RANKIN-SELBERG L-FUNCTIONS FOR $\mathrm{GSpin} \times \mathrm{GL}$ GROUPS

MAHDI ASGARI, JAMES W. COGDELL, AND FREYDOON SHAHIDI

Abstract. We construct an integral representation for the global Rankin-Selberg (partial) L-function $L(s, \pi \times \tau)$ where π is an irreducible globally generic cuspidal automorphic representation of a general spin group (over an arbitrary number field) and τ is one of a general linear group, generalizing the works of Gelbart, Piatetski-Shapiro, Rallis, Ginzburg, Soudry and Kaplan among others. We consider all ranks and both even and odd general spin groups including the quasi-split forms. The resulting facts about the location of poles of $L(s, \pi \times \tau)$ have, in particular, important consequences in describing the image of the Langlands funtorial transfer from the general spin groups to general linear groups.

1. Introduction

The purpose of this article to develop *integral representations* for the Rankin-Selberg convolution L-functions for the globally generic automorphic representations of $\mathrm{GSpin} \times \mathrm{GL}$ groups. We consider all semi-simple ranks of both the general spin groups and the general linear groups, including the case of the quasi-split, non-split, even, general spin groups. Our work in this article corresponds to steps (1), part of (2), and (4) in the "L-function machine" as described in [19, §1.7]: establish a "Basic Identity" for the global zeta integrals and expand them as an Euler product; analyze the meromorphic behavior while we do not deal with the functional equation in this article; and complete the "unramified computation" which relates the local zeta integrals to local Langlands L-functions in the unramified case.

For us the main motivation for this work was its application to classifying the image of the generic functorial transfer from the general spin groups to the general linear groups, even though our results are otherwise of interest as well, just as it has been the case with all the cases of developing integral representations for automorphic L-functions. Indeed in [7] the first and third authors already used some analytic properties of the partial Rankin-Selberg L-function $L^S(s, \pi \times \tau)$ for the description of the image of the generic functorial transfer from GSpin_{2n+1} or GSpin_{2n} to GL_{2n}. Here, π is a globally generic, unitary, cuspidal automprhic representation of the general spin group and τ is one of the general linear group. As usual, S denotes a finite set of places, including the Archimedean places, outside of which all the data is unramified. See Proposition 9.2 for the precise statement.

Our results here are restricted to the globally generic representations, and their application is to the functorial transfer of the globally generic automorphic representations. However, in recent years there have been great strides in establishing functorial transfer of arbitrary (not necessarily generic) automorphic representations from the classical groups and their similitude versions, as well as unitary groups, to the general linear groups. In particular we mention J. Arthur's endoscopic classification of representations of special orthogonal and symplectic groups [2] and the works of Y. Cai, S. Friedberg, D. Ginzburg, D. Gourevitch and E. Kaplan on global functoriality for non-generic representations [12, 13, 10, 11, 22]. Arthur's book does not consider the case of general spin groups, even though Arthur's methods, with appropriate modifications, would be applicable to the general spin groups. On the other hand, the works of Y. Cai et al mentioned above for the non-generic representations do cover the general spin groups. The generic transfer for the general spin groups was established earlier by the first and third named authors [6, 7].

S. Gelbart, I. I. Piatetski-Shapiro and S. Rallis gave the original methods of constructing integral representations that produce the Rankin-Selberg L-functions for $G \times \mathrm{GL}_n$ in [18], where $G = \mathrm{SO}_{2n+1}$, SO_{2n}, or Sp_{2n}. The integrals in each case look different and they needed substantially different methods to deal with each, calling them *Method A*, *Method B*, and *Method C*, respectively. As we only deal with groups of Dynkin types B and D, we will focus on the first two methods, which we briefly recall below.

In Method A, one takes a globally generic cuspidal automorphic representation π of $\mathrm{SO}_{2n+1}(\mathbb{A})$ and a cuspidal representation τ of $\mathrm{GL}_n(\mathbb{A})$. We consider GL_n as the Levi factor of the Siegel parabolic subgroup in SO_{2n} (i.e., "doubling the number of variables") and construct an Eisenstein series on it. One then embeds SO_{2n} in SO_{2n+1} and integrates a Whittaker function of π against a Fourier coefficient of the Eisenstein series. Here, the integral is over the adelic points of SO_{2n} modulo its rational points. Gelbart, Piatetski-Shapiro and Rallis then study this integral by "unfolding" it and writing it as an Euler product. They then compute the local integral at an unramified finite place v, which will turn out to be expressed as a quotient of the local L-function $L(s, \pi_v \times \tau_v)$ and the exterior square L-function $L(s, \tau, \wedge^2)$. They prove this by considering the decomposition of a certain symmetric algebra and use some results of Ton-That [38, 39] along with the Caselman-Shalika formula. A similar construction can be done for $\mathrm{SO}_{2n} \times \mathrm{GL}_{n-1}$ as well, with an embedding of $\mathrm{SO}_{2(n-1)+1}$ inside SO_{2n}, with the symmetric square L-function $L(s, \tau, \mathrm{Sym}^2)$ replacing the exterior square L-function.

In Method B, the roles of cuspidal representation π and the Eisenstein series constructed from τ are switched, so the cuspidal representation is on the smaller group SO_{2n} while the Eisenstein series is on the larger group SO_{2n+1}, coming from an induced representation from the Siegel parabolic of SO_{2n+1}. While the analysis in Method B is somewhat different, a similar unramified computation can be done. The result will be again the local Rankin-Selberg L-function $L(s, \pi \times \tau)$, divided by the symmetric square L-function $L(s, \tau, \mathrm{Sym}^2)$. Again, one can also consider the case of $\mathrm{SO}_{2n} \times \mathrm{GL}_n$ using Method B and again the exterior square L-function $L(s, \tau, \wedge^2)$ appears. While [18] mostly focuses on the split groups, they point out that the methods work for quasi-split groups as well, and even double covers of special orthogonal groups, i.e., the spin groups. (They also cover the Rankin-Selberg construction for $\mathrm{Sp}_{2n} \times \mathrm{GL}_n$ in their Method C as we mentioned above.)

D. Ginzburg [20] generalized Method A from $\mathrm{SO}_{2n+1} \times \mathrm{GL}_n$, resp. $\mathrm{SO}_{2n} \times \mathrm{GL}_{n-1}$, to the case of $\mathrm{SO}_{2n+1} \times \mathrm{GL}_m$ with $m \leq n$, resp. $\mathrm{SO}_{2n} \times \mathrm{GL}_m$ with $m \leq n-1$. The idea here is that one proceeds similarly as above, using τ on GL_m to construct an Eisenstein series on SO_{2m} and embeds SO_{2m} inside SO_{2m+1}. However, one then "pads" the integral with some unipotent integrals in order to produce the zeta integral that gives the Rankin-Selberg L-function for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_m$. It is clear that $m \leq n$ is necessary for this process to work. Again, there is also a similar procedure for $\mathrm{SO}_{2n} \times \mathrm{GL}_m$ with $m \leq n-1$. In order to do the unramified computations, Ginzburg uses an inductive argument that reduces the proof to that of the case of $m = n$, resp. $m = n-1$. Fortunately, the results of Ton-That mentioned above for decomposing the symmetric algebra is still available for any rank and one could replace the inductive argument with the direct decomposition the symmetric algebra in this case, as we do in our work for the general spin groups (see below).

However, when one considers generalizing Method B for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_m$ with $m > n$, a serious obstacle appears because the decomposition of the symmetric algebra appears to be much more complicated and not so easily tractable. D. Soudry [34] showed how to resolve this issue by finding a certain "duality" between the Rankin-Selberg L-function for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_m$ with $m > n$ (in Method B) and that for $\mathrm{SO}_{2m} \times \mathrm{GL}_n$ with $m > n$ (in Method A). E. Kaplan [24] then extended this work to the case of $\mathrm{SO}_{2n} \times \mathrm{GL}_m$ with $m \geq n$. Kaplan also considers the quasi-split, non-split, forms that exist in the case of even special orthogonal groups.

In this article we give a construction of an integral representations for Rankin-Selberg L-functions $L(s, \pi \times \tau)$ where π is an irreducible globally generic (i.e., having a Whttaker model with respect to a generic character) cuspidal automorphic representation of a general spin groups and τ is an irreducible cuspidal automorphic representation of a general linear group. See Thoerem 4.2 and Theorem 5.1. We consider both the odd and the even cases, including the quasi-split non-split forms in the even case, and any rank of the general linear group. As such, we are generalizing all the works above, both in Method A and Method B. See the table in Section 7 for the details of the various cases.

Naturally, we follow similar constructions as in the works of Ginzburg, Soudry and Kaplan (and the original works of Gelbart, Piatetski-Shapiro and Rallis). The main difference in the similitude case, in addition to a careful analysis of the unfolding arguments, is the appearance of the *twisted* symmetric/exterior square L-functions of τ. See Theorems 8.1 and 8.2 for the details. We also point out that in the quasi-split non-split case the expressions obtained from the unramified computations in the two theorems resemble the ones in the "opposite parity" case. This phenomenon is to be expected considering the Galois action on the Dynkin diagram in the even case, for example, and it is already present in the work of E. Kaplan for quasi-split SO_{2n}.

As we mentioned above, for the symmetric algebra decomposition we use the results of Ton-That [38, 39], which are available for the special orthogonal and symplectic groups. We carefully study the effect of the presence of the center in the similitude case, cf. Section 8.1. We then combine this with a suitable version of the Casselman-Shalika formula that we review in Section 8.2 in order to complete the unramified computations. As we already mentioned we then use ideas similar to [18, Appendix] to show directly that the two sides of the equations we are claiming as the result of our unramifed computation given the same power series in q^{-s}. It should be possible to accomplish the same goal by arguing inductively as in the works of Ginzburg or Kaplan. As pointed above, there is substantially difference analysis in Methods A and B and therefore the unramifed computations also look significantly different. That is why we do the two methods in two separate theorems in Section 8.3 even though the final expressions for the local unramified integrals look similar in Theorems 8.1 and 8.2.

In order to effect the duality argument of Soudry, mentioned above, for our cases we need to use some results about the γ-factors for the groups involved. The constructions and analysis of the γ-factors is part of the "L-function machine" that we have not studied in this article. However, fortunately E. Kaplan, J. F. Lau and B. Liu have studied them for exactly the cases we need in [28]. As such we have simply used their result in the only place where we need to invoke the γ-factors in this article, namely the generalization of Soudry's duality argument. Finally, we point out that in the quasi-split case in Method B, we also need to invoke a certain uniqueness result which is fortunately also provided by [28] as part of their work on local descent from general spin groups.

We note that $\mathrm{GSpin}_5 \cong \mathrm{GSp}_4$ and therefore our results, in particular, cover the Rankin-Selberg product L-functions for $\mathrm{GSp}_4 \times \mathrm{GL}_1$, $\mathrm{GSp}_4 \times \mathrm{GL}_2$ and $\mathrm{GSp}_4 \times \mathrm{GL}_3$ for generic representations. These cases have been studied extensively over the years. The twist by GL_1 essentially amounts to the construction of the standard L-function of GSp_4 while the twist by GL_2 was studied by Novodvorsky [31, §3] and Soudry [33]. Bump [8, §3.3–3.5] surveys these two cases as well as the twist by GL_3, where he gives a particular embedding of GL_3 in GL_4 and assumes that the representations have trivial central character. We point out that the split group GSpin_6 is isomorphic to $\left\{(A, b) \in \mathrm{GL}_4 \times \mathrm{GL}_1 : \det(A) = b^2\right\}$ (cf. [4, §2.2]). As such, the subgroup of GL_4 that Bump works with for this case corresponds to one of the two (isomorphic, non-conjugate) Siegel Levi subgroups in GSpin_6 and our construction would then agree with Bump's description in [8, §3.5]. (There are also many more works for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ that consider non-generic representations, which our results here do not cover.)

Recently P. Yan and Q. Zhang have considered a Rankin-Selberg integral for a general linear group and a product of two general linear groups in [40]. Their study, in particular, gives another proof of Jacquet's local converse theorem. Since the small rank case of $\mathrm{GL}_2 \times \mathrm{GL}_2$ is very close to the group GSpin_4, their integral for $(\mathrm{GL}_2 \times \mathrm{GL}_2) \times \mathrm{GL}_n$ and our case of $\mathrm{GSpin}_4 \times \mathrm{GL}_n$ are closely related.

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Contents

2. The Preliminaries

2.1. **Notation.** Let k be a number field and let $\mathbb{A} = \mathbb{A}_k$ be the ring of adèles of k. Also, let F be a local field of characteristic zero. Often we have $F = k_v$ for some place v of the number field k.

We consider both odd and even (split and quasi-split) general spin groups defined over k or F. We will use $G = G_{n'} = \mathrm{GSpin}_{n'}$ and $H = H_{m'} = \mathrm{GSpin}_{m'}$ with either

$$\begin{array}{ll} \text{(case A)} & n' = 2n+1, m' = 2m, \quad \text{or} \\ \text{(case B)} & n' = 2n, m' = 2m+1 \end{array}$$

with m and n positive integers.

Later on we will assume that $n' < m'$ and introduce an embedding $G \hookrightarrow H$ with n' and m' of opposite parity.

As in [23, Theorem 4.3.1] there exist surjections $G_{n'} \longrightarrow \mathrm{SO}_{n'}$ and we fix one such surjection and denote it by pr, so that we have

$$\mathrm{pr} : G_{n'} \longrightarrow \mathrm{SO}_{n'}. \tag{2.1}$$

The projection map also gives maps at the level of k-points, F-points, and the adèlic points, all of which will also be denoted pr.

2.2. **Structure of** GSpin **Groups.** There are several constructions one could give for the general spin groups. One construction is via the introduction of a based root datum for each group as in [36, §7.4.1], which we do below. More detailed descriptions can also be found in [6, §2] and [23, §4].

2.2.1. *The root data of* GSpin *groups.* Let $n' \geq 3$. (See Remark 2.16 below.) The based root datum of the split $\mathrm{GSpin}_{n'}$ is given by $(X, R, \Delta, X^\vee, R^\vee, \Delta^\vee)$, where X and $X^\vee$ are $\mathbb{Z}$-modules generated by generators $e_0, e_1, \cdots, e_n$ and $e_0^*, e_1^*, \cdots, e_n^*$, respectively. The pairing

$$\langle\ ,\ \rangle : X \times X^\vee \longrightarrow \mathbb{Z} \tag{2.2}$$

is the standard pairing.

When $n' = 2n+1$ the roots and coroots are given by

$$\begin{aligned} R = R_{2n+1} &= \{\pm(e_i \pm e_j) \,:\, 1 \leq i < j \leq n\} \cup \{\pm e_i \,:\, 1 \leq i \leq n\} &(2.3)\\ R^\vee = R^\vee_{2n+1} &= \left\{\pm(e^*_i - e^*_j) \,:\, 1 \leq i < j \leq n\right\} \cup &(2.4)\\ &\quad \left\{\pm(e^*_i + e^*_j - e^*_0) \,:\, 1 \leq i < j \leq n\right\} \cup \{\pm(2e^*_i - e^*_0) \,:\, 1 \leq i \leq n\} \end{aligned}$$

along with the bijection $R \longrightarrow R^\vee$ given by

$$\begin{aligned} (\pm(e_i - e_j))^\vee &= \pm(e^*_i - e^*_j) &(2.5)\\ (\pm(e_i + e_j))^\vee &= \pm(e^*_i + e^*_j - e^*_0) &(2.6)\\ (\pm e_i)^\vee &= \pm(2e^*_i - e^*_0). &(2.7) \end{aligned}$$

Moreover, we fix the following choice of simple roots and coroots:

$$\begin{aligned} \Delta &= \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_n\}, &(2.8)\\ \Delta^\vee &= \left\{e^*_1 - e^*_2, e^*_2 - e^*_3, \cdots, e^*_{n-1} - e^*_n, 2e^*_n - e^*_0\right\}. &(2.9) \end{aligned}$$

This based root datum determines the group GSpin_{2n+1} uniquely, equipped with a Borel subgroup B containing a maximal torus T.

When $n' = 2n$ we have

$$\begin{aligned} R = R_{2n} &= \{\pm(e_i \pm e_j) \,:\, 1 \leq i < j \leq n\} &(2.10)\\ R^\vee = R^\vee_{2n} &= \left\{\pm(e^*_i - e^*_j) \,:\, 1 \leq i < j \leq n\right\} \cup &(2.11)\\ &\quad \left\{\pm(e^*_i + e^*_j - e^*_0) \,:\, 1 \leq i < j \leq n\right\} \end{aligned}$$

along with the bijection $R \longrightarrow R^\vee$ given by

$$\begin{aligned} (\pm(e_i - e_j))^\vee &= \pm(e^*_i - e^*_j) &(2.12)\\ (\pm(e_i + e_j))^\vee &= \pm(e^*_i + e^*_j - e^*_0). &(2.13) \end{aligned}$$

and

$$\begin{aligned} \Delta &= \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_{n-1} + e_n\}, &(2.14)\\ \Delta^\vee &= \left\{e^*_1 - e^*_2, e^*_2 - e^*_3, \cdots, e^*_{n-1} - e^*_n, e^*_{n-1} + e^*_n - e^*_0\right\}. &(2.15) \end{aligned}$$

Again, this based root datum determines the split group GSpin_{2n} uniquely, equipped with a Borel subgroup B containing a maximal torus T.

Remark 2.16. We should also mention that $\mathrm{GSpin}_0 \cong \mathrm{GL}_1$, $\mathrm{GSpin}_1 \cong \mathrm{GL}_1$ and $\mathrm{GSpin}_2 \cong \mathrm{GL}_1 \times \mathrm{GL}_1$. While some of the notation above make sense for these small rank cases as well, Δ and $\Delta^\vee$ are empty. Finally, for the quasi-split non-split even general spin group GSpin^*_2, associated with a quadratic extension K/k (see below), we have $\mathrm{GSpin}^*_2 \cong \mathrm{Res}_{K/k}\,\mathrm{GL}_1$.

2.2.2. *Abstract group structure of* GSpin *groups.* We proved in [3] that the above is equivalent to a second construction of the split $\mathrm{GSpin}_{n'}$ given as a suitable quotient of $\mathrm{GL}_1 \times \mathrm{Spin}_{n'}$, where $\mathrm{Spin}_{n'}$ is the split, simply-connected, simple, connected group of type B_n if $n' = 2n+1$, or of type D_n if $n' = 2n$. If $n' \geq 3$, we have

$$\mathrm{GSpin}_{n'} \cong (\mathrm{GL}_1 \times \mathrm{Spin}_{n'}) \,/\, \{(1,1), (-1,c)\}, \tag{2.17}$$

where c is a particular element of the center of $\mathrm{Spin}_{n'}$ as follows:

(A) If $n' = 2n+1$, then $Z(\mathrm{Spin}_{n'}) = \{1, c\} \cong \mathbb{Z}/2\mathbb{Z}$,

(B1) If $n' = 2n$ with n even, then $Z(\mathrm{Spin}_{n'}) = \{1, c, z, cz\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and

(B2) If $n' = 2n$ with n odd, then $Z(\mathrm{Spin}_{n'}) = \{1, z, c = z^2, z^3\} \cong \mathbb{Z}/4\mathbb{Z}$,

where $c = \alpha_n^\vee(-1)$ when $n' = 2n+1$ and $c = \alpha_{n-1}^\vee(-1)\alpha_n^\vee(-1)$ when $n' = 2n$, an element of order 2 in all cases.

2.2.3. *Quasi-split forms of even* GSpin. When $n' = 2n$, we also have the quasi-split $\mathrm{GSpin}_{n'}$ groups, which we describe below. They are of type ${}^2D_n$. We refer to [23] and [16] for more details about them.

Let G be a quasi-split form of GSpin_{2n}. We have a fixed Borel subgroup B and a Cartan subgroup $T \subset B$. We fix a *pinning* (or splitting) $(B, T, \{x_\alpha\}_{\alpha\in\Delta})$, where $\{x_\alpha\}$ is a collection of root vectors, one for each simple root of T in B. We also denote the maximal k-split subtorus of T by T_s. Following [23] we give the following two parametrizations of the quasi-split forms of GSpin_{2n}.

First Parametrization. By [36, §16.2], the quasi-split forms of GSpin_{2n} over k are determined by the indexed root data $(X, \Delta, X^\vee, \Delta^\vee, \Delta_0, \nu)$. Here, Δ_0 is empty since the group is quasi-split. Also, ν denotes a Galois action on X and $X^\vee$. The Galois action is either trivial or switches the simple roots $e_{n-1} - e_n$ and $e_{n-1} + e_n$ while keeping all other simple roots fixed. In fact, the nontrivial Galois element acts on X and on $X^\vee$ via

$$\nu(e_i) = \begin{cases} e_0 + e_n, & i = 0, \\ e_i, & 1 \le i \le n-1, \\ -e_n, & i = n, \end{cases} \quad \text{and} \quad \nu(e_i^*) = \begin{cases} e_0^*, & i = 0, \\ e_i^*, & 1 \le i \le n-1, \\ -e_n^* + e_0^*, & i = n. \end{cases} \tag{2.18}$$

Moreover, the k-rational character lattice ${}_kX$ is spanned by $e_1, \ldots, e_{n-1}, e_n + 2e_0$ and the k-rational cocharacter lattice ${}_kX^\vee$ is spanned by $e_0^*, e_1^*, \ldots, e_{n-1}^*$ (cf. [23, §4.3].) In particular, the root system of G relative to T_s is of type B_{n-1} with k-rational simple roots and k-rational simple coroots given by ${}_k\Delta = \{e_1 - e_2, \ldots, e_{n-2} - e_{n-1}, e_{n-1}\}$ and ${}_k\Delta^\vee = \left\{e_1^* - e_2^*, \ldots, e_{n-2}^* - e_{n-1}^*, 2e_{n-1}^* - e_0^*\right\}$.

There is a one-to-one correspondence between

(i) the quasi-split k-groups G with connected component of L-group ${}^LG^0 \cong \mathrm{GSO}_{2n}(\mathbb{C})$

and

(ii) the characters $\mu : \mathrm{Gal}(\overline{k}/k) \longrightarrow S$, where

$$S = \{\sigma \in \mathrm{Aut}(X(T)) : \sigma \text{ permutes } \Delta \text{ via an automorphism of the Dynkin diagram}\}.$$

We have $S \cong \mathbb{Z}/2\mathbb{Z}$ and by class field theory the characters of order two of $\mathrm{Gal}(\overline{k}/k)$ are in bijection with

(iii) the quadratic characters $\mu : k^\times \backslash \mathbb{A}^\times \longrightarrow \{\pm 1\}$.

