otimes\sigma_\ell\\ \otimes(\widetilde{\delta}_q\otimes\omega^{-1})|\det(\cdot)|^{-z_q}\otimes\cdots(\widetilde{\delta}_1\otimes\omega^{-1})|\det(\cdot)|^{-z_1},\end{gathered}\tag{4.21}$$

where δ_i and σ_i are irreducible, unitary, cuspidal automorphic representations, $\sigma_i\cong\widetilde{\sigma}_i\otimes\omega$, and

$$2(a_1+\cdots+a_q)+(b_1+\cdots+b_\ell)=2n.\tag{4.22}$$

Assume that $q>0$. Rearranging if necessary, we may assume $\Re(z_1)\leq\cdots\leq\Re(z_q)<0$. Now, for S a sufficiently large finite set of places, we have

$$\begin{aligned}L^S(s,\pi\times\widetilde{\delta}_1) &= L^S(s,\Pi\times\widetilde{\delta}_1)\\ &= \prod_{i=1}^{q}L^S(s+z_i,\delta_i\times\widetilde{\delta}_1)L^S(s-z_i,\widetilde{\delta}_i\times\widetilde{\delta}_1\otimes\omega^{-1})\\ &\quad\cdot\prod_{i=1}^{\ell}L^S(s,\sigma_i\times\widetilde{\delta}_1).\end{aligned}\tag{4.23}$$

The first term on the right hand side has a pole at $s = 1 - z_1$ which can not be canceled by the other terms because $\Re(1 - z_1 \pm z_i) \geq 1$ and $\Re(1 - z_1) > 1$. Therefore, the left hand side has a pole at $s = 1 - z_1$.

We can apply Theorem 4.18(a) to conclude that $\Re(z_1) \geq 0$. This is a contradiction proving that $q = 0$, i.e., there are no δ_i's. This proves part (a).

So far we have proved that Σ is induced from a representation of the form $\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_\ell$ satisfying $\sigma_i \cong \widetilde{\sigma}_i \otimes \omega$.

Fix $1 \leq j \leq \ell$ and consider

$$L^S(s, \pi \times \widetilde{\sigma}_j) = L^S(s, \Pi \times \widetilde{\sigma}_j) = \prod_{i=1}^{\ell} L^S(s, \sigma_i \times \widetilde{\sigma}_j). \tag{4.24}$$

The right hand side has a pole and hence, so does the left hand side. Moreover, the σ_i's are pairwise inequivalent because otherwise the left hand side of (4.24) would have a pole of higher order. Since $\sigma_i \cong \widetilde{\sigma}_i \otimes \omega$ we can apply Theorem 4.18(b).

We conclude that the L-function $L^S(s, \sigma_i, \wedge^2 \otimes \omega^{-1})$ has a pole at $s = 1$ in the odd case, and the L-function $L^S(s, \sigma_i, \mathrm{Sym}^2 \otimes \omega^{-1})$ has a pole at $s = 1$ in the even case. This proves part (b). □

Corollary 4.25. *The representation Σ is irreducible and $\Pi = \Sigma = \sigma_1 \boxplus \cdots \boxplus \sigma_t$ is an isobaric sum of the σ_i. In particular, the transfer Π of π is unique and $\Pi \cong \widetilde{\Pi} \otimes \omega$ (not just nearly equivalent as in* [AS1, Theorem 1.1]*).*

Proof. The corollary immediately follows from the fact that $r_1 = \cdots = r_t = 0$ and that $\sigma_i \cong \widetilde{\sigma}_i \otimes \omega$. □

We continue to denote by π an irreducible, globally generic, unitary, cuspidal, automorphic representation of $\mathbf{G}(n, \mathbb{A})$. We proved that π has a unique transfer Π, an irreducible, generic, automorphic representation of $\mathrm{GL}(2n, \mathbb{A})$. Moreover, we have shown that $\omega_\Pi = \omega^n \mu$ and $\Pi \cong \widetilde{\Pi} \otimes \omega$, where $\omega = \omega_\pi$ denotes the central character of π, ω_Π denotes the central character of Π, and μ is a quadratic idèle class character.