Therefore, the quasi-split forms of GSpin_{2n} are parametrized by the quadratic idele class characters of k. When μ is nontrivial, we denote the associated quasi-split non-split group by GSpin_{2n}^μ or simply GSpin_{2n}^* when the particular μ is unimportant. We will also denote the quadratic extension of k associated with μ by K^μ/k or simply K/k.

Second Parametrization. For $a \in k^\times$, denote its square class in $k^\times/(k^\times)^2$ by $\underline{a} = a(k^\times)^2$. Let $k(\sqrt{a})$ be the smallest extension of k in which the elements of $\underline{a}$ are squares, so that $k(\sqrt{\underline{a}}) = k(\sqrt{a})$. We then let $\mathrm{GSpin}_{2n}^{\underline{a}} = \mathrm{GSpin}_{2n}^{a}$ denote the quasi-split form of GSpin_{2n} such that the associated map $\mathrm{Gal}(\overline{k}/k) \longrightarrow \{\pm 1\}$ factors through $\mathrm{Gal}\left(k(\sqrt{\underline{a}})/\,k\right) = \mathrm{Gal}\left(k(\sqrt{a})/k\right)$.

2.2.4. *Dual of* GSpin *Groups.* Yet another construction for $\mathrm{GSpin}_{n'}$ is via their dual groups as follows [23, §4.3]:

(A) If $n' = 2n+1$, then the split $G = \mathrm{GSpin}_{n'}$ is the k-split, connected, reductive group having the based root datum dual to GSp_{2n} (so of type B_n). (See [6, §2.3] for the precise description.) Hence, ${}^LG = \mathrm{GSp}_{2n}(\mathbb{C}) \times \mathrm{Gal}(\overline{k}/k)$, with the Galois group acting trivially.

(B) If $n' = 2n$, then the split $G = \mathrm{GSpin}_{n'}$ is the k-split connected, reductive group having based root datum dual to that of GSO_{2n} (so of typle D_n). Here, GSO_{2n} is the connected component of the group GO_{2n} with all groups defined over k. (Again, see [6, §2.3] for the precise description.) Hence, ${}^LG = \mathrm{GSO}_{2n}(\mathbb{C}) \times \mathrm{Gal}(\overline{k}/k)$, with the Galois group acting trivially.

(C) If $n' = 2n$, $G = \mathrm{GSpin}_{2n}^{a}$, the quasi-split group associated with $K = k(\sqrt{a})$ as above, and its L-group can be given by ${}^LG = \mathrm{GSO}_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{k}/k)$, a semi-direct product where the Galois action on $\mathrm{GSO}_{2n}(\mathbb{C})$ is given as follows. If $\gamma \in \mathrm{Gal}(\overline{k}/k)$ with $\gamma|_K = 1$, then the action is trivial. If $\gamma|_K \neq 1$, then the action of γ on $g \in \mathrm{GSO}_{2n}(\mathbb{C})$ is given by jgj^{-1}, where $j = \mathrm{diag}(I_{n-2}, \mathrm{diag}(w,w), I_{n-2})$ and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2.3. **Weyl Groups.** By [23, Lemma 6.2.1], we know that the Weyl group of $G_{n'} = \mathrm{GSpin}_{n'}$ is isomorphic to the Weyl group of $\mathrm{SO}_{n'}$.

When $n' = 2n$, for the split G_{2n}, as for SO_{2n}, we have that the Weyl group $W_{n'} \cong \mathfrak{S}_n \rtimes \{\pm 1\}^{n-1}$. Choose representatives $(p, \underline{\epsilon}) \in \mathfrak{S}_n \rtimes \{\pm 1\}^{n-1}$ as in [23, §6]. Similarly, when $n' = 2n+1$, for the split G_{2n+1}, as for SO_{2n+1}, we have that the Weyl group $W_{n'} \cong \mathfrak{S}_n \rtimes \{\pm 1\}^{n}$. Again, take representatives $(p, \underline{\epsilon}) \in \mathfrak{S}_n \rtimes \{\pm 1\}^{n}$. In both cases, we have the following Weyl actions on the root and coroot lattices of $G_{n'}$:

$$(p, \underline{\epsilon}) \cdot e_i = \begin{cases} e_{p(i)} & i > 0, \epsilon_{p(i)} = 1, \\ -e_{p(i)} & i > 0, \epsilon_{p(i)} = -1, \\ e_0 + \sum\limits_{\epsilon_{p(i)} = -1} e_{p(i)} & i = 0, \end{cases}$$

and

$$(p, \underline{\epsilon}) \cdot e_i^* = \begin{cases} e_{p(i)}^* & i > 0, \epsilon_{p(i)} = 1, \\ e_0^* - e_{p(i)}^* & i > 0, \epsilon_{p(i)} = -1, \\ e_0^* & i = 0. \end{cases}$$

The above assertions for n' even are proved in [23, Lemma 6.2.3]. The assertions in the odd case were intended to be the content of [23, Lemma 13.2.2] although it appears that Lemma 13.2.2 and its proof were copies of Lemma 6.2.3 and its proof without modification. However, a similar calculation gives the above.

3. Unipotent Periods, Parabolic Subgroups, and Embeddings

3.1. **Unipotent Periods.** We recall the following facts from [23, §4,11]:

- The kernel of the projection map pr in (2.1) lies in the center $Z(G_{n'})$. In fact, if we write
$$1 \longrightarrow \{(1,1),(-1,c)\} \longrightarrow \mathrm{GL}_1 \times \mathrm{Spin}_{n'} \longrightarrow G_{n'} \longrightarrow 1,$$
then ker(pr) is the image of the GL_1 factor and so it is central. From this it follows that the action of $G_{n'}$ on itself by conjugation factors through pr.
- If u is a unipotent element of $G_{n'}(\mathbb{A})$ and $g \in G_{n'}(\mathbb{A})$, then $\mathrm{pr}(gug^{-1})$ is unipotent in $\mathrm{SO}_{n'}(\mathbb{A})$ and gug^{-1} is the unique unipotent element of its preimage in $G_{n'}(\mathbb{A})$.
- $\mathrm{pr} : G_{n'} \longrightarrow \mathrm{SO}_{n'}$ induces an isomorphism of unipotent varieties. We may specify unipotent elements of subgroups by their images under pr. This defines coordinates for any unipotent element or subgroup. Hence, we may write $u_{i,j}$ for the (i,j)-entry of $\mathrm{pr}(u)$. In particular, unipotent periods in $G_{n'}$ and $\mathrm{SO}_{n'}$ can be identified. Therefore, any identity or relationship between unipotent periods in $\mathrm{SO}_{n'}$ which is proved only by conjugation or "swapping" (root exchange) extends to $G_{n'}$.

Following the notation of [23], if G is any reductive algebraic group defined over k, U is a unipotent subgroup of G, and ψ_U is a character of $U(k)\backslash U(\mathbb{A})$, we define

$$\varphi^{(U,\psi_U)}(g) = \int\limits_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\psi_U^{-1}(u)\, du. \tag{3.1}$$

This unipotent integral is a Fourier coefficient (cf. [21, 34]). Here, one can take φ to be an automorphic form on G.

3.2. **Parabolic Subgroups.** There are two types of parabolic subgroups that play a role.

3.2.1. *The Maximal Parabolic Subgroups P_ℓ.* For $1 \le \ell < n$ if $n' = 2n+1$ or $1 \le \ell < n-1$ if $n' = 2n$, let P_ℓ be the standard maximal parabolic subgroup of $G_{n'}$ with Levi isomorphic to $\mathrm{GL}_\ell \times \mathrm{GSpin}_{n'-2\ell}$.

If $\gamma \in \mathrm{GL}_\ell(k)$ with k a field, we let $\gamma^\wedge$ denote the image of γ under the isomorphism of $\mathrm{GL}_\ell \times \mathrm{GSpin}_{n'-2\ell}$ with the Levi component of P_ℓ. (Note that the character and cocharacter lattices of $\mathrm{GSpin}_{n'-2\ell}$ are sublattices of X and $X^\vee$, spanned by the generators $e_0, e_{\ell+1}, \cdots, e_n$ and $e_0^*, e_{\ell+1}^*, \cdots, e_n^*$, respectively.) Since there is a corresponding parabolic subgroup of $SO_{n'}$ we can write

$$\gamma^\wedge = \begin{pmatrix} \gamma & & \\ & I_{n'-2\ell} & \\ & & \gamma^* \end{pmatrix} \in SO_{n'}(k)$$

which we can identify with its lift to $G_{n'}(k)$ if γ is a unipotent or a Weyl group element.

3.2.2. *The Siegel Parabolic Subgroup P_n.* This is a standard maximal parabolic subgroup with Levi isomorphic to $\mathrm{GL}_n \times \mathrm{GL}_1$. When $G_{n'}$ is split with $n' = 2n$, there are two parabolic subgroups $P = MU$ with Levi subgroup $M \cong \mathrm{GL}_n \times \mathrm{GL}_1$. Following [23, §6], we denote by P_n the one in which one deletes the root $e_{n-1} + e_n$ and the coroot $e_{n-1}^* + e_n^* - e_0^*$.

If we consider $G_{n'}$ with $n' = 2n+1$, then there is a unique standard parabolic subgroup $P = MU$ with $M \cong \mathrm{GL}_n \times \mathrm{GL}_1$. We denote this parabolic by P_n as well.

In either case, the subgroup $\mathfrak{S}_n \subset W_{n'}$ is isomorphic to W_M, the Weyl group of M. Also, $\mathrm{pr}\,(\mathrm{GL}_n \times \mathrm{GL}_1) = \mathrm{GL}_n$ and $\ker(\mathrm{pr}) = \mathrm{im}(e_0^*)$.

3.2.3. *The Parabolic Subgroups Q_ℓ.* For $G_{n'}$ split with $n' = 2n$, and for $1 \leq \ell < n-1$, we let $Q_\ell = L_\ell N_\ell$ denote the standard parabolic subgroup with Levi

$$L_\ell \cong \mathrm{GL}_1^\ell \times G_{2n-2\ell}$$

and unipotent radical given by

$$N_\ell = \{u \,|\, u_{i,j} = 0 \text{ for } i > \ell \text{ with } i < j \leq 2n - i\}\,.$$

For $n' = 2n+1$ and $1 \leq \ell < n$, we let $Q_\ell = L_\ell N_\ell$ be the parabolic subgroup with Levi

$$L_\ell \cong \mathrm{GL}_1^\ell \times G_{2n+1-2\ell}$$

and unipotent radical given by

$$N_\ell = \{u \,|\, u_{i,j} = 0 \text{ for } i > \ell \text{ with } i < j \leq 2n+1 - i\}\,.$$

3.2.4. *Stabilizers.* First, consider the parabolic $Q_\ell \subset G_{n'}$ with $n' = 2n$ even. In this case, a general character of N_ℓ is of the form (cf. [23, Remark 9.1.2])

$$\psi_0\left(c_1 u_{1,2} + \cdots + c_{\ell-1} u_{\ell-1,\ell} + d_1 u_{\ell,\ell+1} + \cdots + d_{2n-2\ell} u_{\ell,2n-\ell}\right),$$

where ψ_0 is a fixed non-trivial additive character (of k, $\mathbb{A}$, or F, depending on the context).

The Levi L_ℓ acts on the space of characters and over an algebraically closed field there is an open orbit, consisting of all those elements with $c_i \neq 0$ for all i and $\underline{d}\, J\, {}^t\underline{d} \neq 0$, where $\underline{d} = (d_1, d_2, \ldots, d_{2n-2\ell})$ and J is the matrix with 1's on the skew diagonal and zeros elsewhere. Over a general field k two such elements are in the same k-orbit if and only if the values of $\underline{d}\, J\, {}^t\underline{d}$ are in the same square class.

Let Ψ_ℓ be the character of N_ℓ defined by

$$\Psi_\ell(u) = \psi_0\left(u_{1,2} + \cdots + u_{\ell-1,\ell} + u_{\ell,n} - u_{\ell,n+1}\right).$$

Then one can see ([23, Remark 9.1.2]) that

- the stabilizer $L_\ell^{\Psi_\ell}$ has two connected components,
- the connected component of the identity $\left(L_\ell^{\Psi_\ell}\right)^0 \cong G_{2n-2\ell-1}$,
- there is an "obvious" choice of isomorphism $\iota : G_{2n-2\ell-1} \longrightarrow \left(L_\ell^{\Psi_\ell}\right)^0$ having the following property: If $\{e_i^* : i = 0, 1, \ldots, n\}$ is the basis for the cocharacter lattice of G_{2n}, and $\{\bar{e}_i^* : i = 0, 1, \ldots, n-\ell-1\}$ is the basis for $G_{n-\ell-1}$, then

$$\iota \circ \bar{e}_i^* = \begin{cases} e_0^* & i = 0, \\ e_{\ell+1}^* & i = 1, \ldots, n-\ell-1. \end{cases}$$

Note that it follows from $\iota \circ \bar{e}_0^* = \bar{e}_0^*$ that the induced map between $\bar{e}_0^*\,(\mathrm{GL}_1)$ in G and $\bar{e}_0^*\,(\mathrm{GL}_1)$ in H is the identity (and not, for example, the inversion map).

Next, consider $Q_\ell \subset G_{n'}$ with $n' = 2n+1$ odd. In this case, the general character of N_ℓ is of the form

$$\theta(u) = \psi_0\left(c_1 u_{1,2} + \cdots + c_{\ell-1} u_{\ell-1,\ell} + d_1 u_{\ell,\ell+1} + \cdots + d_{2n+1-2\ell} u_{\ell,2n+1-\ell}\right).$$

The Levi L_ℓ acts on the space of characters and over an algebraically closed field there is an open orbit, consisting of all those elements with $c_i \neq 0$ for all i and $\underline{d}\, J\, {}^t\underline{d} \neq 0$, where $\underline{d} = (d_1, d_2, \ldots, d_{2n+1-2\ell})$ and J is the matrix with 1's on the skew diagonal and zeros elsewhere. Over a general field k two such elements are in the same k-orbit if and only if the two values of $\underline{d}\, J\, {}^t\underline{d}$ are in the same square class. With θ as above, define $\mathrm{Inv}(\theta)$ to be the square class of $\underline{d}\, J\, {}^t\underline{d}$. The character θ is said to be in general position if $c_i \neq 0$ for $i = 1, \ldots, \ell - 1$ and $\mathrm{Inv}(\theta) \neq 0$.

By [23, Lemma 16.1.7] we have that if θ is in general position, then its stabilizer in L_ℓ, namely L_ℓ^θ, has two components and $\left(L_\ell^\theta\right)^0 \cong G_{2n-2\ell}^{\mathrm{Inv}(\theta)}$.

For $a \in k^\times$ let Ψ_ℓ^a be the character of N_ℓ defined by

$$\Psi_\ell^a(u) = \psi_0\left(u_{1,2} + \cdots + u_{\ell-1,\ell} + u_{\ell,n} + \frac{a}{2} u_{\ell,n+2}\right).$$

Then the orbit of Ψ_ℓ^a is determined by the square class of a. The character θ is in the same orbit as Ψ_ℓ^1. For each $a \in k^\times$, we have $\left(L_\ell^{\Psi_\ell^a}\right)^0 \cong G_{2n-2\ell}^{\underline{a}}$, where $\underline{a}$ is the square class of a.

We note here that [23] states the above assertions only in the case of even n because only this case is needed there. However, the assertions are valid for all n.

Remark 3.2 (*Notaion for the parabolic subgroups of* [21]*).* We note that [21] uses a different notation for its parabolic subgroups. However, while we have switched the notation for the parabolics to match both [23] and what seems to us more standard notation, the notation for the unipotent radicals do agree with those of [21]. As this caused confusion at one point, let us make this explicit.

In [21], the parabolic P_ℓ is that preserving a maximal isotropic flag of length ℓ [21, p. 42]. Its Levi decomposition is $P_\ell = M_\ell N_\ell$ with $M_\ell \cong (\mathrm{GL}_1)^\ell \times h(W_{m,\ell})$ in their notation. They then define a character ψ_ℓ or $\psi_{\ell,\alpha}$ which agrees with our Ψ_ℓ^a and then have $\mathrm{Stab}_{h(W_{m,\ell})}(\psi_{\ell,\alpha}) \cong G$. Therefore,

$$P_\ell^{GRS} = Q_\ell^{ACS} \quad M_\ell^{GRS} = L_\ell^{ACS} \quad N_\ell^{GRS} = N_\ell^{ACS}. \tag{3.3}$$

In [21] the parabolic Q_ℓ is that preserving a maximal isotropic subgroup of dimension ℓ [21, pp. 65–66]. Its Levi decomposition is $Q_\ell = D_\ell U_\ell$ with $D_\ell \cong \mathrm{GL}_\ell \times h(W_{m,\ell})$ in their notation [21, p. 81]. Therefore,

$$Q_\ell^{GRS} = P_\ell^{ACS} \quad D_\ell^{GRS} = M_\ell^{ACS} \quad U_\ell^{GRS} = U_\ell^{ACS}. \tag{3.4}$$

Here, the labels GRS refer to the subgroups in [21] and the labels ACS refer to the corresponding ones in this paper.

4. Global Integrals I (Case B)

In this section we translate [21, §10.3] into the context of GSpin groups. This corresponds to Method B in the original work of Gelbart and Piatetski-Shapiro [18, Part B], which dealt with the special orthogonal and general linear groups with equal (or nearly equal) ranks. As such, we refer to the integrals of this section as Case B. (See the table in Section 7 for a summary of various cases.) Also, recall Remark 3.2 on our notation for the parabolic subgroups.

We begin with a number field k and a k-vector space V of dimension $\dim V = m' \geq 3$, where m' can be even or odd. We take a non-degenerate quadratic form on V and let $h(V) =$

$\mathrm{SO}(V) = \mathrm{SO}_{m'}$ denote the special orthogonal group of V. We let $H = H_{m'} = \mathrm{GSpin}_{m'}$ be the associated GSpin group that covers $h(V)$, so associated to a quadratic space of dimension m'. We assume that $h(V)$ and H are split so that $m = [\frac{1}{2}\dim V]$ is the Witt index of V. (This is Assumption 2.1 of [21].) Therefore, either $m' = 2m$ or $m' = 2m+1$. Recall that we have fixed a projection

$$\mathrm{pr} : H_{m'} \longrightarrow h(V)$$

that induces an isomorphism of unipotent varieties and allows us to identify unipotent periods on H and $h(V)$. (See [23].)

Following [23], we take an integer ℓ such that $1 \leq \ell < m$ and let $Q_\ell \subset H$ be the standard parabolic subgroup of H, where $Q_\ell = L_\ell N_\ell$ with $L_\ell \cong \mathrm{GL}_1^\ell \times H_{m'-2\ell}$ and the unipotent radical N_ℓ is as in Section 3.2.3. We consider the following characters of N_ℓ :

$$\Psi_\ell(u) = \psi_0\left(u_{1,2} + \cdots + u_{\ell-1,\ell} + u_{\ell,2m} - u_{\ell,2m+1}\right) \text{ if } m' = 2m,$$

or

$$\Psi_\ell^a = \psi_0\left(u_{1,2} + \cdots + u_{\ell-1,\ell} + u_{\ell,2m} + \frac{a}{2}u_{\ell,2m+2}\right) \text{ if } m' = 2m+1,$$

where $a \in k^\times$. We let

$$G = \mathrm{Stab}_{L_\ell}\left(\Psi_\ell\right)^0 \cong \mathrm{GSpin}_{2m-2\ell-1} \text{ if } m' = 2m,$$

by [23, Section 9.1] or

$$G = \mathrm{Stab}_{L_\ell}\left(\Psi_\ell^a\right)^0 \cong \mathrm{GSpin}_{2m-2\ell}^a \text{ if } m' = 2m+1$$

by [23, Lemma 16.1.7], which allows for the case of G being quasi-split. In what follows, we will simply use Ψ_ℓ in either case and suppress the $a \in k^\times$.

In [23], the authors do not consider the case $\ell = 0$ which would correspond to the construction of Gelbart and Piatetski-Shapiro [18, Part B]. When $\ell = 0$ there is no Ψ_ℓ and we just "restrict". We are able to "pull back" the embedding in [21, p. 43]. The embedding proceeds as follows. When $\ell = 0$ we take

$$w_0 = y_a = e_{\tilde{m}} + (-1)^{m'+1}\frac{a}{2}e_{-\tilde{m}} \in SO_{m'}$$

in the notation of [21]. Then $y_a \in W_{m,\tilde{m}-1}$ and hence in $W_{m,\ell}$ for all ℓ including $\ell = 0$. We have $(y_a, y_a) = (-1)^{m'+1}a$ with $a \in k^\times$. Then we take G to be $pr^{-1}(y_a^\perp) \subset H$. In this case,

$$G \cong \mathrm{GSpin}_{2m-1} \text{ if } m' = 2m$$

or

$$G \cong \mathrm{GSpin}_{2m}^a \text{ if } m' = 2m+1.$$

(Note in [21] in the case $B_{m'}$ they take a vector ℓ_0 such that $V = X \oplus \langle \ell_0 \rangle \oplus X^\vee$ with $(\ell_0, \ell_0) = 1$. The vector y_a should play the same role as their ℓ_0. Note that in either case, the larger group H is of type $B_{m'}$.) The statements below now hold in the $\ell = 0$ case.

Let $P_m = M_m U \subset H$ denote the Siegel parabolic subgroup of H. We made a choice of P_m in Section 3 and we have $M_m \cong \mathrm{GL}_m \times \mathrm{GL}_1$. We fix the isomorphism between $\mathrm{GL}_m \times \mathrm{GL}_1$ and M_m as in [12, §1], where the isomorphism is denoted by i_M. (The choice of this isomorphism matters later, such as in (4.2), where we use ω^{-1}, not ω.) With this isomorphism fixed, we have $\mathrm{pr} : M_m \longrightarrow \mathrm{GL}_m$ with $\ker(\mathrm{pr}) = \mathrm{im}(e_0^*) = Z(H)^0$. Let τ be a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$ and let η be an idele class character of $\mathrm{GL}_1(\mathbb{A}) = \mathbb{A}^\times$. We form the following induced representation (with s being replaced by $s - 1/2$ later on below):

$$\rho = \rho_{\tau,\eta,s} = \mathrm{Ind}_{P_m(\mathbb{A})}^{H(\mathbb{A})} \left(\tau|\det|^s \otimes \eta\right).$$

For $\mathrm{Re}(s) \gg 0$ we can choose a section $f_s \in V_\rho$, the space of ρ, form the Eisenstein series

$$E(h, f_s) = \sum_{\delta \in P_m(k)\backslash H(k)} f_s(\delta h),$$

continue as a function of s, and form, as in (3.1), the unipotent period

$$E^{(N_\ell,\Psi_\ell)}(g, f_s),$$

which is naturally an automorphic form on $G(\mathbb{A}) = (\mathrm{Stab}_{L_\ell} \Psi_\ell)^0 (\mathbb{A})$. Note that in the $\ell = 0$ case we interpret as above, i.e., $G \hookrightarrow H$ and simple restriction from H to G.

Take $G \cong \mathrm{GSpin}_{m'-2\ell-1}$ and $H \cong \mathrm{GSpin}_{m'}$ as above with the embedding $G \hookrightarrow H$. Let (π, V_π) be a cuspidal automorphic representation of $G(\mathbb{A})$ with central character ω_π and denote by ω the idele class character such that for $a \in \mathbb{A}^\times$, we have

$$\pi(e_0^*(a)) = \omega(a)\,\mathrm{Id}_{V_\pi}. \tag{4.1}$$

Let (τ, V_τ) be a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. Let f_s be a K-finite section (with K denoting the maximal compact at the Archimedean place) in the space of

$$\rho_{\tau,\omega^{-1},s-1/2} = \mathrm{Ind}_{P_m(\mathbb{A})}^{H(\mathbb{A})} \left(\tau|\det|^{s-1/2} \otimes \omega^{-1}\right). \tag{4.2}$$

For $\varphi \in V_\pi$, a cusp form on $G(\mathbb{A})$, consider

$$\mathcal{L}(\varphi, f_s) = \int_{Z(\mathbb{A})G(k)\backslash G(\mathbb{A})} \varphi(g) E^{(N_\ell,\Psi_\ell)}(g, f_s)\, dg,$$

where $Z = Z(G)^0$ is the identity component of the center of G. (When $\ell = 0$ we would just restrict and there is no unipotent period.)

Lemma 4.1. *The integral $\mathcal{L}(\varphi, f_s)$ converges absolutely and uniformly in vertical strips in $\mathbb{C}$ away from poles of the Eisenstein series. Therefore, it defines a meromorphic function on $\mathbb{C}$.*

Proof. This is essentially basic estimates on Siegel sets in G. It is verified in the SO case in [21] and the same proof works for GSpin. □

Theorem 4.2. *Let the notation be as above.*

(i) *Assume that $\mathcal{L}(\varphi, f_s)$ is not identically zero. Then π is globally generic (for a suitable ψ-Whittaker model).*

(ii) *If $\mathrm{Re}(s) \gg 0$, then we have an identity*

$$\mathcal{L}(\varphi, f_s) = \int_{N_G(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi^\psi(g) \int_{N_\ell(\mathbb{A})\cap\beta^{-1}P_m(\mathbb{A})\beta\backslash N_\ell(\mathbb{A})} f_s^{(Z_m,\psi)}(\beta u g)\,\Psi_\ell(u)^{-1}\, du\, dg,$$

where

- *$N_G \subset G$ is a standard maximal unipotent subgroup of G,*
- *$\psi = \psi_{N_G}$ is the standard Whittaker character on N_G obtained from ψ_0 as in* [21, p. 289],

- $W^{\psi}_{\varphi}(g)$ *is the corresponding* ψ*-Whittaker function of* φ,
- $\beta = \beta_{\ell,a}$ *is a product of a certain Weyl group element and a (rational) "diagonal" element in* H*, or more precisely, in a unipotent subgroup* $J \subset N_\ell$ *described in Lemma 4.5 below,*
- Z_m *is the maximal unipotent subgroup of* $\mathrm{GL}_m \subset P_m \subset H$ *and* ψ *is the standard Whittaker character of* N_m*, and*
- *we have*
$$f_s^{(Z_m,\psi)}(h) = \int\limits_{Z_m(k)\backslash Z_m(\mathbb{A})} f_s(\hat{z}h)\psi^{-1}(z)\,dz,$$
with $\hat{z}$ *the image of* $z \in \mathrm{GL}_m$ *in* H.

Proof. We prove this in the $\ell \neq 0$ case. The $\ell = 0$ case follows as in [18]. The proof involves several steps as we detail below, establishing several intermediate results along the way.

Step 1. The integral converges for all s, because

- φ is rapidly decreasing (mod Z),
- E is of moderate growth (mod Z),
- $E^{(N_\ell,\Psi_\ell)}$ is a compact integration.

Step 2. For $\mathrm{Re}(s) \gg 0$, i.e., in the realm of absolute convergence of the Eisenstein series, we replace E by its absolutely convergent series. Then,
$$E^{(N_\ell,\Psi_\ell)}(h, f_s) = \int\limits_{N_\ell(k)\backslash N_\ell(\mathbb{A})} \sum_{\delta \in P_m(k)\backslash H(k)} f_s(\delta u h)\Psi_\ell^{-1}(u)\,du.$$

Step 3. We factor the sum through the double cosets of $P_m\backslash H/P_\ell$, where P_ℓ is the maximal parabolic subgroup of H with Levi isomorphic to $\mathrm{GL}_\ell \times \mathrm{GSpin}_{m'-2\ell}$.

Lemma 4.3. *The coset representatives for* $P_m(k)\backslash H(k)/P_\ell(k)$ *are as in* [21, pp. 70-71]*, or* [21, p. 285]*, i.e.,* $\epsilon_r \in W_H = W_{\mathrm{SO}_{m'}}$ *for* $0 \le r \le \ell$*, with*

$$\mathrm{pr}(\epsilon_r) = \begin{pmatrix} I_r & & \\ & 0 & I_{m-\ell} \\ & I_{\ell-r} & 0 \end{pmatrix}^{\wedge} \begin{pmatrix} I_r & & & & \\ & & & I_{\ell-r} & \\ & & I_{m'-2\ell} & & \\ & I_{\ell-r} & & & \\ & & & & I_r \end{pmatrix} w_b^{\ell-r},$$

where the $\wedge$ *notation is defined in 3.2.1, taken for the group* H*, and where* w_b *is as in* [21, pp. 70–71]*, so an auxiliary Weyl group element.*

Proof. We know that $W_H \cong W_{\mathrm{SO}_{m'}}$ and that $\ker(\mathrm{pr}) \subset Z(H) \subset Z(T)$, where T is the maximal torus in H. From [21] we know that
$$\mathrm{SO}_{m'} = \bigsqcup_{r=0}^{\ell} \mathrm{pr}\left(P_m(k)\right)\mathrm{pr}(\epsilon_r)\mathrm{pr}\left(P_\ell(k)\right).$$

Since $\ker(\mathrm{pr}) \subset Z(H) \subset P_m$, we have

$$H(k) = \bigsqcup_{r=0}^{\ell} P_m(k)\epsilon_r P_\ell(k).$$

□

Using this decomposition, we can partially unfold the Eisenstein series:

$$E^{(N_\ell,\Psi_\ell)}(h, f_s) = \sum_{r=0}^{\ell} \int_{N_\ell(k)\backslash N_\ell(\mathbb{A})} \sum_{\delta\in P_\ell^{(r)}} f_s\left(\epsilon_r \delta u h\right) \Psi_\ell^{-1}(u)\, du,$$

where $P_\ell^{(r)} = P_\ell \cap \epsilon_r^{-1} P_m \epsilon_r$.

The authors of [21] make $P_\ell^{(r)}$ explicit in their equation (4.19)–(4.20) or (4.22)–(4.23). If we write $P_\ell^{(r)} = M_\ell^{(r)} \ltimes U_\ell^{(r)}$, then the unipotent part $U_\ell^{(r)}$ agrees with [21] and the $\mathrm{GSpin}_{m'}/\mathrm{SO}_{m'}$ difference is in $M_\ell^{(r)}$. We will need the Levi part of this decomposition when $r = 0$.

Step 4. Now, we factor the innermost sum through the double cosets $P_\ell^{(r)}\backslash P_\ell/R_\ell$, where $R_\ell = R_{\ell,a} = \mathrm{Stab}_{L_\ell}(\Psi_\ell^a) \ltimes N_\ell = G \ltimes N_\ell$. The authors of [21] compute the representatives for these double cosets for $\mathrm{SO}_{m'}$. In the case of $\mathrm{SO}_{m'}$ they are of the form

$$\begin{pmatrix} \epsilon & & \\ & \gamma & \\ & & \epsilon^* \end{pmatrix}$$

with ϵ running through a set of representatives for $W_{\mathrm{GL}_r\times\mathrm{GL}_{\ell-r}}\backslash W_{\mathrm{GL}_\ell}$ and

$$\gamma = \begin{cases} I_{m'-2\ell} & \text{if } m' = 2m, \text{ or } m' = 2m+1 \text{ and } a \neq t^2, \\ \begin{pmatrix} I_{m'-\ell-1} & & & & \\ & 1 & & & \\ & \pm t & 1 & & \\ & -\frac{t^2}{2} & \mp t & 1 & \\ & & & & I_{m'-\ell-1} \end{pmatrix} & \text{if } m' = 2m+1 \text{ and } a = t^2. \end{cases}$$

Note that these are either Weyl group representatives, which we choose representatives for in H, or Weyl group representatives times unipotents, which have unique lifts to H. So we obtain the following.

Lemma 4.4. *The representatives $\{\eta\}$ for $P_\ell^{(r)}(k)\backslash P_\ell(k)/R_\ell(k)$ are either Weyl group representatives or Weyl group representatives times unipotents and so are uniquely determined by*

$$\mathrm{pr}(\eta) = \begin{pmatrix} \epsilon & & \\ & \gamma & \\ & & \epsilon^* \end{pmatrix} \in \mathrm{SO}_{m'}(k)$$

as above.

Then we can further unfold the Eisenstein series

$$E^{(N_\ell,\Psi_\ell)}(h,f_s) =$$
$$\sum_{r=0}^{\ell} \sum_{\eta\in P_\ell^{(r)}(k)\backslash P_\ell(k)/R_\ell(k)} \int_{N_\ell(k)\backslash N_\ell(\mathbb{A})} \sum_{\delta\in R_\ell(k)\cap\eta^{-1}P_\ell^{(r)}(k)\eta\backslash P_\ell(k)} f_s(\epsilon_r\eta\delta uh)\Psi_\ell^{-1}(u)\,du.$$

Step 5. We next describe some decompositions. Since $R_\ell = G\cdot N_\ell$, we have

$$R_\ell\cap\eta^{-1}P_\ell^{(r)}\eta = \left(G\cap\eta^{-1}M_\ell^{(r)}\eta\right)\cdot\left(N_\ell\cap\eta^{-1}P_\ell^{(r)}\eta\right)$$

and

$$(R_\ell\cap\eta^{-1}P_\ell^{(r)}\eta)\backslash R_\ell = \left((G\cap\eta^{-1}M_\ell^{(r)}\eta)\backslash G\right)\cdot\left((N_\ell\cap\eta^{-1}P_\ell^{(r)}\eta)\backslash N_\ell\right).$$

We utilize this decomposition inside the N_ℓ integration. For fixed r and η the inner integration becomes

$$\int_{N_\ell(k)\backslash N_\ell(\mathbb{A})} \sum_{\delta_2\in(G(k)\cap\eta^{-1}M_\ell^{(r)}(k)\eta)\backslash G(k)} \sum_{\delta_1\in(N_\ell(k)\cap\eta^{-1}P_\ell^{(r)}(k)\eta)\backslash N_\ell(k)} f_s(\epsilon_r\eta\delta_1\delta_2uh)\Psi_\ell^{-1}(u)\,du.$$

We next interchange the u integral and the δ_2 sum. We can do this, as in [21], by the absolute convergence for $\mathrm{Re}(s)\gg 0$. Note that any modulus character will be 1 on δ_2 since it is rational, and since $\delta_2\in G(k)$ it stabilizes the character Ψ_ℓ. After interchanging, we can collapse the N_ℓ integration and the δ_1 summation to obtain

$$\sum_{\delta\in(G(k)\cap\eta^{-1}M_\ell^{(r)}(k)\eta)\backslash G(k)} \int_{(N_\ell(k)\cap\eta^{-1}P_\ell^{(r)}(k)\eta)\backslash N_\ell(\mathbb{A})} f_s(\epsilon_r\eta u\delta h)\Psi_\ell^{-1}(u)\,du.$$

Step 6. We next fix δ and factor the N_ℓ integration as

$$\int_{(N_\ell(k)\cap\eta^{-1}P_\ell^{(r)}(k)\eta)\backslash N_\ell(\mathbb{A})} f_s(\epsilon_r\eta u\delta h)\Psi_\ell^{-1}(u)\,du =$$
$$\int_{(N_\ell(\mathbb{A})\cap\eta^{-1}P_\ell^{(r)}(\mathbb{A})\eta)\backslash N_\ell(\mathbb{A})} \int_{(N_\ell(k)\cap\eta^{-1}P_\ell(k)\eta)\backslash(N_\ell(\mathbb{A})\cap\eta^{-1}P_\ell^{(r)}(\mathbb{A})\eta)} f_s(\epsilon_r\eta u'u\delta h)\Psi_\ell^{-1}(u'u)\,du'\,du.$$

Lemma 4.5. *If $r>0$, then there exists a unipotent k-group $J\subset N_\ell\cap\eta^{-1}P_\ell^{(r)}\eta$ such that*

- *Ψ_ℓ is non-trivial on $J(\mathbb{A})$ and*
- *$\epsilon_r\eta J\eta^{-1}\epsilon_r^{-1}\subset U_m$, the unipotent radical of P_m.*

Proof. This exists in $\mathrm{SO}_{m'}$ by [21, page 287 or the proof of Proposition 5.1]. However, this is a unipotent element and the unipotent varieties are the same for $\mathrm{SO}_{m'}$ and H. Hence, the group theoretic statements remain true in H.

The characters $\psi_{\ell,\alpha}$ of [21, (3.10)] and Ψ_ℓ^a of [23, Definition 16.1.9] are equal, so we get the first statement. ∎

So if $r > 0$, since $\epsilon_r \eta J \eta^{-1} \epsilon_r^{-1} \subset U_m$ and the Eisenstein series is induced from P_m, for $j \in J(\mathbb{A})$, we see that

$$\begin{aligned} f_s\left(\epsilon_r \eta j u'' u \delta h\right) &= f_s\left(\epsilon_r \eta j \eta^{-1} \epsilon_r^{-1} \epsilon_r \eta u'' u \delta h\right) \\ &= f_s\left(\epsilon_r \eta u'' u \delta h\right) \end{aligned}$$

and

$$\int_{J(k)\backslash J(\mathbb{A})} \Psi_\ell^{-1}(j)\, dj = 0.$$

Therefore, the unipotent (N_ℓ, Ψ_ℓ) integration is zero except in the case of $r = 0$. Hence, we have the following.

Proposition 4.6.

$$\begin{aligned} E^{(N_\ell,\Psi_\ell)}(h, f_s) &= \sum_{\eta \in P_\ell^{(0)}\backslash P_\ell(k)/R_\ell(k)} \ \sum_{\delta \in (G(k)\cap \eta^{-1} M_\ell^{(0)} \eta)\backslash G(k)} \\ & \int_{(N_\ell(k)\cap \eta^{-1} P_\ell^{(0)}(k)\eta)\backslash N_\ell(\mathbb{A})} f_s\left(\epsilon_0 \eta u \delta h\right) \Psi_\ell^{-1}(u)\, du. \end{aligned}$$

Step 7. At this point we consider two cases:

Case I: m' is even, or m' is odd and $a \neq t^2$ is not a square. In this case there is only one η, namely $\eta = I_{m'}$.

Case II: m' is odd and $a = t^2$ is a square. In this case there are three choices for η, namely, $\eta = I_{m'}$, η_+, and η_-, where

$$\eta_\pm = \begin{pmatrix} I_\ell & & \\ & \gamma_\pm & \\ & & I_\ell \end{pmatrix}$$

and

$$\gamma_\pm = \begin{pmatrix} I_{m'-\ell-1} & & & & \\ & 1 & & & \\ & \pm t & 1 & & \\ & -\frac{t^2}{2} & \mp t & 1 & \\ & & & & I_{m'-\ell-1} \end{pmatrix}.$$

Lemma 4.7. *The above are still the coset representatives for $P_\ell^{(0)}(k)\backslash P_\ell(k)/R_\ell(k)$.*

Proof. This lemma follows from the agreement of the unipotent varieties for GSpin and SO along with the fact that the differences between GSpin and SO will lie in $P_\ell(k)$ and $R_\ell(k)$. □

We can now eliminate Case II and $\eta = \eta_\pm$.

Proposition 4.8. *In the case of $m' = 2m+1$ and $\eta = \eta_\pm$, the contributions of these terms to the unipotent period of the Eisenstein series vanish.*

Proof. We let $\eta = \eta_\pm \neq I_{m'}$. We have $N_\ell(k) \cap \eta^{-1} P_\ell^{(0)}(k)\eta = \eta^{-1}\left(N_\ell(k) \cap P_\ell^{(0)}\right)\eta$. Therefore,

$$\int_{(N_\ell(k)\cap\eta^{-1}P_\ell^{(0)}(k)\eta)\backslash N_\ell(\mathbb{A})} f_s\left(\epsilon_0 \eta u \delta h\right) \Psi_\ell^a(u)^{-1}\, du =$$

$$\int_{(N_\ell(\mathbb{A})\cap\eta^{-1}P_\ell^{(0)}(\mathbb{A})\eta)\backslash N_\ell(\mathbb{A})} \int_{(N_\ell(k)\cap P_\ell^{(0)}(k))\backslash(N_\ell(\mathbb{A})\cap P_\ell^{(0)}(\mathbb{A}))} f_s(\epsilon_0 v\eta u\delta h)\Psi_\ell^0(\eta^{-1}v\eta u)^{-1}\, dv\, du.$$

In terms of matrices, as in [23], using the agreement of unipotent varieties,

$$N_\ell \cap P_\ell^{(0)} = \left\{ v = \begin{pmatrix} z & 0 & 0 & y & 0 \\ & I_{m-\ell} & 0 & 0 & y' \\ & & 1 & 0 & 0 \\ & & & I_{m-\ell} & 0 \\ & & & & z^* \end{pmatrix} : z \in Z_\ell \right\}.$$

Note that

$$\Psi_\ell^a(\eta^{-1}v\eta) = \psi(z_{1,2} + \cdots + z_{\ell-1,\ell}) = \psi_\ell(z)$$

and this is independent of y. If we conjugate this past ϵ_0, we obtain

$$\epsilon_0\left(N_\ell \cap P_\ell^{(0)}\right) = \left\{ \epsilon_0 v \epsilon_0^{-1} = \hat{z}'_\ell = \begin{pmatrix} I_{m-\ell} & x \\ & \zeta \end{pmatrix}^\wedge \right\} = \hat{Z}'_\ell,$$

where for $g \in \mathrm{GL}_t$, we set $\hat{g}$ to be the lift of g into the Levi $\mathrm{GL}_t \times \mathrm{GL}_{m'-2t}$ of P_t. Note that even though we have not specified ζ, as in [21, p. 287] we know that $\zeta \in Z_\ell$. In these coordinates

$$\Psi_\ell^a(\eta^{-1}v\eta) = \psi(\zeta_{1,2} + \cdots + \zeta_{\ell-1,\ell}) = \psi(z'_{m-\ell+1,m-\ell+2} + \cdots + z'_{m-1,m}).$$

If we denote this character of Z'_ℓ by $\psi^0_{Z'_\ell}$ and

$$f_s^{(Z'_\ell,\psi^0_{Z'_\ell})}(h) = \int_{Z'_\ell(k)\backslash Z'_\ell(\mathbb{A})} f_s(\hat{z}'_\ell h)\psi^0_{Z'_\ell}(z'_\ell)^{-1}\, dz'_\ell,$$

then we have

$$\int_{N_\ell(k)\cap P_\ell^{(0)}(k)\backslash N_\ell(\mathbb{A})\cap P_\ell^{(0)}(\mathbb{A})} f_s\left(\epsilon_0 v\eta u\delta h\right) \Psi_\ell^0\left(\eta^{-1}v\eta\right)^{-1}\, dv = f_s^{(Z'_\ell,\psi^0_{Z'_\ell})}\left(\epsilon_0\eta u\delta h\right)$$

and finally

$$\int_{\left(N_\ell(k)\cap\eta^{-1}P_\ell^{(0)}(k)\eta\right)\backslash N_\ell(\mathbb{A})} f_s\left(\epsilon_0\eta u\delta h\right)\Psi_\ell^a(u)^{-1}\, du = \int_{\left(N_\ell(\mathbb{A})\cap\eta^{-1}P_\ell^{(0)}(\mathbb{A})\eta\right)\backslash N_\ell(\mathbb{A})} f_s^{(Z'_\ell,\psi^0_{Z'_\ell})}\left(\epsilon_0\eta u\delta h\right)\Psi_\ell^a(u)^{-1}\, du.$$

However, in the unipotent period $f_s^{(Z'_\ell,\psi^0_{Z'_\ell})}(\epsilon_0\eta u h)$, as an inner integral we have the constant term of f_s along the unipotent radical of the standard parabolic subgroup of GL_m

which corresponds to the partition $(m-\ell,\ell)$ of m. Since we are inducing from a cuspidal representation τ of $\mathrm{GL}_m(\mathbb{A})$ and $1 \le \ell \le m$, we have that this constant term is $\equiv 0$. Hence,

$$\int\limits_{(N_\ell(k)\cap\eta^{-1}P_\ell^{(0)}(k)\eta)\backslash N_\ell(\mathbb{A})} f_s\left(\epsilon_0\eta u\delta h\right)\Psi_\ell^a(u)^{-1}\,du \equiv 0$$

as desired. □

Now, in either case I or Case II we are reduced to $\eta = I_{m'}$. As a consequence of the previous proposition, we have the following corollary.