Furthermore, Theorem 4.20 gives an "upper bound" for the image of the transfer from $\mathbf{G}(n)$ groups to $\mathrm{GL}(2n)$. Combining this with the "lower bound" provided by Hundley and Sayag (when each n_i is even) in [HS1, HS2] and Ginzburg, Rallis, and Soudry [GRS] (when some n_i is odd), cf. [HS1, Remark 2.1.1 (3)], we obtain the full description of the image of this transfer. We summarize all these results as follows.

Theorem 4.26. *Let k be a number field and let $\mathbb{A} = \mathbb{A}_k$ be the ring of adèles of k. Assume that $n \geq 1$ is an integer. Denote by $\mathbf{G}(n)$ either the split $\mathrm{GSpin}(2n+1)$, the split $\mathrm{GSpin}(2n)$, or one of the non-split quasi-split groups $\mathrm{GSpin}^*(2n)$ (associated with a quadratic extension K/k.) Let π be an irreducible, globally generic, cuspidal, automorphic representation of $\mathbf{G}(n, \mathbb{A})$ with central character $\omega = \omega_\pi$. Then π has a unique functorial transfer to an automorphic representation Π of $\mathrm{GL}(2n, \mathbb{A})$ associated with the L-homomorphism ι described in* [AS1] *(the split case) and Section 3 (the quasi-split non-split case). The transfer Π satisfies*

$$\Pi \cong \widetilde{\Pi} \otimes \omega.$$

Moreover,

$$\omega_\Pi = \omega_\pi^n \mu,$$

where μ is a quadratic idèle class character which is trivial in the split case and non-trivial in the quasi-split non-split case. In fact, μ defines the quadratic extension of k over which *(the non-split)* $\mathrm{GSpin}^*(2n)$ *splits. (The triviality or non-triviality of* $\omega_\Pi\omega^{-n}$ *can tell apart the split and quasi-split non-split cases.)*

The automorphic representation Π *is an isobaric sum of the form*

$$\Pi = \mathrm{Ind}\,(\Pi_1 \otimes \cdots \otimes \Pi_t) = \Pi_1 \boxplus \cdots \boxplus \Pi_t,$$

where each Π_i *is a unitary, cuspidal, automorphic representation of* $\mathrm{GL}(n_i, \mathbb{A})$ *such that for* T *a sufficiently large finite set of places of* k *containing the archimedean places, the partial* L*-function* $L^T(s, \Pi_i, \wedge^2 \otimes \omega^{-1})$ *has a pole at* $s = 1$ *in the odd case and* $L^T(s, \Pi_i, \mathrm{Sym}^2 \otimes \omega^{-1})$ *has a pole at* $s = 1$ *in the even case (both split and quasi-split non-split cases). We have* $\Pi_i \ncong \Pi_j$ *if* $i \neq j$ *and* $n_1 + \cdots + n_t = 2n$.

Conversely, any automorphic representation Π *of* $\mathrm{GL}(2n, \mathbb{A})$ *satisfying the above conditions is a functorial transfer of some irreducible, globally generic, cuspidal, automorphic representation* π *of* $\mathbf{G}(n, \mathbb{A})$.

Again we should point out that the partial L-functions in the statement above may be replaced by the completed L-functions (cf. Remark 4.19).

5. Applications

5.1. **Local Representations at the Ramified Places.** The local components of the automorphic representation $\Pi = \otimes_v \Pi_v$ are well understood for the archimedean v as well as those non-archimedean v outside of the finite set S through our construction of the candidate transfer. However, the converse theorem tells us nothing about Π_v for $v \in S$. Having proved Theorem 4.26 we can now get some information for these places as well. This shows that while we did not have control over places $v \in S$, the automorphic representation Π does indeed turn out to have the right local components in S.