Corollary 4.9.

$$E^{(N_\ell,\Psi_\ell)}(h,f_s) = \sum_{\delta\in(G(k)\cap M_\ell^{(0)}(k))\backslash G(k)} \int\limits_{(N_\ell(k)\cap P_\ell^{(0)}(k))\backslash N_\ell(\mathbb{A})} f_s\left(\epsilon_0 u\delta h\right)\Psi_\ell(u)^{-1}\,du.$$

Step 8. If we take this last expression for the Fourier coefficient of the Eisenstein series and insert it into our global integral we arrive at the expression

$$\begin{aligned}\mathcal{L}(\varphi,f_s) &= \int\limits_{Z(\mathbb{A})G(k)\backslash G(\mathbb{A})} \varphi(g)E^{(N_\ell,\Psi_\ell)}(g,f_s)\,dg \\ &= \int\limits_{Z(\mathbb{A})G(k)\backslash G(\mathbb{A})} \varphi(g)\left(\sum_{\delta\in(G(k)\cap M_\ell^{(0)}(k))\backslash G(k)} \int\limits_{(N_\ell(k)\cap P_\ell^{(0)}(k))\backslash N_\ell(\mathbb{A})} f_s\left(\epsilon_0 u\delta g\right)\Psi_\ell(u)^{-1}\,du\right)dg.\end{aligned}$$

Since $\delta \in G(k)$ and φ is automorphic, we can bring φ into the sum over δ and replace $\varphi(g)$ by $\varphi(\delta g)$. Then we can absorb the δ sum into the integral over $G(\mathbb{A})$.

Proposition 4.10.

$$\mathcal{L}(\varphi,f_s) = \int\limits_{Z(\mathbb{A})\left(G(k)\cap M_\ell^{(0)}(k)\right)\backslash G(\mathbb{A})} \varphi(g) \int\limits_{(N_\ell(k)\cap P_\ell^{(0)}(k))\backslash N_\ell(\mathbb{A})} f_s\left(\epsilon_0 u g\right)\Psi_\ell(u)^{-1}\,du\,dg.$$

As noted above, if we conjugate the unipotent past ϵ_0 we obtain

$$\epsilon_0\left(N_\ell\cap P_\ell^{(0)}\right)\epsilon_0^{-1} = \left\{\epsilon_0 u\epsilon_0^{-1} = \hat{z}'_\ell = \begin{pmatrix} I_{m-\ell} & x \\ & \zeta\end{pmatrix}^\wedge\right\} = \hat{Z}'_\ell,$$

where for $g \in \mathrm{GL}_t$, we denote by $\hat{g}$ the lift of g into the Levi $\mathrm{GL}_t \times \mathrm{GSpin}_{m'-2t}$ of P_t. If we transfer the character $\Psi_\ell = \Psi_\ell^a$ to Z'_ℓ, we find

$$\Psi_\ell^a\left(\epsilon_0^{-1}\hat{z}'_\ell\epsilon\right) = \psi\left(\zeta_{1,2}+\cdots+\zeta_{\ell-1,\ell}+\frac{a}{2}x_{m-\ell,1}\right)$$

and so on the group Z'_ℓ we set

$$\Psi_{Z'_\ell}\left(z'_\ell\right) = \Psi^a_{Z'_\ell}\left(z'_\ell\right) = \psi\left(\frac{a}{2}z'_{m-\ell,m-\ell+1}+z'_{m-\ell+1,m-\ell+2}+\cdots+z'_{m-1,m}\right).$$

Then if we set

$$f_s^{(Z'_\ell,\Psi_{Z'_\ell})}(h) = \int\limits_{Z'_\ell(k)\backslash Z'_\ell(\mathbb{A})} f_z(\hat{z}'_\ell)\Psi_{Z'_\ell}(z'_\ell)^{-1}\,dz'_\ell = \int\limits_{Z'_\ell(k)\backslash Z'_\ell(\mathbb{A})} f_z(\hat{z}'_\ell)\Psi_\ell(\epsilon_0^{-1}z'_\ell\epsilon_0)^{-1}\,dz'_\ell,$$

then part of the integral that appears in the global integral is the unipotent period

$$\int\limits_{(N_\ell(k)\cap P_\ell^{(0)}(k))\backslash(N_\ell(\mathbb{A})\cap P_\ell^{(0)}(\mathbb{A}))} f_s\left(\epsilon_0 u' u g\right)\Psi_\ell(u')^{-1}\,du' = f_s^{(Z'_\ell,\Psi_{Z'_\ell})}(\epsilon_0 u g)$$

and the integral representation itself becomes

$$\mathcal{L}(\varphi, f_s) = \int\limits_{Z(\mathbb{A})\left(G(k)\cap M_\ell^{(0)}(k)\right)\backslash G(\mathbb{A})} \varphi(g) \int\limits_{(N_\ell(\mathbb{A})\cap P_\ell^{(0)}(\mathbb{A}))\backslash N_\ell(\mathbb{A})} f_s^{(Z'_\ell,\Psi_{Z'_\ell})}(\epsilon_0 u g)\,\Psi_\ell(u)^{-1}\,du\,dg.$$

Next we turn to the Whittaker coefficient. Recall that in Step 1 we commented on convergence issues and steps 2 through 8 only deal with the Eisenstein series and have nothing to do with the dg integration. The purpose in these steps was to unfold and simplify the Eisenstein series and show that certain terms vanish. In step 9 below we show that integral factors through a Whittaker-Fourier coefficient of φ.

Step 9. We now have to analyze $G(k)\cap M_\ell^{(0)}(k)$. Recall that

$$G \cong \begin{cases} \mathrm{GSpin}_{2(m-\ell)-1} & \text{if } m' = 2m, \\ \mathrm{GSpin}^a_{2(m-\ell)} & \text{if } m' = 2m+1. \end{cases}$$

As in [21], we have $G \cap M_\ell^{(0)} \subset P_{G,m-\ell}$, the "Siegel" parabolic subgroup of G. (We placed "Siegel" in quotes since in the case of $a \neq t^2$. we did not define it as such.) If we consider the cases, then we have the following:

Case (i): $m' = 2m$ even. In this case $G \cap M^{(0)} \subset P_{G,m-\ell-1}$, the Siegel parabolic of G. The Levi of the Siegel is then $\mathrm{GL}_{m-\ell-1} \times \mathrm{GL}_1$. We may have $G\cap M^{(0)} = \mathrm{GL}_{m-\ell-1} \ltimes U_{G,m-\ell-1}$ or $G \cap M^{(0)} = P_{G,m-\ell-1}$.

Case (ii): $m' = 2m+1$ and $a \neq t^2$. In this case we have that $G \cap M^{(0)} \subset P_{G,m-\ell-1}$. This parabolic subgroup of G has Levi subgroup given by $\mathrm{GL}_{m-\ell-1} \times \mathrm{GSpin}^a_2$. Again, $G\cap M^{(0)} = \mathrm{GL}_{m-\ell-1} \ltimes U_{G,m-\ell-1}$.

Case (iii): $m' = 2m+1$ and $a = t^2$ a square. We still have $G \cap M^{(0)} \subset P_{G,m-\ell}$. (This case does not appear in [21] since it was not needed there.)

On the other hand, the actual form of the Levi part of $G\cap M^{(0)}$ does not play a role. What one needs is the following. Write $G \cap M^{(0)} = P' = M'U'$. Then U' is the same as in [21] since it is unipotent.

Define

$$C = C_{G,m-\ell} = \{u' \in U' : u' e_m = e_m\}.$$

Lemma 4.11. *We have*

- $\epsilon_0 C_{G,m-\ell}\epsilon_0^{-1} \subset U_{H,m}$, *the unipotent radical of the Siegel parabolic* H.
- $C_{G,m-\ell}$ *normalizes* N_ℓ *and commutes with* $N_\ell \cap P_\ell^{(0)}$.
- $C_{G,m-\ell}$ *preserves* Ψ_ℓ.

Proof. Note that $C_{G,m-\ell}$ has these properties in the SO case, so by the equality of unipotent varieties and lifting of Weyl elements remains true in the GSpin case. □

Therefore, this unipotent subgroup leaves $f_s^{(Z'_\ell,\Psi_{Z'_\ell})}(\epsilon_0 ug)\,\Psi_\ell(u)^{-1}$ invariant. So we can factor it through and take this unipotent period of φ. Note that since this is not the unipotent radical of a parabolic subgroup, there is no reason for this period of φ to vanish. Then

$$\mathcal{L}(\varphi, f_s) = \int\limits_{P'(k)C(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi^{(C,\mathbf{1})}(g) \int\limits_{(N_\ell(\mathbb{A})\cap P_\ell^{(0)}(\mathbb{A}))\backslash N_\ell(\mathbb{A})} f_s^{(Z'_\ell,\Psi_{Z'_\ell})}(\epsilon_0 ug)\,\Psi_\ell(u)^{-1}\,du dg.$$

Step 10. We next need the following Lemma.

Lemma 4.12. $C\backslash P' \cong P^1_{m-\ell} \subset \mathrm{GL}_{m-\ell}$, *the mirabolic subgroup of* $\mathrm{GL}_{m-\ell}$.

Proof. We can see this in a similar way as in [21, p. 288–289]. There are obvious similitude analogs of [21, (10.16) and (10.17)] which give the elements of $P^1_{m-\ell}$. The similitude analogs would only have the similitude character in the term d^* in the notation of [21, (10.16) and (10.17)] (so that the element is indeed in the GSpin group) and otherwise the same formulas apply. We could then give the isomorphism by sending the cosets of C in $C\backslash P'$ to their corresponding elements in $P^1_{m-\ell}$ as in [21, p. 289]. □

Since $\varphi^{(C,\mathbf{1})}(g)$ is left invariant under both $C(\mathbb{A})$, by taking period, and $P'(k)$, since φ is cuspidal, we can use the process of Piatetski-Shapiro and Shalika to Fourier expand along $C\backslash P' \cong P^1_{m-\ell}$. Note that $P^1_{m-\ell} \cong \mathrm{GL}_{m-\ell-1} \ltimes k^{m-\ell-1}$ and the $\mathrm{GL}_{m-\ell-1}$ lies in the Levi of the parabolic subgroup of G with Levi $\mathrm{GL}_{m-\ell-1} \times \mathrm{GL}_1$ if $m' = 2m$ is even and $\mathrm{GL}_{m-\ell-1} \times \mathrm{GSpin}_2^a$ if $m' = 2m+1$ is odd. The $F^{m-\ell-1}$ represents a unipotent subgroup. So this construction should lift from $\mathrm{SO}_{2(m-\ell)-1}$ (resp. $\mathrm{SO}^a_{2(m-\ell)}$) to G. Then, according to [21] we have

$$\varphi^{(C,\mathbf{1})}(g) = \sum_{d \in Z_{m-\ell-1}(k)\backslash \mathrm{GL}_{m-\ell-1}(k)} W_\varphi^\psi\left(\begin{pmatrix} I_\ell & & \\ & d & \\ & & 1\end{pmatrix}^{\wedge} g\right) = \sum_{q \in N_G(k)\backslash P'(k)} W_\varphi^\psi(qg),$$

where

$$W_\varphi^\psi(g) = \int\limits_{N_G(k)\backslash N_G(\mathbb{A})} \varphi(vg)\psi(v)^{-1}\,dv$$

and $\psi = \psi_{N_G}$ is the standard Whittaker character on N_G.

If we replace $W_\varphi^\psi(g)$ with its Fourier expansion and unfold the sum over $N_G(k)\backslash P'(k)$ against the integral over $P'(k)C(\mathbb{A})\backslash G(\mathbb{A})$, we obtain

$$\mathcal{L}(\varphi, f_s) = \int\limits_{N_G(k)C(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi^\psi(g) \int\limits_{N_\ell(\mathbb{A})\cap P_\ell^{(0)}(\mathbb{A})\backslash N_\ell(\mathbb{A})} f_s^{(Z'_\ell,\Psi_{Z'_\ell})}(\epsilon_0 ug)\,\Psi_\ell(u)^{-1}\,du dg.$$

Step 11. We next factor the dg integration through $N_G(k)\backslash N_G(\mathbb{A})$. The Whittaker function will return $\psi_{N_G}(u')$. So we first obtain

$$\mathcal{L}(\varphi, f_s) = \int_{N_G(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi^\psi(g) \int_{N_G(k)C(\mathbb{A})\backslash N_G(\mathbb{A})} \psi_{N_G}(u') \int_{N_\ell(\mathbb{A})\cap P_\ell^{(0)}(\mathbb{A})\backslash N_\ell(\mathbb{A})} f_s^{(Z'_\ell,\Psi_{Z'_\ell})}(\epsilon_0 uu'g)\,\Psi_\ell(u)^{-1}\,dudu'dg.$$

Now $u \in N_\ell(\mathbb{A})$ and $u' \in N_G(\mathbb{A})$. Since $G = L_{\ell,a}$ lies in the Levi subgroup L_ℓ, we know that u' normalizes u and does not change Ψ_ℓ. Therefore, we can interchange the (compact) u and u' integrations and write the argument of f_s as

$$f_s^{(Z'_\ell,\Psi_{Z'_\ell})}(\epsilon_0 u'ug)\,.$$

Following [21], we decompose $\epsilon_0 u'\epsilon_0^{-1} = z'u''$ with $z' \in Z_m$, the maximal unipotent of $\mathrm{GL}_m \in H$ and $u'' \in U_m$, the unipotent radical of the Siegel parabolic $P_m \subset H$.

Lemma 4.13. *If we write $\epsilon_0 u'\epsilon_0^{-1} = z'u''$ as above, then*

- $Z_m = Z'_\ell \cdot \left\{z' : \epsilon_0^{-1}z'\epsilon_0 \in N_G\right\}$
- $\psi_{N_G}\left(\epsilon_0^{-1}z'\epsilon_0\right) = \Psi_m^a|_{\{z':\epsilon_0^{-1}z'\epsilon_0\in N_G\}}$ *for Ψ_m^a some non-degenerate character of Z_m.*

Proof. This is true in SO and, by the agreement of unipotent varieties and lifts of Weyl elements (ϵ_0), it also holds in GSpin. □

Therefore

$$\int_{N_G(k)C(\mathbb{A})\backslash N_G(\mathbb{A})} \psi_{N_G}(u')f_s^{(Z'_\ell,\Psi_{Z'_\ell})}(\epsilon_0 u'ug)\;du' = f_s^{(Z_m,\Psi_m^a)}(\epsilon_0 ug)\,,$$

where Ψ_m^a is a non-degenerate character of Z_m and

$$\mathcal{L}(\varphi, f_s) = \int_{N_G(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi^\psi(g) \int_{N_\ell(\mathbb{A})\cap P_\ell^{(0)}(\mathbb{A})\backslash N_\ell(\mathbb{A})} f_s^{(Z_m,\Psi_m^a)}(\epsilon_0 ug)\,\Psi_\ell(u)^{-1}\,dudg.$$

When $G = \mathrm{GSpin}_{2m}$ it may be occasionally useful, particularly in the local context, to consider an arbitrary Whittaker character ψ of N_G for the Whittaker model of π. One can do this by making Ψ_m^a or β explicit (each determining the other).

Step 12. Choose an element $d_a \in T_m \subset \mathrm{GL}_m$ which conjugates Ψ_m^a to the standard character ψ_m of Z_m, i.e.,

$$\Psi_m^a\left(d_a x d_a^{-1}\right) = \psi_m(z), \quad z \in Z_m.$$

This all takes place in the GL_m Levi of the Siegel parabolic in $\mathrm{SO}_{m'}$ and so the same is true in $\mathrm{GSpin}_{m'} = H$. Then

$$f_s^{(Z_m,\Psi_m^a)}(h) = f_s^{(Z_m,\psi_m)}\left(\hat{d}_a h\right).$$

So if we let

$$\beta = \beta_{\ell,a} = \hat{d}_a\epsilon_0,$$

then

$$\mathcal{L}(\varphi, f_s) = \int_{N_G(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi^\psi(g) \int_{N_\ell(\mathbb{A})\cap P_\ell^{(0)}(\mathbb{A})\backslash N_\ell(\mathbb{A})} f_s^{(Z_m,\psi_m)}(\beta ug)\,\Psi_\ell(u)^{-1}\,dudg.$$

We finally note that

$$N_\ell \cap P^{(0)} = N_\ell \cap P_\ell \cap \epsilon_0^{-1} P_m \epsilon_0 = N_\ell \cap \beta^{-1} P_m \beta$$

since $\beta = \hat{d}_a \epsilon_0$ and $\hat{d}_a \in T_m \subset \mathrm{GL}_m \subset P_m$. Then

$$\mathcal{L}(\varphi, f_s) = \int_{N_G(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi^\psi(g) \int_{N_\ell(\mathbb{A})\cap\beta^{-1}P_m(\mathbb{A})\beta\backslash N_\ell(\mathbb{A})} f_s^{(Z_m,\psi_m)}(\beta ug)\,\Psi_\ell(u)^{-1}\,dudg.$$

This finishes the proof of Theorem 4.2 □

5. Global Integrals II (Case A)

In this section we translate [21, §10.4] into the GSpin context. This corresponds to Method A in the original work of Gelbart and Piatetski-Shapiro [18, Part B], which dealt with the special orthogonal and general linear groups with equal (or nearly equal) ranks. As such, we refer to the integrals of this section as Case A. (See the table in Section 7 for a summary of various cases.) We again note that we follow [23] in the labeling of various parabolic subgroups (cf. Remark 3.2).

We still take $H = \mathrm{GSpin}_{m'}$ to be the larger group. Now, (π, V_π) will be a cuspidal representation of $H(\mathbb{A})$ and H will be either split or quasi-split. The group $G = \mathrm{GSpin}_{n'}$ will be the smaller group, split, and having the opposite parity to H.

We have that H is the GSpin cover of an orthogonal group $\mathrm{SO}_{m'}(V)$ with V a quadratic space. Let $\tilde{m}$ be the Witt index of V. Hence,

$$\tilde{m} = \begin{cases} m & \text{if } m' = 2m+1, \\ m & \text{if } m' = 2m \text{ with } H \text{ split}, \\ m-1 & \text{if } m' = 2m \text{ with } H \text{ quasi-split}. \end{cases}$$

Let $0 \le \ell$ be such that $\ell < \tilde{m}$ if m' is even and $\ell < \tilde{m} - 1$ if m' is odd. Let $Q_\ell = L_\ell \ltimes N_\ell$ be the parabolic subgroup of H with Levi of the form $(\mathrm{GL}_1)^\ell \times \mathrm{GSpin}_{m'-2\ell}$. Let N_ℓ be its unipotent radical. Let Ψ_ℓ be a character of N_ℓ so that $\mathrm{Stab}_{L_\ell}(\Psi_\ell)$ is a split $\mathrm{GSpin}_{m'-2\ell-1}$. If H is split, we can take the character Ψ_ℓ from the previous section.

When H is quasi-split, we find a character Ψ_ℓ of N_ℓ such that $\mathrm{Stab}_{L_\ell}(\Psi_\ell)$ is split and the GSpin cover of the $\mathrm{SO}(W_{m,\ell} \cap w_0^\perp)$ of [21]. This is done in the same as the Ψ_ℓ from the previous section for $a = 1$. In [23] the authors do not consider this case since for descent they can always take the larger group H to be split. Note that if H is quasi-split, so an even GSpin, then $\mathrm{SO}(W_{m,\ell} \cap w_0^\perp)$ is an odd SO and so automatically split. Hence, it does not matter which anisotropic vector we take.

Let $G = \mathrm{Stab}_{L_\ell}(\Psi_\ell)$. Then $G = \mathrm{GSpin}_{m'-2\ell-1}$. We let $n' = m' - 2\ell - 1$. If $m' = 2m+1$ is odd, then $n' = 2n$ with $n = m - \ell$ and if $m' = 2m$ is even, then $n' = 2m - 2\ell - 1 = 2n + 1$ with $n = m - \ell - 1$. In either case G is split.

Again, when $\ell = 0$ there is no Ψ_ℓ and we just "restrict" and the statements below hold in the $\ell = 0$ case as well.

Let $P_G = M_G \ltimes U_G$ be the Siegel parabolic subgroup of G. Then $M_G = \mathrm{GL}_n \times \mathrm{GL}_1$ with the GL_1 factor being the connected component of the center of G. Let τ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ and let

$$\rho = \rho_{\tau,\omega_\pi^{-1},s} = \mathrm{Ind}_{P_G}^{G} \left(\tau|\det|^s \otimes \omega_\pi^{-1}\right) \tag{5.1}$$

so that ρ_s transforms by ω_π^{-1} under the connected component of the center of G. (Again, we will replace s by $s-1/2$ below.) Let f_s be a K-finite holomorphic section of ρ and form the Eisenstein series

$$E(g, f_s) = \sum_{\delta \in P_G(k)\backslash G(k)} f_s(\delta g)$$

which is absolutely convergent, uniformly on compact subsets, for $\mathrm{Re}(s) \gg 0$.

For $\varphi \in V_\pi$, a cusp form on $H(\mathbb{A})$, define

$$\mathcal{L}(\varphi, f_s) = \int\limits_{G(k)Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi^{(N_\ell,\Psi_\ell)}(g) E(g, f_s)\, dg,$$

where $Z = Z(G)^0$ is the identity component of the center of G as in the earlier case.

Similarly to Lemma 4.1, the integral $\mathcal{L}(\varphi, f_s)$ converges absolutely and uniformly in vertical strips in $\mathbb{C}$ away from poles of the Eisenstein series and hence it defines a meromorphic function on $\mathbb{C}$.

Theorem 5.1. *Let the notaion be as above.*

(i) *If $\mathcal{L}(\varphi, f_s)$ is not identically zero, then π is globally generic with respect to a certain "standard" Whittaker character ψ. (For a precise description of the character ψ of N_G is all cases we refer to* [34] *and* [24]*.)*

(ii) *For* $\mathrm{Re} s \gg 0$ *we have an identity*

$$\mathcal{L}(\varphi, f_s) = \int\limits_{N_G(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} \int\limits_{\mathcal{X}(\mathbb{A})} W_\varphi^\psi(\lambda\delta_\ell g) f_s^{(Z_n,\psi_n)}(g)\, d\lambda\, dg,$$

where W_φ^ψ is the appropriate Whittaker function of φ, $\mathcal{X}$ is isomorphic to a unipotent subgroup of GL_m *if $m' = 2m+1$, resp.* GL_{m-1} *if $m' = 2m$, and is of the form*

$$\mathcal{X} = \left\{ \begin{pmatrix} I_n & \\ \lambda & I_\ell \end{pmatrix}^\wedge \right\}, \tag{5.2}$$

the element

$$\delta_\ell = \begin{pmatrix} 0 & I_n \\ I_\ell & 0 \end{pmatrix}^\wedge \tag{5.3}$$

is a Weyl group element of GL_m*, resp.* GL_{m-1}*, as above, and $f_s^{(Z_n,\Psi_n)}$ is the same unipotent period that appears in Theorem 4.2 (so Z_n is the maximal unipotent subgroup of* GL_n*.)*

Proof. For $\mathrm{Re}(s) \gg 0$ the Eisenstein series $E(g, f_s)$ is an absolutely convergent series. We replace it by its definition:

$$\mathcal{L}(\varphi, f_s) = \int\limits_{G(k)Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi^{(N_\ell,\Psi_\ell)}(g) \sum_{\delta \in P_G(k)\backslash G(k)} f_s(\delta g)\, dg.$$

Since φ is automorphic and $G = \mathrm{Stab}(\Psi_\ell)$, we can move the summation outside $\varphi^{(N_\ell,\Psi_\ell)}$

$$\mathcal{L}(\varphi, f_s) = \int\limits_{G(k)Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{\delta\in P_G(k)\backslash G(k)} \varphi^{(N_\ell,\Psi_\ell)}(\delta g) f_s(\delta g)\, dg$$

and collapse the sum against the integral to obtain

$$\mathcal{L}(\varphi, f_s) = \int\limits_{P_G(k)Z(\mathbb{A})\backslash G(\mathbb{A})} \left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,\mathbf{1})}(g) f_s(g)\, dg.$$

We now come to a crucial result.

Proposition 5.2. *We have*

$$\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,\mathbf{1})}(g) = \sum_{\gamma\in Z_n(k)\backslash \mathrm{GL}_n(k)} \int\limits_{\mathcal{X}(\mathbb{A})} W_\varphi^\psi\left(\hat{\gamma}\lambda\delta_\ell g\right)\, d\lambda.$$

Proof. We will prove this Proposition in Section 6 after we review the process of "root exchange". The analogous result for the special orthogonal groups is [21, Theorem 7.3]. □

With this, we have

$$\mathcal{L}(\varphi, f_s) = \int\limits_{M_G(k)U_G(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{\gamma\in Z_n(k)\backslash \mathrm{GL}_n(k)} \int\limits_{\mathcal{X}(\mathbb{A})} W_\varphi^\psi\left(\hat{\gamma}\lambda\delta_\ell g\right) f_s(g)\, d\lambda\, dg,$$

where U_G is the unipotent radical of the Siegel parabolic P_G as in above (5.1). (In particular, we have $Z_n U_G = N_G$, the maximal unipotent subgroup of G.)

Now, for $\gamma \in Z_n(k)\backslash \mathrm{GL}_n(k)$, we have that $\hat{\gamma}$ normalizes the group $\mathcal{X}$. This is true whether we are in the SO context or the GSpin context because it is taking place in the GL_m or GL_{m-1} Levi subgroups.

We claim that $\delta_\ell^{-1}\hat{\gamma}\delta_\ell$ is a general element of $M_G(k)/Z(k) = \mathrm{GL}_n(k)$. Again this is true because it takes palce in the GL_m, resp. GL_{m-1}, Levi subgroup.

So we now move $\hat{\gamma}$ to the right and then collapse the sum over $Z_n(k)\backslash \mathrm{GL}_n(k)$ with the integral to obtain

$$\mathcal{L}(\varphi, f_s) = \int\limits_{Z_n(k)U_G(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} \int\limits_{\mathcal{X}(\mathbb{A})} W_\varphi^\psi\left(\lambda\delta_\ell g\right) f_s(g)\, d\lambda\, dg.$$

If we now integrate over $Z_n(k)\backslash Z_n(\mathbb{A})$, then $W_\varphi^\psi\left(\lambda\delta_\ell g\right)$ will produce a character ψ_n of Z_n. We then integrate this unipotent period for f_s and obtain

$$\mathcal{L}(\varphi, f_s) = \int\limits_{Z_n(k)U_G(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} \int\limits_{\mathcal{X}(\mathbb{A})} W_\varphi^\psi\left(\lambda\delta_\ell g\right) f_s^{(Z_n,\psi_n)}(g)\, d\lambda\, dg.$$

Finally, it follows from $Z_n U_G = N_G$ that

$$\mathcal{L}(\varphi, f_s) = \int\limits_{N_G(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} \int\limits_{\mathcal{X}(\mathbb{A})} W_\varphi^\psi\left(\lambda\delta_\ell g\right) f_s^{(Z_n,\psi_n)}(g)\, d\lambda\, dg.$$

This finishes the proof of Theorem 5.1. □

6. Constant Terms

We now translate [21, Section 7] into the GSpin context.

6.1. **The Process of Root Exchanges.** We need the process of "root exchanges" of [21]. A version of this occurs in [23] as well, but the Euler product expansion of Theorem 5.1 requires the more elaborate version of [21]. Let us work in the following context.

Let H be a connected reductive algebraic k-group, such as one of our GSpin groups. For our purposes we can have it be quasi-split. Let $U < G$ be a maximal unipotent k-subgroup. Suppose $C < U$ is a k-subgroup of U, and $\psi = \psi_C$ is a non-trivial character of $C(k)\backslash C(\mathbb{A})$. Suppose we have two other k-subgroups X and Y of U such that the following six axioms hold:

(1) X and Y normalize C.

(2) $X \cap C$ is normal in X and $(X \cap C)\backslash X$ is abelian, and similarly $Y \cap C$ is normal in Y and $(Y \cap C)\backslash Y$ is abelian.

(3) When $X(\mathbb{A})$ and $Y(\mathbb{A})$ act on $C(\mathbb{A})$ by conjugation, they preserve ψ_C.

(4) ψ_C is trivial on $(X \cap C)(\mathbb{A})$ and $(Y \cap C)(\mathbb{A})$.

(5) The commutator $(X, Y) \subset C$. (Recall that $(x, y) = x^{-1}y^{-1}xy$ and (X, Y) denotes the subgroup generated by all the commutators.)

Note that these five conditions imply that, for a fixed $y \in Y(\mathbb{A})$, the map $x \mapsto \psi_C((x, y))$ defines a character of $X(\mathbb{A})$, trivial on $(X \cap C)(\mathbb{A})$ and similarly, for a fixed $x \in X(\mathbb{A})$, the map $y \mapsto \psi_C((x, y))$ defines a character of $Y(\mathbb{A})$ which is trivial on $(Y \cap C)(\mathbb{A})$. (This is checked in [21, §7.1].) Finally

(6) The pairing of $(X \cap C)(\mathbb{A})\backslash X(\mathbb{A}) \times (Y \cap C)(\mathbb{A})\backslash Y(\mathbb{A})$ given by $(x, y) \mapsto \psi_C((x, y))$ is bilinear and non-degenerate and identifies

$$(Y \cap C)(k)\backslash Y(k) \simeq [(X(k)(X \cap C)(\mathbb{A}))\backslash X(\mathbb{A})]^\wedge$$

and

$$(X \cap C)(k)\backslash X(k) \simeq [(Y(k)(Y \cap C)(\mathbb{A}))\backslash Y(\mathbb{A})]^\wedge .$$

Now let $B = CY = YC$, $D = CX = XC$ and $A = CXY$. Extend ψ_C to a character ψ_B of $B(\mathbb{A})$ trivial on $B(k)$ by making it trivial on $Y(\mathbb{A})$ and to a character ψ_D of $D(\mathbb{A})$ trivial on $D(k)$ by making it trivial on $X(\mathbb{A})$.

Proposition 6.1 (Root exchange). *Let f be an automorphic function on $H(\mathbb{A})$ which is smooth and of uniform moderate growth. Then*

$$\int_{B(k)\backslash B(\mathbb{A})} f(v)\psi_B(v)^{-1}\, dv = \int_{(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})} \int_{D(k)\backslash D(\mathbb{A})} f(uy)\psi_D(u)^{-1}\, du dy,$$

where the right hand side converges in the sense that

$$\int_{(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})} \left| \int_{D(k)\backslash D(\mathbb{A})} f(uyh)\psi_D(u)^{-1}\, du \right| dy < \infty$$

uniformly in any compact subset of $H(\mathbb{A})$.

Proof. The proof of this proposition in [21] depends only on the six axioms above and the properties of smooth automorphic functions of uniform moderate growth. As G is connected reductive, the results of [29], as used by [21], hold. The proposition follows as in [21]. □

The proof of the proposition has the following corollary that is used in the proof of Proposition 5.2.

Corollary 6.2. *Let f be a smooth automorphic function of uniform moderate growth on $H(\mathbb{A})$. Then there exist smooth, uniform moderate growth, automorphic functions $f_1, \ldots, f_r$ and Schwartz functions $\phi_1, \ldots, \phi_r \in \mathcal{S}((Y \cap C)(\mathbb{A}))\backslash Y(\mathbb{A})$ such that for all $y \in (Y \cap C)(\mathbb{A})\backslash Y(\mathbb{A})$ we have*

$$\int_{D(k)\backslash D(\mathbb{A})} f(uy)\psi_D^{-1}(u)\ du = \sum_{i=1}^{r} \phi_i(y) \int_{D(k)\backslash D(\mathbb{A})} f_i(uy)\psi_D^{-1}(u)\ du.$$

Proof. The proof in [21] uses the theory of smooth automorphic functions of uniform moderate growth from [29] as well as the theorem of Dixmier and Malliavin from [17], both of which hold in our context. The result follows as in [21]. □

6.2. **Proof of Proposition 5.2.** We keep the notation from Section 5. We will use repeatedly that a cusp form $\varphi \in V_\pi$ on $H(\mathbb{A})$ is smooth and of uniform moderate growth, so that we may use Section 6.1. The proof will be an induction, and for the induction we will need the following subgroups.

Recall that if $m' = 2m + 1$ is odd, then $n + \ell = m - 1$ (by assumption) and $\mathrm{GL}_{n+\ell} \subset H$ is the Levi subgroup of the parabolic subgroup of H with Levi subgroup $\mathrm{GL}_{n-\ell} \times H_3$ where H_3 is the split GSpin_3. If $m' = 2m$ is even then $n + \ell = m - 1$ and GL_m is part of the Levi of the parabolic subgroup with Levi $\mathrm{GL}_{n+\ell} \times H_2$, where H_2 denotes the (split or quasi-split) GSpin(2). In either case, for $\gamma \in \mathrm{GL}_{n+\ell}$ we let $\gamma^\wedge$ denote the lift of γ to an element of H. Since there is a corresponding parabolic subgroup of $\mathrm{SO}_{m'}$ we can follow [21] and write

$$\gamma^\wedge = \begin{pmatrix} \gamma & & \\ & I_{m'-2(n+\ell)} & \\ & & \gamma^* \end{pmatrix} \in \mathrm{SO}_{m'}$$

which we can identify with its lift to $H_{m'}$ if γ is unipotent or a Weyl group element. Note that $m' - 2(n + \ell)$ is either equal to 3 or 2.

We have defined

$$\mathcal{X} = \left\{ \begin{pmatrix} I_n & \\ \lambda & I_\ell \end{pmatrix}^\wedge \right\}$$

and we set

$$\mathcal{X}^{(i)} = \left\{ \begin{pmatrix} I_n & \\ \lambda & I_\ell \end{pmatrix}^\wedge \in \mathcal{X} \,:\, \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_\ell \end{pmatrix} \text{ with } \lambda_j = 0 \text{ for } j \neq \ell - i \right\}$$

and

$$\mathcal{X}_i = \left\{ \begin{pmatrix} I_n & \\ \lambda & I_\ell \end{pmatrix}^\wedge \in \mathcal{X} \,:\, \lambda_{\ell-i} = \cdots = \lambda_\ell = 0 \right\}$$

Note that for each i, $\mathcal{X}_{i-1} = \mathcal{X}_i \mathcal{X}^{(i)}$ where we set $\mathcal{X}_{-1} = \mathcal{X}$. In what follows we will also need the groups

$$U^i_{n+\ell} = \left\{ \begin{pmatrix} I_{n+\ell-i} & * \\ & z \end{pmatrix}^{\wedge} \,:\, z \in Z_i \right\} \cdot U_{n+\ell}. \subset N_{n+\ell}$$

By definition we have

$$\varphi^{(N_\ell,\Psi_\ell)}(g) = \int\limits_{N_\ell(k)\backslash N_\ell(\mathbb{A})} \varphi(vg)\Psi_\ell^{-1}(v)\ dv$$

and then

$$(\varphi^{(N_\ell,\Psi_\ell)})^{(U_G,1)}(g) = \int\limits_{U_G(k)\backslash U_G(\mathbb{A})} \left(\int\limits_{N_\ell(k)\backslash N_\ell(\mathbb{A})} \varphi(vrg)\Psi_\ell^{-1}(v)\ dv \right)\ dr.$$

As φ is automorphic on H, it is left invariant by $\delta_\ell \in H(k)$. Thus

$$\varphi(vrg) = \varphi(\delta_\ell vrg) = \varphi(\delta_\ell vr\delta_\ell^{-1}\delta_\ell g).$$

The authors of [21] analyze the conjugates $\delta_\ell N_\ell \delta_\ell^{-1}$ and $\delta_\ell U_G \delta_\ell^{-1}$ in terms of matrices in $\mathrm{SO}_{m'}$. To this end, let

$$s(z;u,a,d,e;x,y) = \begin{pmatrix} I_n & & x & d & y \\ u & z & a & e & d' \\ & & I_{m'-2(n+\ell)} & a' & x' \\ & & & z^* & \\ & & & u' & I_n \end{pmatrix}$$

where $z \in Z_\ell$, the maximal unipotent subgroup of GL_ℓ. With this notation,

$$\begin{aligned} \delta_\ell U_G \delta_\ell^{-1} &= \{ s(I_\ell;0,0,0,0,;x^\circ,y) \in H \,:\, \delta_\ell^{-1} s(I_\ell;0,0,0,0;x^\circ,y)\delta_\ell \in U_G \} \\ &= \left\{ \begin{pmatrix} I_n & & x^\circ & 0 & y \\ 0 & I_\ell & 0 & 0 & 0 \\ & & I_{m'-2(n+\ell)} & 0 & x^{\circ\prime} \\ & & & I_\ell & \\ & & & 0 & I_n \end{pmatrix} \right\} \end{aligned}$$

where there is a condition on x° to make sure this comes from an element of U_G, and

$$\begin{aligned} \delta_\ell N_\ell \delta_\ell^{-1} &= \{ s(z;u,a,d,e;0,0) \,:\, z \in Z_\ell \} \\ &= \left\{ \begin{pmatrix} I_n & & 0 & d & 0 \\ u & z & a & e & d' \\ & & I_{m'-2(n+\ell)} & a' & 0 \\ & & & z^* & \\ & & & u' & I_n \end{pmatrix} \right\}. \end{aligned}$$

Let

$$S = \delta_\ell U_G N_\ell \delta_\ell^{-1}$$

and let the character Ψ_S be the character of S obtained as follows: we extend the character Ψ_ℓ of $N_\ell(k)\backslash N_\ell(\mathbb{A})$ to $U_G N_\ell$ by making it trivial on $U_G(\mathbb{A})$ and then set $\Psi_S(s) = \Psi_\ell(\delta_\ell^{-1} s \delta_\ell)$.

Let

$$\tilde{S} = \{s \in S : u = 0, z = I_\ell\}$$
$$= \left\{ \begin{pmatrix} I_n & & x^\circ & d & y \\ 0 & I_\ell & a & e & d' \\ & & I_{m'-2(n+\ell)} & a' & x^{\circ\prime} \\ & & & I_\ell & \\ & & & 0 & I_n \end{pmatrix} \right\}.$$

Note that $\mathcal{X} \subset S$ consists of the lower triangular elements of S, that is, $\mathcal{X} = \{s(0; u, 0, 0, 0; 0, 0)\}$.

Lemma 6.3. *Ψ_S is trivial on $\mathcal{X}$.*

Proof. By definition $\Psi_S(s) = \Psi_\ell(\delta_\ell^{-1} s \delta_\ell)$. We have $\delta_\ell = \begin{pmatrix} 0 & I_n \\ I_\ell & 0 \end{pmatrix}^\wedge$ and $\mathcal{X} = \left\{ \begin{pmatrix} I_n & 0 \\ \lambda & I_\ell \end{pmatrix}^\wedge \right\}$. Letting $\tilde{\lambda} = \begin{pmatrix} I_n & 0 \\ \lambda & I_\ell \end{pmatrix}$ we find that $\Psi_S(\tilde{\lambda}) = \Psi_\ell(\delta_\ell^{-1} \tilde{\lambda} \delta_\ell)$. However

$$\delta_\ell^{-1} \tilde{\lambda} \delta_\ell = \left(\begin{pmatrix} 0 & I_\ell \\ I_n & 0 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ \lambda & I_\ell \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_\ell & 0 \end{pmatrix} \right)^\wedge = \begin{pmatrix} I_\ell & \lambda \\ 0 & I_n \end{pmatrix}^\wedge$$

and

$$\Psi_\ell \left(\begin{pmatrix} I_\ell & \lambda \\ 0 & I_n \end{pmatrix}^\wedge \right) = \begin{cases} \psi_0(\lambda_{\ell,m} - \lambda_{\ell,m+1}) & \text{if } m' = 2m, \\ \psi_0(\lambda_{\ell,m} + \frac{a}{2}\lambda_{\ell,m+2}) & \text{if } m' = 2m+1. \end{cases}$$

by the definitions in Section 4. If $m' = 2m+1$, then $n + \ell = m - 1$ and so $\lambda_{\ell,m} = \lambda_{\ell,m+2} = 0$. If $m' = 2m$, then $n + \ell = m - 1$ and again $\lambda_{\ell,m} = \lambda_{\ell,m+1} = 0$. (For these conditions on the relation between n, ℓ, m see the second paragraph of Section 6.2.) Hence $\Psi_S(\tilde{\lambda}) = 1$. □

With this notation we can write

$$\begin{aligned}
(\varphi^{(N_\ell, \Psi_\ell)})^{(U_G,1)}(g) &= \int\limits_{U_G(k)\backslash U_G(\mathbb{A})} \left[\int\limits_{N_\ell(k)\backslash N_\ell(\mathbb{A})} \varphi(vrg) \Psi_\ell^{-1}(v)\, dv \right] dr \\
&= \int\limits_{S(k)\backslash S(\mathbb{A})} \varphi(s\delta_\ell g) \Psi_S^{-1}(s)\, ds \\
&= \int\limits_{Z_\ell(k)\backslash Z_\ell(\mathbb{A})} \int\limits_{\mathcal{X}(k)\backslash \mathcal{X}(\mathbb{A})} \int\limits_{\tilde{S}(k)\backslash \tilde{S}(\mathbb{A})} \varphi(z\lambda\tilde{s}\delta_\ell g) \Psi_S^{-1}(\tilde{s}) \Psi_{n+\ell}^{-1}(z)\, d\tilde{s}\, d\lambda\, dz.
\end{aligned} \tag{6.1}$$

Since $\mathcal{X} = \mathcal{X}_0 \mathcal{X}^{(0)}$ we can decompose the integral over $\mathcal{X}$ as

$$\int\limits_{\mathcal{X}(k)\backslash \mathcal{X}(\mathbb{A})} = \int\limits_{\mathcal{X}_0(k)\backslash \mathcal{X}_0(\mathbb{A})} \int\limits_{\mathcal{X}^{(0)}(k)\backslash \mathcal{X}^{(0)}(\mathbb{A})} .$$

Then we obtain an inner integral of the form

$$\int\limits_{\mathcal{X}^{(0)}(k)\backslash \mathcal{X}^{(0)}(\mathbb{A})} \int\limits_{\tilde{S}(k)\backslash \tilde{S}(\mathbb{A})} \varphi(z\lambda_0\lambda^{(0)}\tilde{s}\delta_\ell g) \Psi_S^{-1}(\tilde{s}) \Psi_{n+\ell}^{-1}(z)\, d\tilde{s} d\lambda^{(0)}$$

or

$$\int_{\mathcal{X}^{(0)}(k)\backslash\mathcal{X}^{(0)}(\mathbb{A})}\int_{\tilde{S}(k)\backslash\tilde{S}(\mathbb{A})}\left[\varphi(z\lambda_0\lambda^{(0)}\tilde{s}\delta_\ell g)\Psi_{n+\ell}^{-1}(z)\right]\Psi_S^{-1}(\tilde{s})\ d\tilde{s}d\lambda^{(0)}.$$

We would now like to perform a root exchange on this integral. To this end, let $J = \{s(I_\ell;0,0,0,0;x,y) \in H\} \supset \delta_\ell U_G \delta_\ell^{-1}$ and $J_0 = J \cap S = \delta_\ell U_G \delta_\ell^{-1}$. Now in the setting of §6.1, set $C = \tilde{S}$, $Y = \mathcal{X}^{(0)}$, and $X = J$ so that $B = \tilde{S}\mathcal{X}^{(0)}$, $D = \tilde{S}J = U_{n+\ell}$, and $A = \tilde{S}\mathcal{X}^{(0)}J$. The authors of [21] verify that these satisfy (1) – (6) of §6.1. Applying Proposition 6.1 to this integral then gives

$$\begin{aligned}
&\int_{\mathcal{X}^{(0)}(k)\backslash\mathcal{X}^{(0)}(\mathbb{A})}\int_{\tilde{S}(k)\backslash\tilde{S}(\mathbb{A})}\left[\varphi(z\lambda_0\lambda^{(0)}\tilde{s}\delta_\ell g)\Psi_{n+\ell}^{-1}(z)\right]\Psi_S^{-1}(\tilde{s})\ d\tilde{s}d\lambda^{(0)}\\
&\qquad=\int_{\mathcal{X}^{(0)}(\mathbb{A})}\int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})}\left[\varphi(uz\lambda_0\lambda^{(0)}\delta_\ell g)\Psi_{n+\ell}^{-1}(z)\right]\Psi_{n+\ell}^{-1}(u)\ dud\lambda^{(0)}\\
&\qquad=\int_{\mathcal{X}^{(0)}(\mathbb{A})}\int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})}\varphi(uz\lambda_0\lambda^{(0)}\delta_\ell g)\Psi_{n+\ell}^{-1}(zu)\ dud\lambda^{(0)}
\end{aligned}$$

with convergence of

$$\int_{\mathcal{X}^{(0)}(\mathbb{A})}\left|\int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})}\varphi(uz\lambda_0\lambda^{(0)}\delta_\ell g)\Psi_{n+\ell}^{-1}(zu)\ du\right|d\lambda^{(0)}$$

uniformly on compact subsets.