Proposition 5.1. *Let* π_v *be an irreducible, admissible, generic representation of* $\mathbf{G}(n, k_v)$*, where* v *is a non-archimedean place of* k*. Assume that* π_v *appears as the local component of a globally generic cuspidal representation of* $\mathbf{G}(n, \mathbb{A}_k)$*. Then, there exists a unique irreducible, admissible, generic representation* Π_v *of* $\mathrm{GL}(2n, k_v)$ *which is the local transfer of* π_v *in the following sense:*

If π_v *is the local component at* v *of any globally generic cuspidal automorphic representation* π*, then, as the name suggests,* Π_v *is the local component at* v *of the unique transfer* Π *of* π *to* $\mathrm{GL}(2n, \mathbb{A}_k)$ *whose existence and uniqueness we established in Theorem 4.26.*

To be more explicit, let π_v *be of the form*

$$\pi_v \cong \mathrm{Ind}\left(\pi_{1,v}|\det|^{b_{1,v}} \otimes \cdots \otimes \pi_{r,v}|\det|^{b_{r,v}} \otimes \pi_{0,v}\right),$$

where each $\pi_{i,v}$ *is a tempered representation of* $\mathrm{GL}(n_i, k_v)$*,* $b_{1,v} > \cdots > b_{r,v}$ *and* $\pi_{0,v}$ *is a tempered, generic representation of some smaller* $\mathbf{G}(m, k_v)$ *with* $n_1 + \cdots + n_r + m = n$*. Denote the central character of* π_v *by* ω_v*. Then* Π_v *is of the form*

$$\begin{aligned} \Pi_v \quad \cong \quad & \mathrm{Ind}\left((\pi_{1,v}|\det|^{b_{1,v}} \otimes \cdots \otimes \pi_{r,v}|\det|^{b_{r,v}} \otimes \Pi_{0,v} \otimes \right. \\ & \left. (\widetilde{\pi}_{r,v} \otimes \omega_v)|\det|^{-b_{r,v}} \otimes \cdots \otimes (\widetilde{\pi}_{1,v} \otimes \omega_v)|\det|^{-b_{1,v}}\right). \end{aligned} \tag{5.2}$$

Here, $\Pi_{0,v}$ *appears only if* $m > 0$*. It is a tempered representation of* $\mathrm{GL}(2m, k_v)$ *which is the local transfer of* $\Pi_{0,v}$*.*

In particular, if S is the non-empty finite set of non-archimedean places in [AS1, Thm. 1.1.] or Theorem 3.5, then the local components Π_v, $v \in S$, are uniquely determined by the corresponding local components π_v.

Proof. The argument proceeds the same way as in the proof of [AS2, Prop. 2.5.], which proved the analogous result for the case of $\mathrm{GSp}(4) = \mathrm{GSpin}(5)$. We briefly mention the steps for completeness.

Given π_v, let π be as in the statement of the proposition and let Π be its unique transfer to $\mathrm{GL}(2n, \mathbb{A}_k)$. The fact that Π is an isobaric sum of unitary, cuspidal representations of general linear groups, Theorem 4.26, implies that every local component of Π is generic and full induced from tempered representations in the sense of the Langlands Classification. In particular, so is Π_v.

The first step is to show that

$$\gamma(s, \pi_v \times \rho_v, \psi_v) = \gamma(s, \Pi_v \times \rho_v, \psi_v)$$

for every supercuspidal representation ρ_v of $\mathrm{GL}(a, k_v)$. To do this we "embed" the local representation ρ_v in a unitary, cuspidal representation ρ of $\mathrm{GL}(a, \mathbb{A})$ whose other local components are unramified [Sh3, Prop. 5.1] and apply the converse theorem with $S' = S - \{v\}$ just as in [CKPSS2, Prop. 7.2]. Moreover, by multiplicativity of the γ-factors we obtain the equality for ρ_v in the discrete series, as well.

Next, assume that π_v is tempered. We claim that Π_v is also tempered. Here again the main tool is multiplicativity of the γ-factors and the proof is exactly as in [CKPSS2, Lemma 7.1]. In addition to several applications of multiplicativity, it involves information about generic unitary representations of the general linear groups, with the only different ingredient in the general spin case being the so-called tempered L-function conjecture, that the local L-functions for tempered representations are holomorphic for $\Re(s) > 0$. For the general spin groups, this is available in [A], and even more generally in [HO]. This proves the Proposition for $r = 0$.