Moreover, by Corollary 6.2 there are smooth automorphic functions $\varphi_1,\dots,\varphi_r$ of uniform moderate growth on $H(\mathbb{A})$ and Schwartz functions $\phi_1,\dots,\phi_r$ on $\mathcal{X}^{(0)}(\mathbb{A})$ such that

$$\begin{aligned}
&\int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})}\varphi(uz\lambda_0\lambda^{(0)}\delta_\ell g)\Psi_{n+\ell}^{-1}(zu)\ du\\
&\qquad=\sum_{i=1}^{r}\phi_i(z\lambda_0\lambda^{(0)}\delta_\ell g)\int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})}\varphi_i(uz\lambda_0\lambda^{(0)}\delta_\ell g)\Psi_{n+\ell}^{-1}(zu)\ du.
\end{aligned}$$

If we insert this in the result of our root exchange we have

$$\begin{aligned}
&\int_{\mathcal{X}^{(0)}(\mathbb{A})}\int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})}\varphi(uz\lambda_0\lambda^{(0)}\delta_\ell g)\Psi_{n+\ell}^{-1}(zu)\ dud\lambda^{(0)}\\
&\quad=\int_{\mathcal{X}^{(0)}(\mathbb{A})}\left[\sum_{i=1}^{r}\phi_i(z\lambda_0\lambda^{(0)}\delta_\ell g)\int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})}\varphi_i(uz\lambda_0\lambda^{(0)}\delta_\ell g)\Psi_{n+\ell}^{-1}(zu)\ du\right]d\lambda^{(0)}\\
&\quad=\int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})}\left[\sum_{i=1}^{r}\int_{\mathcal{X}^{(0)}(\mathbb{A})}\phi_i(z\lambda_0\lambda^{(0)}\delta_\ell g)\varphi_i(uz\lambda_0\lambda^{(0)}\delta_\ell g)\Psi_{n+\ell}^{-1}(zu)\ d\lambda^{(0)}\right]du
\end{aligned}$$

so if we set

$$\varphi_0(u\lambda_0 z\delta_\ell g) = \sum_{i=1}^{r} \int_{\mathcal{X}^{(0)}(\mathbb{A})} \phi_i(z\lambda_0\lambda^{(0)}\delta_\ell g)\varphi_i(uz\lambda^{(0)}\lambda_0 z\delta_\ell g)\ d\lambda^{(0)},$$

then we have

$$\int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})} \varphi_0(uz\lambda_0\delta_\ell g)\Psi_{n+\ell}^{-1}(zu)\ du.$$

If we return now to (6.1), noting the decomposition of integration right after (6.1), and insert this we have

$$\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \int_{Z_\ell(k)\backslash Z_\ell(\mathbb{A})} \int_{\mathcal{X}_0(k)\backslash \mathcal{X}_0(\mathbb{A})} \int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})} \varphi_0(uz\lambda_0\delta_\ell g)\Psi_{n+\ell}^{-1}(zu)\, du\, d\lambda_0\, dz.$$

We next use the decompositions $\mathcal{X}_0 = \mathcal{X}_1\mathcal{X}^{(1)}$ and $Z_\ell = Z_{\ell-1}Z_\ell^{(1)}$, where

$$Z_\ell^{(1)} = \left\{ \begin{pmatrix} I_{\ell-1} & * \\ 0 & 1 \end{pmatrix} \right\} \subset Z_\ell.$$

To this end, we write $\lambda_0 = \lambda_1\lambda^{(1)}$ and, with abuse of notation, $z = z_\ell = z_{\ell-1}z^{(1)} = zz^{(1)}$ and decompose our integrals as

$$\int_{\mathcal{X}_0(k)\backslash \mathcal{X}_0(\mathbb{A})} = \int_{\mathcal{X}_1(k)\backslash \mathcal{X}_1(\mathbb{A})} \int_{\mathcal{X}^{(1)}(k)\backslash \mathcal{X}^{(1)}(\mathbb{A})}$$

and

$$\int_{Z_\ell(k)\backslash Z_\ell(\mathbb{A})} = \int_{Z_{\ell-1}(k)\backslash Z_{\ell-1}(\mathbb{A})} \int_{Z_\ell^{(1)}(k)\backslash Z_\ell^{(1)}(\mathbb{A})} .$$

For $h \in H(\mathbb{A})$ set

$$I_1(\varphi_0, \Psi_{n+\ell})(h) = \int_{\mathcal{X}^{(1)}(k)\backslash \mathcal{X}^{(1)}(\mathbb{A})} \int_{Z_\ell^{(1)}(k)\backslash Z_\ell^{(1)}(\mathbb{A})} \int_{U_{n+\ell}(k)\backslash U_{n+\ell}(\mathbb{A})} \varphi_0(uz^{(1)}\lambda^{(1)}h)\Psi_{n+\ell}^{-1}(uz^{(1)})\, du\, dz^{(1)}\, d\lambda^{(1)}$$

so that

$$\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \int_{Z_{\ell-1}(k)\backslash Z_{\ell-1}(\mathbb{A})} \int_{\mathcal{X}_1(k)\backslash \mathcal{X}_1(\mathbb{A})} I_1(\varphi_0, \Psi_{n+\ell})(z\lambda_1\delta_\ell g)\Psi_{n+\ell}^{-1}(z)\, d\lambda_1\, dz.$$

At this point, we apply another root exchange (or swap in the languate of [23]) in I_1. In the language of Section 6.1 we set

$$C = U_{n+\ell}Z^{(1)} \qquad Y = \mathcal{X}^{(1)} \qquad X = \left\{ \begin{pmatrix} I_n & & x \\ & I_{\ell-1} & \\ & & 1 \end{pmatrix}^{\wedge} \right\}$$

$$\psi_C = \Psi_{n+\ell}|_C \qquad B = CY \qquad D = CX = U_{n+\ell}^1.$$

Applying this to $I_1(\varphi_0, \Psi_{n+\ell})$ gives

$$I_1(\varphi_0, \Psi_{n+\ell})(h) = \int\limits_{\mathcal{X}^{(1)}(\mathbb{A})} \int\limits_{U^1_{n+\ell}(k)\backslash U^1_{n+\ell}(\mathbb{A})} \varphi_0(u^1\lambda^{(1)}h)\Psi^{-1}_{n+\ell}(u^1)\ du^1\ d\lambda^{(1)}.$$

We insert this into the above formula for $(\varphi^{(N_\ell,\Psi_\ell)})^{(U_G,1)}(g)$ and interchange the order of integration to obtain

$$\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \int\limits_{\mathcal{X}^{(1)}(\mathbb{A})} \int\limits_{Z_{\ell-1}(k)\backslash Z_{\ell-1}(\mathbb{A})} \int\limits_{\mathcal{X}_1(k)\backslash \mathcal{X}_1(\mathbb{A})} \int\limits_{U^1_{n+\ell}(k)\backslash U^1_{n+\ell}(\mathbb{A})} \varphi_0(u^1\lambda_1 z\lambda^{(1)})\Psi^{-1}_{n+\ell}(u^1z)\, du^1\, d\lambda_1\, dz\, d\lambda^{(1)}.$$

We next apply Corollary 6.2 (essentially Dixmier–Malliavin [17]) to write

$$\int\limits_{U^1_{n+\ell}(k)\backslash U^1_{n+\ell}(\mathbb{A})} \varphi_0\left(u^1(\lambda_1 z\lambda^{(1)}\delta_\ell g)\right)\Psi^{-1}(u^1z)\, du$$

$$= \sum_{I=1}^{r} \phi_{0,i}(\lambda_1 z\lambda^{(1)}\delta_\ell g) \int\limits_{U^1_{n+\ell}(k)\backslash U^1_{n+\ell}(\mathbb{A})} \varphi_{0,i}\left(u^1\lambda_1 z\lambda^{(1)}\delta_\ell g\right)\Psi^{-1}_{n+\ell}(u^1z)\, du^1.$$

Note that the $\varphi_{0,i}$ are smooth and of uniform moderate growth (in fact in $W(\varphi)$) and the $\phi_{0,i}$ are Schwartz functions. If we let

$$\varphi_1\left(u^1\lambda_1 zh\right) = \sum_{I=1}^{r} \int\limits_{\mathcal{X}^{(1)}(\mathbb{A})} \varphi_{0,i}\left(u^1\lambda_1 z\lambda^{(1)}h\right)\phi_{0,i}\left(\lambda_1 z\lambda^{(1)}h\right)\, d\lambda^{(1)},$$

then by Corollary 6.2 we have

$$\int\limits_{\mathcal{X}^{(1)}(\mathbb{A})} \int\limits_{U^1_{n+\ell}(k)\backslash U^1_{n+\ell}(\mathbb{A})} \varphi_0\left(u^1(\lambda_1 z\lambda^{(1)}\delta_\ell g)\right)\Psi^{-1}(u^1z)\, du\, d\lambda^{(1)}$$

$$= \int\limits_{U^1_{n+\ell}(k)\backslash U^1_{n+\ell}(\mathbb{A})} \varphi_1(u^1(\lambda_1 z\delta_\ell g))\Psi^{-1}_{n+\ell}(u^1z)\ du^1$$

and so

$$(\varphi^{(N_\ell,\Psi_\ell)})^{(U_G,1)}(g) = \int\limits_{Z_{\ell-1}(k)\backslash Z_{\ell-1}(\mathbb{A})} \int\limits_{\mathcal{X}_1(k)\backslash \mathcal{X}_1(\mathbb{A})} \int\limits_{U^1_{n+\ell}(k)\backslash U^1_{n+\ell}(\mathbb{A})} \varphi_1(u^1\lambda_1 z\delta_\ell g)\Psi^{-1}_{n+\ell}(u^1z)\ du^1\ d\lambda_1\ dz.$$

Remark 6.2. In [21] they keep this integral, as well as the original integral involving φ. Then this would be

$$(\varphi^{(N_\ell,\Psi_\ell)})^{(U_G,1)}(g) = \int\limits_{\mathcal{X}^{(1)}(\mathbb{A})} \int\limits_{\mathcal{X}^{(0)}(\mathbb{A})} \int\limits_{Z_{\ell-1}(k)\backslash Z_{\ell-1}(\mathbb{A})} \int\limits_{\mathcal{X}_1(k)\backslash \mathcal{X}_1(\mathbb{A})} \int\limits_{U^1_{n+\ell}(k)\backslash U^1_{n+\ell}(\mathbb{A})} \varphi(u^1\lambda_1 z\lambda^{(0)}\lambda^{(1)}\delta_\ell g)\Psi^{-1}(u^1z)\, du^1\, d\lambda_1\, dz\, d\lambda^{(0)}\, d\lambda^{(1)}.$$

They perform $\varphi \mapsto \varphi_0 \mapsto \varphi_1$ to obtain convergence and they want the original integral with φ and the integrations over the $\mathcal{X}^{(i)}$ to obtain an Euler product for decomposable data (i.e., φ decomposable) and need the full adelic integrals over the $\mathcal{X}^{(i)}$.

We next do an induction to either remove the various $\mathcal{X}_i$ integrations or replace them with full adelic integrations over the $\mathcal{X}^{(i)}$.

6.2.1. *The Induction.* For $1 \le i < \ell$ set

$$Z_{\ell-i} = \left\{ z_{\ell-i} = \begin{pmatrix} z & \\ & I_i \end{pmatrix} \in Z_\ell \right\} \subset H.$$

Assume, by induction, that for $1 \le i \le \ell - 2$ we have:

$$(i) \qquad \left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \int\limits_{\mathcal{X}^{(i,\dots,1,0)}(\mathbb{A})} \int\limits_{Z_{\ell-i}(k)\backslash Z_{\ell-i}(\mathbb{A})} \int\limits_{\mathcal{X}_i(k)\backslash\mathcal{X}_i(\mathbb{A})} \int\limits_{U^i_{n+\ell}(k)\backslash U^i_{n+\ell}(\mathbb{A})} \varphi\left(u^i \lambda_i z_{\ell-i} \lambda^{(\prime)} \delta_\ell g\right) \Psi^{-1}(u^i z_{\ell-i})\, du^i\, d\lambda_i\, dz_{\ell-i}\, d\lambda^{(\prime)},$$

where $\lambda^{(\prime)} \in \mathcal{X}^{(i,\dots,1,0)} = \mathcal{X}^{(i)}\mathcal{X}^{(i-1)}\cdots\mathcal{X}^{(1)}\mathcal{X}^{(0)}$ and $d\lambda^{(\prime)} = d\lambda^{(i)}\cdots d\lambda^{(0)}$, and

$$(ii) \qquad \left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \int\limits_{Z_{\ell-i}(k)\backslash Z_{\ell-i}(\mathbb{A})} \int\limits_{\mathcal{X}_i(k)\backslash\mathcal{X}_i(\mathbb{A})} \int\limits_{U^i_{n+\ell}(k)\backslash U^i_{n+\ell}(\mathbb{A})} \varphi_i\left(u^i \lambda_i z_{\ell-i} \delta_\ell g\right) \Psi^{-1}_{n+\ell}(u^i z_{\ell-i})\, du^i\, d\lambda_i\, dz_{\ell-i}.$$

with $\varphi_i \in W(\varphi)$.

We then repeat the above process to obtain the next step, formulas [21, (7.29)–(7.30)]. For $i = \ell - 2$ these formulas give

$$(i) \qquad \left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \int\limits_{\mathcal{X}(\mathbb{A})} \int\limits_{U^{\ell-1}_{n+\ell}(k)\backslash U^{\ell-1}_{n+\ell}(\mathbb{A})} \varphi\left(u^{\ell-1}\lambda\delta_\ell g\right) \Psi^{-1}(u^{\ell-1})\, du^{\ell-1}\, d\lambda,$$

and

$$(ii) \qquad \left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \int\limits_{U^{\ell-1}_{n+\ell}(k)\backslash U^{\ell-1}_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(u^{\ell-1}\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u^{\ell-1})\, du^{\ell-1}.$$

Lemma 6.4. *Let* $p \in P^1_{n-1,1}(\mathbb{A}) \subset \mathrm{GL}_n(\mathbb{A})$. *(This is the notation of* [21] *for the mirabolic subgroup of* GL_n.*) Then*

$$\int\limits_{\mathcal{X}(\mathbb{A})} \int\limits_{U^{\ell-1}_{n+\ell}(k)\backslash U^{\ell-1}_{n+\ell}(\mathbb{A})} \varphi\left(u^{\ell-1}\hat{p}\lambda\delta_\ell g\right) \Psi^{-1}(u^{\ell-1})\, du^{\ell-1}\, d\lambda = \int\limits_{U^{\ell-1}_{n+\ell}(k)\backslash U^{\ell-1}_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(u^{\ell-1}\hat{p}\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u^{\ell-1})\, du^{\ell-1}.$$

Proof. The proof proceeds in the same way as in [21]. □

Remark 6.3. In their derivation, the authors of [21] only keep track of $\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(e)$ and so g does not appear in their formulas.

For $x \in \mathbb{A}^n$ let $n_\ell(x) = \begin{pmatrix} I_n & x \\ & 1 \end{pmatrix}^\wedge \in N_{n+\ell}(\mathbb{A})$. Set

$$\phi_g(x) = \int\limits_{U_{n+\ell}^{\ell-1}(k)\backslash U_{n+\ell}^{\ell-1}(\mathbb{A})} \varphi_{\ell-1}\left(u^{\ell-1} n_\ell(x)\delta_\ell g\right) \Psi_{n+\ell}^{-1}(u^{\ell-1})\, du$$

so that

$$\phi_g(0) = \left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g).$$

This is a smooth function on $\mathbb{A}^n$ which is left invariant under k^n. We can write its Fourier expansion along $k^n\backslash\mathbb{A}^n$ and then evaluate at $x=0$. The general character of $\mathbb{A}^n$ which is trivial on k^n is of the form $\psi({}^t\eta\cdot x)$ for $\eta \in k^n$. By abelian Fourier analysis

$$\phi_g(x) = \sum_{\eta\in k^n} a_\eta(\phi_g)\psi({}^t\eta\cdot x)$$

where

$$a_\eta(\phi_g) = \int\limits_{k^n\backslash\mathbb{A}^n} \phi_g(x)\psi^{-1}({}^t\eta\cdot x)\; dx$$

The zero Fourier coefficient. The Fourier coefficient corresponding to the trivial character $\eta = 0$ is:

$$a_0(\phi_g) = \int\limits_{k^n\backslash\mathbb{A}^n}\ \int\limits_{U_{n+\ell}^{\ell-1}(k)\backslash U_{n+\ell}^{\ell-1}(\mathbb{A})} \varphi_{\ell-1}\left(u^{\ell-1} n_\ell(x)\delta_\ell g\right) \Psi_{n+\ell}^{-1}(u^{\ell-1})\, du\, dx.$$

Note that $\{n_\ell(x) \mid x\in\mathbb{A}^n\}\cdot U_{n+\ell}^{\ell-1} = U_{n+\ell}^{\ell} \simeq N_\ell \ltimes U_n$, where we view N_ℓ as being in the GSpin part of the Levi subgroup of $P_n \simeq (\mathrm{GL}_n\times \mathrm{GSpin})\ltimes U_n$, and $\Psi_{n+\ell}$ on $U_{n+\ell}^{\ell-1}$ equals ψ_ℓ on N_ℓ, so that this 0-Fourier coefficient gives

$$\int\limits_{k^n\backslash\mathbb{A}^n}\ \int\limits_{U_{n+\ell}^{\ell-1}(k)\backslash U_{n+\ell}^{\ell-1}(\mathbb{A})} \varphi_{\ell-1}\left(u^{\ell-1} n_\ell(x)\delta_\ell g\right) \Psi_{n+\ell}^{-1}(u^{\ell-1})\, du\, dx = \left(\varphi_{\ell-1}^{(U_n,1)}\right)^{(N_\ell,\psi_\ell)}(\delta_\ell g).$$

The non-zero Fourier coefficients. If $\eta\neq 0$, then we can write ${}^t\eta = (0,\dots,0,1)\gamma$ where $\gamma\in P_{n-1}^1(k)\backslash\mathrm{GL}_n(k)$ with $P_{n-1,1}^1 = \mathrm{Stab}_{\mathrm{GL}_n}(0,\dots,0,1)$ is the mirabolic subgroup of GL_n. We can manipulate this Fourier coefficient into

$$\begin{aligned} a_\eta(\phi_g) &= \int\limits_{k^n\backslash\mathbb{A}^n} \phi_g(n_\ell(x))\psi^{-1}({}^t\eta\cdot x)\, dx \\ &= \int\limits_{k^n\backslash\mathbb{A}^n}\ \int\limits_{U_{n+\ell}^{\ell-1}(k)\backslash U_{n+\ell}^{\ell-1}(\mathbb{A})} \varphi_{\ell-1}\left(u^{\ell-1} n_\ell(x)\delta_\ell g\right) \Psi_{n+\ell}^{-1}(u^{\ell-1}n_\ell(\gamma x))\, du\, dx. \end{aligned}$$

Since $\varphi_{\ell-1}$ is still automorphic, it is left invariant under $\hat\gamma$ since $\gamma\in\mathrm{GL}_n(k)$. Then

$$\varphi_{\ell-1}\left(un_\ell(x)\delta_\ell g\right) = \varphi_{\ell-1}\left(\hat\gamma u n_\ell(x)\delta_\ell g\right) = \varphi_{\ell-1}\left(\hat\gamma u\hat\gamma^{-1}\hat\gamma n_\ell(x)\delta_\ell g\right) = \varphi_{\ell-1}\left(\hat\gamma u\hat\gamma^{-1} n_\ell(\gamma x)\hat\gamma\delta_\ell g\right).$$

So the η-Fourier coefficient of ϕ_g is then

$$a_\eta(\phi_g) = \int\limits_{k^n\backslash\mathbb{A}^n} \phi_g(n_\ell(x))\psi^{-1}({}^t\eta\cdot x)\,dx$$
$$= \int\limits_{k^n\backslash\mathbb{A}^n}\ \int\limits_{U^{\ell-1}_{n+\ell}(k)\backslash U^{\ell-1}_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(\hat\gamma u\hat\gamma^{-1}n_\ell(\gamma x)\hat\gamma\delta_\ell g\right)\Psi^{-1}_{n+\ell}(un_\ell(\gamma x))\,du\,dx.$$

Now perform a change of variables $u\mapsto\hat\gamma^{-1}u\hat\gamma$ and $x\mapsto\gamma^{-1}x$ to obtain that the η-Fourier coefficient of ϕ_g is given by

$$a_\eta(\phi_g) = \int\limits_{k^n\backslash\mathbb{A}^n} \phi_g(n_\ell(x))\psi^{-1}({}^t\eta\cdot x)\,dx$$
$$= \int\limits_{k^n\backslash\mathbb{A}^n}\ \int\limits_{U^{\ell-1}_{n+\ell}(k)\backslash U^{\ell-1}_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(un_\ell(x)\hat\gamma\delta_\ell g\right)\Psi^{-1}_{n+\ell}(un_\ell(x))\,du\,dx.$$

We once again write $\{n_\ell(x)\mid x\in\mathbb{A}^n\}\cdot U^{\ell-1}_{n+\ell}=U^\ell_{n+\ell}$ and this becomes

$$a_\eta(\phi_g) = \int\limits_{k^n\backslash\mathbb{A}^n} \phi_g(n_\ell(x))\psi^{-1}({}^t\eta\cdot x)\,dx = \int\limits_{U^{\ell}_{n+\ell}(k)\backslash U^{\ell}_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(u\hat\gamma\delta_\ell g\right)\Psi^{-1}_{n+\ell}(u)\,du\,dx,$$

where ${}^t\eta=(0,\dots,0,1)\gamma$

If we now combine these expressions, we have

$$\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \phi_g(0) = \sum_\eta a_\eta(\phi_g)$$
$$= \left(\varphi^{(U_n,1)}_{\ell-1}\right)^{(N_\ell,\psi_\ell)}(\delta_\ell g) + \sum_{\gamma\in P^1_{n-1,1}(k)\backslash \mathrm{GL}_n(k)}\ \int\limits_{U^{\ell}_{n+\ell}(k)\backslash U^{\ell}_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(u\hat\gamma\delta_\ell g\right)\Psi^{-1}_{n+\ell}(u)\,du\,dx.$$

We need to work in the above form for convergence (see the above remark). In order to prove our integrals are Eulerian, we need to express this in terms of our original cusp form φ. This happens Fourier coefficient by Fourier coefficient. For a non-trivial Fourier coefficient we have

$$\begin{aligned}
a_\eta(\phi_g) &= \int_{U^\ell_{n+\ell}(k)\backslash U^\ell_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(u\hat{\gamma}\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u)\, du \\
&= \int_{k^n\backslash \mathbb{A}^n} \int_{U^{\ell-1}_{n+\ell}(k)\backslash U^{\ell-1}_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(un_\ell(x)\hat{\gamma}\delta_\ell g\right) \Psi^{-1}_{n+\ell}(un_\ell(x))\, du\, dx \\
&= \int_{k^n\backslash \mathbb{A}^n} \int_{\mathcal{X}(\mathbb{A})} \int_{U^{\ell-1}_{n+\ell}(k)\backslash U^{\ell-1}_{n+\ell}(\mathbb{A})} \varphi\left(un_\ell(x)\hat{\gamma}\lambda\delta_\ell g\right) \Psi^{-1}_{n+\ell}(un_\ell(x))\, du\, d\lambda\, dx \\
&= \int_{\mathcal{X}(\mathbb{A})} \int_{k^n\backslash \mathbb{A}^n} \int_{U^{\ell-1}_{n+\ell}(k)\backslash U^{\ell-1}_{n+\ell}(\mathbb{A})} \varphi\left(un_\ell(x)\hat{\gamma}\lambda\delta_\ell g\right) \Psi^{-1}_{n+\ell}(un_\ell(x))\, du\, dx\, d\lambda \\
&= \int_{\mathcal{X}(\mathbb{A})} \int_{U^\ell_{n+\ell}(k)\backslash U^\ell_{n+\ell}(\mathbb{A})} \varphi\left(u\hat{\gamma}\lambda\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u)\, du\, dx\, d\lambda
\end{aligned}$$

and the authors of [21] go to great lengths to justify the exchange of the integral over $k^n\backslash\mathbb{A}^n$ and that over $\mathcal{X}(\mathbb{A})$. Along the way, in the successive interchanges, they use

- $\Psi^{-1}_{n+\ell}([x,\lambda]) = \Psi_{n+\ell}([x^{-1},\lambda])$ for $x \in X(\mathbb{A})$ where $X = \left\{ \begin{pmatrix} I_n & 0 & x \\ & 1 & 0 \\ & & 1 \end{pmatrix}^\wedge \in N_{n+\ell} \right\}$
- $\hat{\gamma}[x,\lambda] = [x,\lambda]\hat{\gamma}$
- $\hat{\gamma}x\hat{\gamma}^{-1} \in U^\ell_{n+\ell}(\mathbb{A})$
- $\Psi_{n+\ell}(\hat{\gamma}x\hat{\gamma}^{-1}) = 1.$

These all involve either unipotent matrices of lifts of elements of GL_n and remain valid in the GSpin context.

Similarly, for the 0-Fourier coefficient they show

$$\left(\varphi^{(U_n,1)}_{\ell-1}\right)^{(N_\ell,\psi_\ell)}(\delta_\ell g) = \int_{\mathcal{X}(\mathbb{A})} \left(\varphi^{(U_n,1)}\right)^{(N_\ell,\psi_\ell)}(\lambda\delta_\ell g)\, d\lambda.$$

Note that since φ is assumed to be cuspidal, the first unipotent period $\varphi^{(U_n,1)} \equiv 0$ since U_n is the unipotent radical of the maximal parabolic subgroup P_n. Hence this contribution to $\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g)$ vanishes. Hence

$$\begin{aligned}
\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) &= \sum_{\gamma\in P^1_{n-1,1}(k)\backslash \mathrm{GL}_n(k)} \int_{U^\ell_{n+\ell}(k)\backslash U^\ell_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(u\hat{\gamma}\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u)\, du\, dx \\
&= \sum_{\gamma\in P^1_{n-1,1}(k)\backslash \mathrm{GL}_n(k)} \int_{\mathcal{X}(\mathbb{A})} \int_{U^\ell_{n+\ell}(k)\backslash U^\ell_{n+\ell}(\mathbb{A})} \varphi\left(u\hat{\gamma}\lambda\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u)\, du\, dx\, d\lambda.
\end{aligned}$$

To proceed, they further Fourier expand φ or $\varphi_{\ell-1}$ along the ideas of Piatetski-Shapiro and Shalika.

- If f is an automorphic form on GL_n, to obtain its Fourier expansion, first restrict to the mirabolic $P^1_{n-1,1}$ and expand along its unipotent radical $U_{n-1} = U(P^1_{n-1,1}) \simeq \mathbb{A}^{n-1}$.
- For each such Fourier coefficient, which can now be viewed as a function on
$$P^1_{n-2,1}(k)\backslash P^1_{n-2,1}(\mathbb{A}),$$
expand along the unipotent radical of $P^1_{n-2,1} \simeq \mathbb{A}^{n-2}$.
- Continue inductively until one obtains the sum of the Whittaker function over $Z_n(k)\backslash \mathrm{GL}_n(k)$.

To this end, set
$$n_{\ell+1}(x) = \begin{pmatrix} I_{n-1} & x \\ & 1 \end{pmatrix}^{\wedge} \in N_{n+\ell}(\mathbb{A}), \quad x \in \mathbb{A}^{n-1}$$
and let
$$\phi_{1,\hat{\gamma},g}(x) = \int\limits_{U^{\ell}_{n+\ell}(k)\backslash U^{\ell}_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(u n_{\ell+1}(x)\hat{\gamma}\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u)\, du.$$
Then, exactly as before, when we write the Fourier expansion of $\phi_{1,\hat{\gamma},g}(x)$ in x and evaluate at $x = 0$ we obtain
$$\phi_{1,\hat{\gamma},g}(0) = \sum_{\gamma' \in P^1_{n-2,1}(k)\backslash \mathrm{GL}_{n-1}(k)} \int\limits_{U^{\ell+1}_{n+\ell}(k)\backslash U^{\ell+1}_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(u\hat{\gamma'}\hat{\gamma}\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u)\, du$$
$$+ \left(\varphi_{\ell-1}^{(U_{n-1},1)}\right)^{(N_{\ell+1},\Psi_{\ell+1})} (\hat{\gamma}\delta_\ell g)\,.$$
If, following [21], we denote
$$P^{1,\dots,1}_{n-i,1,\dots,1}(k) = \left\{ \begin{pmatrix} g & x \\ & z \end{pmatrix} \in \mathrm{GL}_n(k) \ : \ z \in Z_i(k) \right\}$$
then, as above, this gives
$$\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \sum_{\gamma \in P^{1,1}_{n-2,1,1}(k)\backslash \mathrm{GL}_n(k)} \int\limits_{U^{\ell+1}_{n+\ell}(k)\backslash U^{\ell+1}_{n+\ell}(\mathbb{A})} \varphi_{\ell-1}\left(u\hat{\gamma}\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u)\, du$$
$$+ \sum_{\gamma \in P^1_{n-1}(k)\backslash \mathrm{GL}_n(k)} \left(\varphi_{\ell-1}^{(U_{n-1},1)}\right)^{(N_{\ell+1},\psi_{\ell+1})} (\hat{\gamma}\delta_\ell g).$$
If we, as above, write this in terms of our original φ, so again interchanging a number of integrals, this becomes
$$\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \sum_{\gamma \in P^{1,1}_{n-2,1,1}(k)\backslash \mathrm{GL}_n(k)} \int\limits_{\mathcal{X}(\mathbb{A})} \int\limits_{U^{\ell+1}_{n+\ell}(k)\backslash U^{\ell+1}_{n+\ell}(\mathbb{A})} \varphi\left(u\hat{\gamma}\lambda\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u)\, du\, d\lambda$$
$$+ \sum_{\gamma \in P^1_{n-1}(k)\backslash \mathrm{GL}_n(k)} \int\limits_{\mathcal{X}(\mathbb{A})} \left(\varphi^{(U_{n-1},1)}\right)^{(N_{\ell+1},\psi_{\ell+1})} (\hat{\gamma}\lambda\delta_\ell g)\ d\lambda.$$

Once again, by the cuspidality of φ, we see $\varphi^{(U_{n-1},1)} \equiv 0$ and so we have

$$\left(\varphi^{(N_\ell,\Psi_\ell)}\right)^{(U_G,1)}(g) = \sum_{\gamma \in P^{1,1}_{n-2,1,1}(k)\backslash \mathrm{GL}_n(k)} \int_{\mathcal{X}(\mathbb{A})} \int_{U^{\ell+1}_{n+\ell}(k)\backslash U^{\ell+1}_{n+\ell}(\mathbb{A})} \varphi\left(u\hat{\gamma}\lambda\delta_\ell g\right) \Psi^{-1}_{n+\ell}(u)\, du\, d\lambda.$$

Continuing in this fashion inductively, we arrive at the statement of Proposition 5.2.

7. Euler Product Expansion

As a consequence of Theorems 4.2 and 5.1 we can now obtain an Euler product expansion for the global integrals therein, which will in turn allow us to relate the integrals to the generic Rankin-Selberg L-functions for GSpin $\times$ GL (with arbitrary rank and including the quasi-split forms).

Recall that $\mathbb{A} = \mathbb{A}_k$ and we have the embedding $G = G_{n'} \hookrightarrow H = H_{m'}$ in all cases. We can summarize the various cases from Sections 4 and 5 in the table below. In each case, we also indicate the Rankin-Selberg L-function the integral will produce (see below).

	case A: Sec.5 integrals	**case B: Sec.4 integrals**
odd	$G = \mathrm{GSpin}_{2n} \hookrightarrow H = \mathrm{GSpin}_{2m+1}$ $m \geq n, (\ell = m - n)$ $L(s, \pi \times \tau)$ for $H \times \mathrm{GL}_n$ (G split, H split)	$G = \mathrm{GSpin}_{2n+1} \hookrightarrow H = \mathrm{GSpin}_{2m}$ $m > n, (\ell = m - n - 1)$ $L(s, \pi \times \tau)$ for $G \times \mathrm{GL}_m$ (G split, H split)
even	$G = \mathrm{GSpin}_{2n+1} \hookrightarrow H = \mathrm{GSpin}_{2m}$ $m > n, (\ell = m - n - 1)$ $L(s, \pi \times \tau)$ for $H \times \mathrm{GL}_n$ (G split, H quasi-split)	$G = \mathrm{GSpin}_{2n} \hookrightarrow H = \mathrm{GSpin}_{2m+1}$ $m \geq n, (\ell = m - n)$ $L(s, \pi \times \tau)$ for $G \times \mathrm{GL}_m$ (G quasi-split, H split)

Remark 7.1. When we do the so-called "unramified computation" later, following the ideas of Soudry [34, §12] there will be a certain duality, to be made more precise later, between the diagonal entries of this table. The case of $\mathrm{GSpin}_{2a+1} \times \mathrm{GL}_b$ will be related to the case of $\mathrm{GSpin}_{2b} \times \mathrm{GL}_a$. In the local setting, both will give a Rankin-Selberg L-function of degree $2ab$ (in q^{-s}). We will discuss this in Section 8.

Remark 7.2. We note, as it is pointed out in [18, page 58], that when $H = \mathrm{SO}_{2m+1}$, Method A with $m = n$ gives the $\mathrm{SO}_{2m+1} \times \mathrm{GL}_m$ case while Method B when $G = \mathrm{SO}_{2n+1}$ with $m = n+1$ gives $\mathrm{SO}_{2n+1} \times \mathrm{GL}_{n+1}$. On the other hand, when $H = \mathrm{SO}_{2m}$, Method A with $n = m-1$ gives the $\mathrm{SO}_{2m} \times \mathrm{GL}_{m-1}$ case while Method B when $G = \mathrm{SO}_{2n}$ with $m = n$ gives the $\mathrm{SO}_{2n} \times \mathrm{GL}_n$ case. These are consistent with our choice of names for (case A) and (case B). In [18, Part B], Gelbart and Piatetski-Shapiro write down the details of the first and the last of these four cases. (They also include their Method C that treats the symplectic groups and remark that they could also be applied to Spin groups.)

Our goal in this section is to factor the right hand sides of the basic identities in Theorems 4.2 and 5.1 as products of local zeta integrals (over all the places of k). Given that the adelic domains of integration factor over the places of k, we need to show that we may choose the data in the integrands that also factor and that the resulting local zeta integrals converge absolutely in some right half plane, not depending on the place of k, and that the Euler product converges in that half plane. Fortunately, the same proofs as in the case of special orthogonal groups in [34] and [24] work without the need for much modification; the existence of the nontrivial center in the general spin groups makes little difference for the purposes of the results in this section. Therefore, we only briefly review the steps below.

Let $\pi = \otimes_v \pi_v$ be an irreducible, cuspidal, globally ψ-generic, automorphic representation of $G(\mathbb{A})$ in (case B), resp. of $H(\mathbb{A})$ in (case A). Choose a decomposable $\varphi \in V_\pi$ so that the corresponding Whittaker function $W_\varphi = W_\varphi^\psi$ in the Whittaker model $\mathcal{W}(\pi, \psi)$ is a product of local Whittaker functions

$$W_\varphi(x) = \prod_v W_v(x_v),$$

with each W_v a nonzero function in the local Whittaker model $\mathcal{W}(\pi_v, \psi_v)$. In fact, since $\mathcal{W}(\pi, \psi) \neq \{0\}$ by assumption, the map $\varphi \mapsto W_\varphi$ is an isomorphism onto $\bigotimes_v' \mathcal{W}(\pi_v, \psi_v)$.

Similarly, let $\tau = \otimes \tau_v$ be an irreducible, cuspidal, automorphic representation of $\mathrm{GL}_m(\mathbb{A})$ in (case B), resp. of $\mathrm{GL}_n(\mathbb{A})$ in (case A), and assume that f_s in (4.2), resp. (5.1), is a decomposable section. When we apply the GL-Whittaker coefficient on τ with respect to the character ψ^{-1} to it, we can write

$$f_s^{(Z_m, \psi)}(y) = \prod_v f_{v,s}(y_v; I_m), \quad \text{(case B)},$$

resp.

$$f_s^{(Z_n, \psi_n)}(y) = \prod_v f_{v,s}(y_v; I_n), \quad \text{(case A)}.$$

Here $f_{s,v}$ is a K-finite holomorphic section in $\rho_{\tau_v,s}$ taking values in the local Whittaker model $\mathcal{W}(\tau_v, \psi_v^{-1})$ of τ_v. For a fixed y_v we denote the corresponding Whittaker function in the Whittaker model of τ_v by $x \mapsto f_{\tau_v,s}(y_v; x)$.

We choose a finite set S of places of k, containing all the places at infinity, outside of which all data are unramified. For $v \notin S$, we take $W_v = W_v^0$ to be the unique Whittaker function such that its value at the identity element is 1 and take the function $x \mapsto f_{\tau_v,s}(I; x)$ be the unique spherical and normalized Whittaker function in the corresponding Whittaker model of τ_v. Again, denote this unique section by $f_{v,s}^0$.

Theorem 7.1. *Let $\mathcal{L}(\varphi, f_s)$ be as in Theorem 4.2, resp. Theorem 5.1.*

(i) *With a choice of data as above, for* $\mathrm{Re}(s) \gg 0$ *we have*

$$\mathcal{L}(\varphi, f_s) = \prod_v \mathcal{L}_v(W_v, f_{v,s}),$$

where the local factor is given by

$$\mathcal{L}_v(W_v, f_{v,s}) = \tag{7.3}$$

$$\begin{cases} \int\limits_{N_G(k_v)Z_G(k_v)\backslash G(k_v)} W_v(g) \int\limits_{N_\ell(k_v)\cap\beta^{-1}P_m(k_v)\beta\backslash N_\ell(k_v)} f_{v,s}\,(\beta ug; I_m)\,\Psi_{\ell,v}(u)^{-1}\,du\,dg, & \text{(caseB)}, \\ \int\limits_{N_G(k_v)Z_G(k_v)\backslash G(k_v)} f_{v,s}(g; I_n) \int\limits_{\mathcal{X}(k_v)} W_v(\lambda\delta_\ell g)\,d\lambda\,dg, & \text{(caseA)}. \end{cases} \tag{7.4}$$

Here, each local factor $\mathcal{L}_v$ converges absolutely in a fixed right half plane independent of the v and continues to a meromorphic function on the complex plane.

(ii) *When v is finite, we can choose the data W_v and $f_{v,s}$ such that $\mathcal{L}_v(W_v, f_{v,s})$ is identically 1 as a function of s.*

(iii) *When v is infinite, given $s_0 \in \mathbb{C}$ we can choose the data W_v and $f_{v,s}$ such that $\mathcal{L}_v(W_v^{\psi_v}, f_{v,s})$ is holomorphic and nonzero in a neighborhood of s_0.*

Proof. As we mentioned already the proof follows exactly as in the case of special orthogonal groups and the existence of the large center in the case of the general spin groups does not make much of a difference for this proof.

For (i) one starts out by estimating the Whittaker functions $W_v(g)$ in (case B). For the non-archimedean v this is done exactly as in [34, §2] by estimating the Whittaker function by a "gauge", originally introduced by Jacquet, Piatetski-Shapiro, and Shalika in the case of the general linear groups. These estimates result in the convergence of the local integrals for $\mathrm{Re}(s) \geq s_0$, with $s_0 \in \mathbb{R}$ only depending on the groups and the embeddings, and not the place v as in [34, §4] for the non-archimedean v.

For the archimedean places v again we follow Sourdy as in [34, §3] for the estimates, where one also appeals to results of Dixmier and Malliavin [17]. The convergence then follows as in [34, §5] for the archimedean v. While Soudry focuses on the odd case in [34] the even case is also covered in [24]. The analogous results for (case A) are already covered by Ginzburg [20] and also reviewed by [24] in the even case. Since in the case of the general spin groups we already divide by the center in the domains of the integrals, no modifications in the above proofs are necessary and we can conclude (i).

We should note here that the results we cite above show that region of convergence of the local integrals depend only on the representations, and not on the data W_v and $f_{v,s}$. The dependence on the local representations is through their exponents which can be uniformly bounded. Therefore, when $\mathrm{Re}(s)$ is sufficiently large we have convergence that is valid for all v. Similarly part (ii) proceeds as in [34, §6].

Also, part (iii) follows as in [34, §7]. We need to choose K-finite data and show that the integral admits meromorphic continuation which is continuous in the input data. Here, the argument of [35] applies to (case A) and (case B) where we can follow [27, p. 402] in the spit case. The non-split, quasi-split case for the special orthogonal groups (and for GSpin groups) has not been yet appeared in a published paper and, as in the SO case in [27], we assume it for the GSpin groups. □

8. The Unramified Computations

In this section we compute the local zeta integrals for "unramified data". This local analysis allows us to relate $\mathcal{L}_v(s, W_v, f_{v,s})$ in Theorem 7.1 to the local L-functions when $v \notin S$, $W_v = W_v^0$ and $f_{v,s} = f_{v,s}^0$ and is usually referred to as the "unramfied computation".

In the split case with n equal, or nearly equal to m, the unramfied computation was already done for the odd and even special orthogonal groups in [18, Appendix]. More precisely, Gelbart, Piatetski-Shapiro, and Rallis worked out the cases of $\mathrm{SO}_{2n+1} \times \mathrm{GL}_n$ and $\mathrm{SO}_{2n} \times \mathrm{GL}_{n-1}$ in (case A) and the cases of $\mathrm{SO}_{2n+1} \times \mathrm{GL}_{n+1}$ and $\mathrm{SO}_{2n} \times \mathrm{GL}_n$ in (case B), even though the emphasis in [18, Appendix] is on the first and last of these four cases. Their method combines the Casselman-Shalika formula with a decomposition of the symmetric algebra of polynomials defined on complex matrices of appropriate size on which the (connected components) of the Langlands duals of the special orthogonal group and the general linear group act. They then show that the local zeta integral with unramified data is a quotient of L-functions, by explicitly calculating the integral and the local L-functions as series in q^{-s}, where q, as usual, denotes the cardinality of the residue field of the local non-Archimedean field. For the symmetric algebra decomposition they invoke some results of Ton-That [38, 39].

The unramified computation in the case of $\mathrm{SO}_{2n+1} \times \mathrm{GL}_m$ with $m \leq n$ was then carried out by Ginzburg in [20] as well as the case of $\mathrm{SO}_{2n} \times \mathrm{GL}_m$ with $m \leq n-1$, where he uses a certain inductive argument to reduced the proof for the more general m to those of $m = n$ or $m = n-1$ in the odd and even cases, respectively.

Instead of extending Ginzburg's inductive argument to the general spin groups, we have followed the original approach of Gelbart, Piatetski-Shapiro, and Rallis mentioned above in (case A). This is possible because Ton-That's results on the decomposition of the symmetric algebra are fortunately available for $m \leq n$, resp. $m \leq n-1$, in the odd, resp. even, cases (and not just $m = n$ and $m = n-1$ respectively, which is what was used in [18, Appendix]).

However, in (case B) when the rank of the general linear group is larger than the rank of the SO groups the decomposition of the symmetric algebra becomes too complicated to be helpful. Soudry [34, §12] then showed how to deal with the case of $\mathrm{SO}_{2n+1} \times \mathrm{GL}_m$ with $m > n$ by relating the local Rankin-Selberg L-functions in this case to that of $\mathrm{SO}_{2m} \times \mathrm{GL}_n$, where we already have the (case A) results. E. Kaplan has also extended Soudry's method to the case of $\mathrm{SO}_{2n} \times \mathrm{GL}_m$ with $m \geq n$ as well as considering the quasi-split forms of SO_{2n} [24]. Kaplan also suggests modifications of the method in [25]. Both Soudry and Kaplan use a certain uniqueness result that is fortunately now available for the general spin groups as well thanks to [28]. This allows us to apply Soudry's ideas in (case B) for the general spin groups.