Now consider the case of $r > 0$. Apply the case of $r = 0$ to $\pi_v = \pi_{0,v}$ and take the resulting tempered representation of $\mathrm{GL}(2m, k_v)$ to be $\Pi_{0,v}$. To show that this representation satisfies the requirements of the proposition we use the converse theorem again. Let $T = \{w\}$ consist of a single non-archimedean place $w \neq v$ at which π is unramified and consider the global representation Π' of $\mathrm{GL}(2n, \mathbb{A})$ whose local components are the same as those of Π outside of T and Π'_w is the irreducible, induced representations on the right hand side of (5.2). We can apply the converse theorem, Theorem 3.11, to Π' and T because $L(s, \Pi' \times \tau) = L(s, \pi \times \tau)$ and similarly the contragredients as well as the ϵ-factors for all τ. The conclusion is that Π' is a transfer of π (outside of T) and by the uniqueness of the transfer, Theorem 4.26, we have $\Pi'_v \cong \Pi_v$ for $v \in S$. This completes the proof. □

5.2. **Ramanujan Estimates.** Following [CKPSS2], we introduce the following notation. Let $\Pi = \otimes_v \Pi_v$ be a unitary, cuspidal, automorphic representation of $\mathrm{GL}(m, \mathbb{A}_k)$. For each place v the representation Π_v is unitary generic and can be written as a full induced representation

$$\Pi_v \cong \mathrm{Ind}\,(\Pi_{1,v} |\det|^{a_{1,v}} \otimes \cdots \otimes \Pi_{r,v} |\det|^{a_{r,v}}) \tag{5.3}$$

with $a_{1,v} > \cdots > a_{r,v}$ and each $\Pi_{i,v}$ tempered.

Definition 5.4. *We say Π satisfies $H(\theta_m)$ with $\theta_m \geq 0$ if for all places v we have*

$$-\theta_m \leq a_{i,v} \leq \theta_m.$$

The classification of the generic unitary dual of $\mathrm{GL}(m)$, [Td, V], trivially gives $H(1/2)$. The best result currently known for a general number field is $\theta_m = 1/2 - 1/(m^2 + 1)$ proved in [LRS] with a few better results known for small values of m and over $\mathbb{Q}$. The Ramanujan conjecture for $\mathrm{GL}(m)$ demands $H(0)$.

Similarly, if $\pi = \otimes_v \pi_v$ is a unitary, generic, cuspidal, automorphic representation of $\mathbf{G}(n, \mathbb{A}_k)$ each π_v can be written as a full induced representation

$$\pi_v \cong \mathrm{Ind}\left(\pi_{1,v}|\det|^{b_{1,v}} \otimes \cdots \otimes \pi_{r,v}|\det|^{b_{r,v}} \otimes \tau_v\right), \tag{5.5}$$

where each $\pi_{i,v}$ is a tempered representation of some $\mathrm{GL}(n_i, k_v)$ and τ_v is a tempered, generic representation of some $\mathbf{G}(m, k_v)$ with $n_1 + \cdots + n_t + m = n$.

Definition 5.6. *We say π satisfies $H(\theta_n)$ with $\theta_n \geq 0$ if for all places v we have*

$$-\theta_m \leq b_{i,v} \leq \theta_m.$$

Again, we would have the bound $H(1)$ trivially as a consequence of the classification of the generic unitary dual and the Ramanujan conjecture demands $H(0)$.

Proposition 5.7. *Let k be a number field and assume that all the unitary, cuspidal, automorphic representations of $\mathrm{GL}(m, \mathbb{A}_k)$ satisfy $H(\theta_m)$ for $2 \leq m \leq 2n$. Then any globally generic, unitary, cuspidal, automorphic representation π of $\mathbf{G}(n, \mathbb{A}_k)$ satisfies $H(\theta)$, where $\theta = \max\limits_{2\leq i\leq 2n} \theta_i$. In fact, if π transfers to a non-cuspidal representation $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_t$, then π satisfies the possibly better bound of $H(\theta)$ where $\theta = \max\{\theta_{n_1}, \theta_{n_2}, \ldots, \theta_{n_t}\}$. Here, Π_i is a unitary, cuspidal representation of $\mathrm{GL}(n_i, \mathbb{A}_k)$.*

Proof. The argument is exactly the same as the proof of [AS2, Theorem 3.3] and we do not repeat it here. Note that our Proposition 5.1 is used for the ramified non-archimedean places. □

Corollary 5.8. *Every globally generic, unitary, cuspidal, automorphic representation π of $\mathbf{G}(n, \mathbb{A}_k)$ satisfy*

$$H\left(\frac{4n^2 - 1}{2(4n^2 + 1)}\right).$$

If π transfers to a non-cuspidal, automorphic representation

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_t,$$

then we can replace n with the size of the largest GL *block appearing, resulting in a better estimate.*

Proof. This is immediate if we combine Proposition 5.7 with the $\mathrm{GL}(m)$ estimate of $1/2 - 1/(m^2 + 1)$. □

We should remark that for small values of n it is possible to obtain better estimates because much better estimates are available for small general linear groups (and also for $k = \mathbb{Q}$). For an example, see [AS2, §3.1]

Corollary 5.9. *The Ramanujan conjecture for the unitary, cuspidal representations of $\mathrm{GL}(m, \mathbb{A}_k)$ for $m \leq 2n$ implies the Ramanujan conjecture for the generic spectrum of $\mathbf{G}(n, \mathbb{A}_k)$.*

Proof. This is an immediate corollary of Proposition 5.7 where all the θ's are zero. □

5.3. **Image of Kim's exterior square.** H. Kim proved the exterior square transfer of automorphic representations from $\mathrm{GL}(4, \mathbb{A}_k)$ to $\mathrm{GL}(6, \mathbb{A}_k)$ [K2, H2]. A. Raghuram and the first author gave a complete cuspidality criterion for this transfer, determining when the image of this transfer is not cuspidal [AR]. A natural question about the image of this transfer is which automorphic representations of $\mathrm{GL}(6, \mathbb{A}_k)$ are indeed in the image of this transfer. We can now answer this question as an application of our Theorem 4.26.

Proposition 5.10. *Let Π be a cuspidal, automorphic representation of* $\mathrm{GL}(6, \mathbb{A}_k)$. *Then there is a cuspidal automorphic representation π of* $\mathrm{GL}(4, \mathbb{A}_k)$ *such that $\Pi = \wedge^2 \pi$ if and only if there is an idèle class character ω such that the partial L-function $L^S(s, \Pi, \mathrm{Sym}^2 \otimes \omega^{-1})$ has a pole at $s = 1$ for S a sufficiently large finite set of places of k including all the archimedean ones.*

Proof. The proposition follows immediately from our Theorem 4.26 if we recall that Kim's exterior square transfer from GL(4) to GL(6) is a special case of the transfer in the split even case of our theorem when $m = 3$, i.e., the transfer from GSpin(6) to GL(6) [AS1, Prop. 7.6].

If we assume that Π is the transfer of π, then we have proved that we can take $\omega = \omega_\pi$, the central character of π. The opposite direction requires the descent method in our cases and would follow from J. Hundley and E. Sayag's "lower bound" result for our transfer [HS1, HS2] because Kim's $\wedge^2$ is a special case of transfer from GSpin(6) to GL(6) as mentioned above. □

Another natural question regarding the image of Kim's exterior square transfer is to determine "the fiber" for each cuspidal Π which is indeed in the image. In other words, determine all representations π such that $\Pi = \wedge^2 \pi$.

A further interesting question would be to explore possible overlaps between various transfers to cuspidal representations of GL(6). As pointed out in [CPSS2, §6] (for the untwisted $\omega = 1$ case) and as it is apparent from our Theorem 4.26 there can be no overlap between the images of transfers from GSpin(7) and quasi-split forms of GSpin(6) (which includes Kim's transfer) to GL(6). However, there may be potential overlaps with the transfer from unitary groups or the Kim-Shahidi transfer [KS1] from $\mathrm{GL}(2) \times \mathrm{GL}(3)$ to GL(6).

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Department of Mathematics, Oklahoma State University, Stillwater, OK 74078–1058, USA
E-mail address: `asgari@math.okstate.edu`

Mathematics Department, Purdue University, West Lafayette, IN 47907, USA
E-mail address: `shahidi@math.purdue.edu`