As mentioned, we consider the unramified computation for the general spin groups, both when the rank of the general spin group is larger and when it is smaller than the rank of the general linear group, following the above works. We also consider the quasi-split case in the even case, following [24, §3.2.1] where a similar argument is given for the quasi-split even special orthogonal groups. Given that the method of proof is somewhat different in (case A) and (case B) as we just explained, we state the results in two separate theorems below (cf. Theorem 8.1 and Theorem 8.2) even though the statements end up being similar.

For this section only, we let F denote a non-Archimedean local field of characteristic zero with ring of integers O_F and the cardinality of the residue field $q = p^f$. We also fix a uniformizer $\varpi \in O_F$. The usual p-adic absolute value on F is denoted by $|\cdot| = |\cdot|_F$, with $|\varpi| = q^{-1}$. Also, let ψ denote an additive character of F which is unramified, i.e., it is trivial on O_F, but non-trivial on $\varpi^{-1}O$.

Recall that we have $G = \mathrm{GSpin}_{n'} \hookrightarrow H = \mathrm{GSpin}_{m'}$. In (case A) we have either $n' = 2n$, $m' = 2m+1$, and $n \le m$, (odd case) or $n' = 2n+1$, $m' = 2m$, and $n < m$ (even case). In the latter case, H may be quasi-split. Let (π, V_π) denote an unramified globally ψ-generic representation of $H(F)$ and let (τ, V_τ) be an unramified representation of $\mathrm{GL}_n(F)$. (By unramified we mean a representation which has fixed vectors under the action of the maximal compact subgroup $H(O_F)$, resp. $\mathrm{GL}_n(O_F)$. Such a representation is then induced from an unramified quasi-character of the torus.) The representation τ is determined by its Frobenius-Hecke or Satake parameter t_τ, a semi-simple conjugacy class in $\mathrm{GL}_n(\mathbb{C})$. Similarly, let t_π denote the parameter of π, a semi-simple conjugacy class in the Langlands dual ${}^L H$. We may take ${}^L H \cong \mathrm{GSp}_{2m}(\mathbb{C})$ when $m' = 2m+1$. When $m' = 2n$, we have ${}^L H \cong \mathrm{GSO}_{2m}(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F)$ when H is split and ${}^L H \cong \mathrm{GSO}_{2m}(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{F}/F)$ when H is quasi-split non-split; cf. §2.2.4(B) and (C).

Similarly, in (case B) we let (π, V_π) denote an unramified globally ψ-generic representation of $G(F)$ and let (τ, V_τ) be an unramified representation of $\mathrm{GL}_m(F)$, with $n' = 2n+1$, $m' = 2m$, and $m > n$ (odd case), or $n' = 2n$, $m' = 2m+1$, and $m \ge n$ (even case). Now, t_τ is a semi-simple conjugacy class in $\mathrm{GL}_m(\mathbb{C})$ and t_π is a semi-simple conjugacy class in ${}^L G$. Again, we may take ${}^L G \cong \mathrm{GSp}_{2n}(\mathbb{C})$ when $n' = 2n+1$ and we have ${}^L G = \mathrm{GSO}_{2n}(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F)$ if $n' = 2n$ with G split and ${}^L G = \mathrm{GSO}_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{F}/F)$ if $n' = 2n$ with G quasi-split non-split; cf. §2.2.4(B) and (C).

We fix a Haar measure on the additive group F with the volume of O_F equal to 1. We can then use this measure to fix a left Haar measure on $G(F)$ and $H(F)$ in such a way that

$$\mathrm{Vol}(u_\alpha(O_F)) = \mathrm{Vol}\left(\{u_\alpha(x) : x \in F, |x| \le 1\}\right) = 1, \tag{8.1}$$

for all roots α in G and H. Here, the image of u_α is the root group associated with α in the ambient reductive group. In particular, we will have $\mathrm{Vol}(K_G) = \mathrm{Vol}(T_G \cap K_G) = 1$ and similarly for H. Here, K_G, resp., K_H, denotes the (fixed choice of a) maximal compact in $G(F)$, resp., $H(F)$, and T_G, reps. T_H, denotes the (fixed) maximal torus in G, resp., H.

We now review some preliminary facts that help us relate the local integrals of Theorem 7.1 to the local L-functions. In the local setting, the local field F will always be the completion of the number field k at a non-Archimedean place v of k and the local representations π and τ above will be the component at v of the corresponding global representations bearing the same names in Sections 4 and 5.

8.1. **Symmetric Algebra Decompositions.** We next recall some preliminary facts about the decomposition of a symmetric algebra that is needed for our results. Note that this analysis is only feasible in (case A) with both parities. Indeed, it is precisely the complications with the analysis of the symmetric algebra decompositions in (case B) that will force us to use an alternative method in that case, as we will see below.

We refer to [38, 39] for the details. While the results of Ton-that are proved for the case when the rank of the classical group is larger than or equal to the rank of GL, Gelbart, Piatetski-Shapiro and Rallis only used them in the case of equal (or almost equal) ranks in [18, Appendix]. We should add that Ton-that's results are for the non-similitude situation. We will use [38] with $m \ge n$ when $m' = 2m+1$ and [39] with $m > n$ when $m' = 2m$ and make the necessary adjustments in the calculations for the similitude situation.

Assume that $m \geq n$. Let $E = \mathbb{C}^{n\times 2m}$ denote the space of $n \times 2m$ complex matrices. Then $\widehat{\mathrm{GL}}_n = \mathrm{GL}_n(\mathbb{C})$ acts on E on the left and $\widehat{H} = \mathrm{GSp}_{2m}(\mathbb{C})$, resp., $\mathrm{GSO}_{2m}(\mathbb{C})$, acts on E on the right. Let $S(E^*)$ denote the symmetric algebra of complex-valued polynomial functions on E. It becomes a $\left(\widehat{H} \times \widehat{\mathrm{GL}}_n\right)$-module via

$$((g_1, g_2) \cdot P)(X) = P\left({}^t g_2 X g_1\right), \quad X \in E, P \in S(E^*), g_1 \in \widehat{H}, g_2 \in \mathrm{GL}_n(\mathbb{C}). \tag{8.2}$$

Our goal is to decompose this module in a way that is useful for our setting. Recall that the group $\widehat{H} = \mathrm{GSp}_{2m}(\mathbb{C})$, resp., $\mathrm{GSO}_{2m}(\mathbb{C})$, is defined as in [6, §2.3], i.e., it is the connected component of the group

$$\left\{g \in \mathrm{GL}_{2m}(\mathbb{C}) : {}^t g J g = \mu(g) J\right\}, \tag{8.3}$$

where the $2m \times 2m$ matrix J is defined via

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & & \\ -1 & & & & & \end{pmatrix}, \quad \text{resp., } J = \begin{pmatrix} & & & & & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ & & 1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & & \\ 1 & & & & & \end{pmatrix}, \tag{8.4}$$

and μ denotes the similitude character. (Notice that the algebraic group defined in (8.3) is connected for the former J, but it has two connected components for the latter J, cf. [6, §2.3].)

Next, we introduce two subalgebras $I(E^*)$ and $H(E^*)$ of $S(E^*)$ such that

$$S(E^*) \cong I(E^*) \otimes H(E^*) \tag{8.5}$$

as $\left(\widehat{H} \times \widehat{\mathrm{GL}}_n\right)$-modules. We let $I(E^*)$ be the subalgebra of all $\mathrm{Sp}_{2n}(\mathbb{C})$-, resp., $\mathrm{SO}_{2n}(\mathbb{C})$-invariant polynomials in $S(E^*)$. Equivalently, $I(E^*)$ is the algebra of polynomials on the space

$$\left\{Y = X\, J\, {}^t X : X \in E\right\},$$

where J is as in (8.4). The action of $\mathrm{GL}_n(\mathbb{C})$ on the space of polynomials of degree i in $I(E^*)$ is given by

$$\begin{aligned} &\mathrm{Sym}^i\left(\wedge^2(g_2)\right), && g_2 \in \mathrm{GL}_n(\mathbb{C}), \text{ or} \\ &\mathrm{Sym}^i\left(\mathrm{Sym}^2(g_2)\right), && g_2 \in \mathrm{GL}_n(\mathbb{C}), \end{aligned} \tag{8.6}$$

respectively, while the action of $\widehat{H} = \mathrm{GSp}_{2m}(\mathbb{C})$, resp., $\mathrm{GSO}_{2m}(\mathbb{C})$, is given simply by

$$\mu(g_1)^i, \quad g_1 \in \widehat{H}, \tag{8.7}$$

in both cases.

Similarly, $H(E^*)$ denotes the subspace of $S(E^*)$ consisting of $\mathrm{Sp}_{2m}(\mathbb{C})$-, resp., $\mathrm{SO}_{2m}(\mathbb{C})$-harmonic polynomials. Equivalently, $H(E^*)$ is isomorphic, as $\mathrm{Sp}_{2m}(\mathbb{C})$-, resp., $\mathrm{SO}_{2m}(\mathbb{C})$-module, to the symmetric algebra of polynomials on the space

$$\left\{X \in E : X\, J\, {}^t X = 0\right\}.$$

Let

$$\delta = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n \tag{8.8}$$

with $k_1 \geq k_2 \geq \ldots k_n \geq 0$. For such a "dominant" δ we define

$$\bar{\delta} = (k_1, \ldots, k_n, 0, \ldots, 0) \in \mathbb{Z}^m. \tag{8.9}$$

Let $H(E^*, \delta)$ be the subspace of $H(E^*)$ consisting of polynomials transforming under $\mathrm{GL}_n(\mathbb{C})$ according to the irreducible finite-dimensional representation

$$\rho_\delta^{\mathrm{GL}_n(\mathbb{C})}(g_2), \quad g_2 \in \mathrm{GL}_n(\mathbb{C}),$$

of highest weight δ. Then, as an $\widehat{H}$-module, $H(E^*, \delta)$ is equivalent to the representation

$$\rho^{\widehat{H}}_{(\bar{\delta};\mathrm{tr}\,\delta)}(g_1), \quad g_1 \in \widehat{H},$$

where

$$\rho^{\widehat{H}}_{(\bar{\delta};\mathrm{tr}\,\delta)}(g_1) = \begin{cases} \mu(g_1)^{\mathrm{tr}\,\delta} \cdot \rho_{\bar{\delta}}^{\mathrm{Sp}_{2m}(\mathbb{C})}\left(\overline{g}_1\right), & \text{if } g_1 \in \mathrm{GSp}_{2m}(\mathbb{C}), \\ \mu(g_1)^{\mathrm{tr}\,\delta} \cdot \rho_{\bar{\delta}}^{\mathrm{SO}_{2m}(\mathbb{C})}\left(\overline{g}_1\right), & \text{if } g_1 \in \mathrm{GSO}_{2m}(\mathbb{C}), \end{cases} \tag{8.10}$$

with

$$\overline{g}_1 = \mu(g_1)^{-1/2} g_1 \in \mathrm{Sp}_{2m}(\mathbb{C}), \text{ resp., } \mathrm{SO}_{2m}(\mathbb{C}). \tag{8.11}$$

We recall here that an irreducible finite-dimensional representation of $\widehat{H} = \mathrm{GSp}_{2m}(\mathbb{C})$, resp., $\mathrm{GSO}_{2m}(\mathbb{C})$, is given as

$$\rho_{((k_1,k_2,\ldots,k_m),k_0)}(g) = \mu(g)^{k_0} \cdot \rho_{(k_1,k_2,\ldots,k_m)}\left(\bar{g}\right), \quad g \in \widehat{H},$$

where $(k_1, k_2, \ldots, k_m) \in \mathbb{Z}^m$ satisfies $k_1 \geq \ldots k_n \geq 0$, $k_0 \in \mathbb{Z}$, and $\rho_{(k_1,k_2,\ldots,k_m)}$ denotes the irreducible finite-dimensional representation of $\mathrm{Sp}_{2m}(\mathbb{C})$, resp., $\mathrm{SO}_{2m}(\mathbb{C})$, of highest weight $(k_1, \ldots, k_n)$. It follows that

$$\mathrm{tr}\,\mathrm{Sym}^r\left(g_1 \otimes g_2\right) = \tag{8.12}$$
$$\begin{cases} \displaystyle\sum_{2i+j=r} \mathrm{tr}\,\mathrm{Sym}^i(\wedge^2 g_2)\mu(g_1)^i \sum_{\substack{\mathrm{tr}\,\delta=j \\ \delta \text{ dominant}}} \mu(g_1)^j \chi_{\bar{\delta}}^{\mathrm{Sp}_{2m}(\mathbb{C})}\left(\overline{g}_1\right) \chi_\delta^{\widehat{\mathrm{GL}}_n}(g_2), & \text{if } \widehat{H} = \mathrm{GSp}_{2m}(\mathbb{C}), \\ \displaystyle\sum_{2i+j=r} \mathrm{tr}\,\mathrm{Sym}^i(\mathrm{Sym}^2 g_2)\mu(g_1)^i \sum_{\substack{\mathrm{tr}\,\delta=j \\ \delta \text{ dominant}}} \mu(g_1)^j \chi_{\bar{\delta}}^{\mathrm{SO}_{2m}(\mathbb{C})}\left(\overline{g}_1\right) \chi_\delta^{\widehat{\mathrm{GL}}_n}(g_2), & \text{if } \widehat{H} = \mathrm{GSO}_{2m}(\mathbb{C}). \end{cases}$$

8.2. **The Casselman-Shalika Formula.** The Casselman-Shalika formula [14] evaluates the normalized spherical Whittaker function of an unramified representation of a connected reductive group. Here, normalized means that we choose the Whittaker function to have the value 1 at the identity as we explained in Section 7. For a split group one can combine the formula with the Weyl character formula (for the dual group) to arrive at the form we state below, as can be found, for example, in [8, §3]. When the group is a quasi-split general spin group, we modify the formula accordingly.

Consider an irreducible, admissible, unramified representation π of the F-points of a split, connected, reductive group G (over F), with a fixed Borel $B = TU_G$, where T is a maximal torus and U_G is the unipotent radical of B. Let χ be an unramified character of U_G and let $W^0 \in \mathcal{W}(\pi, \chi)$ denote a normalized spherical Whittaker function. Then, W^0 is right K_G-invariant and satisfies $W(1) = 1$. Hence, it is completely determined by its values on $T(F) = T_G(F)$ and in fact, as we will recall below, by the dominant elements t in $T(F)/T(O_F)$. Here, dominant means that $|\alpha(t)| \leq 1$ for all simple roots α.

Moreover, there is a parametrization of the irreducible representations of the complex dual group $\widehat{G}$ by the dominant elements t in $T(F)/T(O_F)$. Let t_λ be such a dominant element and assume that it corresponds to the representation whose character χ_λ has highest weight vector λ. Then the Casselman-Shalika formula can be stated as

$$W^0(t_\lambda) = \delta_G(t_\lambda)^{1/2}\chi_\lambda^{\widehat{G}}(t_\pi), \tag{8.13}$$

where δ_G denotes the modulus character of the standard Borel subgroup in G and t_π denotes the semi-simple conjugacy class in the complex dual group parametrizing π. See [37, Prop. 1] for another example of the formulation (8.13).

We recall the explicit form of the t_π for the groups of interest to us. Let $\pi = \mathrm{Ind}_{B_G}^G(\mu)$, where μ is a character of $T_G(F)$. When $G = \mathrm{GL}_m$, we have $\mu = \chi_1 \otimes \cdots \otimes \chi_m$, with χ_i unramified quasi-characters of $F^\times$. Similarly, when $G = \mathrm{GSpin}_{2n+1}$ or GSpin_{2n} (split), we have $\mu = \chi_0 \otimes \chi_1 \otimes \cdots \chi_n$. Here, χ_0 is the pullback of the central character of π to $e_0^*(\mathrm{GL}_1(F))$. Then, a representative t_π for the semi-simple conjugacy class in the dual group corresponding to π is given by

$$t_\pi = \begin{cases} \mathrm{diag}\left(\chi_1(\varpi), \ldots, \chi_m(\varpi)\right) \in \mathrm{GL}_m(\mathbb{C}) & \text{if } G = \mathrm{GL}_m, \\ \mathrm{diag}\left(\chi_1(\varpi), \ldots, \chi_n(\varpi), \chi_n^{-1}\chi_0(\varpi), \ldots, \chi_1^{-1}\chi_0(\varpi)\right) \in \mathrm{GSp}_{2n}(\mathbb{C}) & \text{if } G = \mathrm{GSpin}_{2n+1}, \\ \mathrm{diag}\left(\chi_1(\varpi), \ldots, \chi_n(\varpi), \chi_n^{-1}\chi_0(\varpi), \ldots, \chi_1^{-1}\chi_0(\varpi)\right) \in \mathrm{GSO}_{2n}(\mathbb{C}) & \text{if } G = \mathrm{GSpin}_{2n}. \end{cases} \tag{8.14}$$

Next, assume that $G = \mathrm{GSpin}_{2n}^a$ with a a non-square in $F^\times$ is quasi-split, but not split over F. Let $E = F(\sqrt{a})$ be a quadratic extension of F over which G splits, cf. §2.2.4(C). Then

$$T_G(F) = F^\times \times \left(F^\times\right)^{n-1} \times \mathrm{GSpin}_2^a(F)$$

with $\mathrm{GSpin}_2^a(F) = \left(\mathrm{Res}_{E/F}\,\mathrm{GL}_1\right)(F) = E^\times$. The unramified character μ of $T_G(F)$ can be written as $\mu = \chi_0 \otimes \chi_1 \otimes \cdots \chi_{n-1} \otimes \chi \circ \mathrm{Norm}_{E/F}$, and we may identify $t_\pi \in {}^LG \cong \mathrm{GSO}_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$ with

$$t_\pi = \mathrm{diag}\left(\chi_1(\varpi), \ldots, \chi_{n-1}(\varpi), \begin{pmatrix} \alpha & \beta a \\ \beta & \alpha \end{pmatrix}, \chi_{n-1}^{-1}\chi_0(\varpi), \ldots, \chi_1^{-1}\chi_0(\varpi)\right) \in \mathrm{GL}_{2n}(\mathbb{C}),$$

with $\alpha^2 - a\beta^2 = \chi_0(\varpi)$.

We also let

$$t'_\pi = \mathrm{diag}\left(\chi_1(\varpi), \ldots, \chi_{n-1}(\varpi), \chi_{n-1}^{-1}\chi_0(\varpi), \ldots, \chi_1^{-1}\chi_0(\varpi)\right) \in \mathrm{GSp}_{2(n-1)}(\mathbb{C}). \tag{8.15}$$

Then, the analog of the Casselman-Shalika formula for G becomes

$$W^0(t_\lambda) = \delta_G(t_\lambda)^{1/2}\chi_\lambda^{\mathrm{GSp}_{2(n-1)}}(t'_\pi). \tag{8.16}$$

8.3. **Local Identity.** We now state and prove the main results of this section: the computation of the integrals with unramified data.

Let ω denote the character of $F^\times$ such that $\pi(e_0^*(\lambda)) = \omega(\lambda)\,\mathrm{Id}_{V_\pi}$ with e_0^* is as in Section 2. For $s \in \mathbb{C}$ consider the representation $\tau_s = \tau|\det|^s \otimes \omega^{-1}$ of $M(F) \cong \mathrm{GL}_n(F) \times \mathrm{GL}_1(F)$ in (case A), resp., of $M(F) \cong \mathrm{GL}_m(F) \times \mathrm{GL}_1(F)$ in (case B), and define ρ_s, a representation of $G(F)$, resp., $H(F)$, similar to the global setting in (5.1), resp., (4.2), as follows:

$$\rho_s = \begin{cases} \mathrm{Ind}_{P_n(F)}^{G(F)}\left(\tau_s\right), & \text{(case A)}, \\ \\ \mathrm{Ind}_{P_m(F)}^{H(F)}\left(\tau_s\right), & \text{(case B)}. \end{cases} \tag{8.17}$$

Let W_π^0 be the unique spherical and normalized Whittaker function in the ψ-Whittaker model of π. Also, let f_s^0 be unramified and normalized so that the vector-valued function $b \to f_s^0(e,b)$, taking values in the ψ^{-1}-Whittaker model of τ, is the normalized Whittaker function of the general linear group. Here, e denotes the identity element of $G(F)$ or $H(F)$ as appropriate.

Similar to the global situation, for $\mathrm{Re}(s) \gg 0$, we set

$$\xi(W_\pi^0, f_s^0) = \begin{cases} \displaystyle\int_{N_G(F)Z_G(F)\backslash G(F)} f_s^0(g; I_n) \int_{\mathcal{X}(F)} W_\pi^0(\lambda\delta_\ell g)\, d\lambda\, dg, & \text{(caseA)}, \\ \displaystyle\int_{N_G(F)Z_G(F)\backslash G(F)} W_\pi^0(g) \int_{N_\ell(F)\cap\beta^{-1}P_m(F)\beta\backslash N_\ell(F)} f_s^0\left(\beta u g; I_m\right) \Psi_\ell(u)^{-1}\, du\, dg, & \text{(caseB)}, \end{cases} \tag{8.18}$$

where β is as in Theorem 4.2 and δ_ℓ is as in Theorem 5.1.

As we mentioned before, there is a significant difference in the way unramified computation is carried out in (case A) versus (case B), due to the feasibility of the symmetric algebra decompositions. Therefore, we consider these two cases separately even though the statements of the results are similar. We first consider (case A).

Theorem 8.1. *Let π be an irreducible, admissible, unramified, globally ψ-generic representation of $H(F)$ where $H = \mathrm{GSpin}_{2m+1}$ referred to as the (odd case), or $H = \mathrm{GSpin}_{2m}$, possibly non-split quasi-split, referred to as the (even case). Assume that $n \le m$ in the (odd case), or $n < m$ in the (even case). Let τ be an irreducible, admissible, unramified, globally ψ^{-1}-generic representation of $\mathrm{GL}_n(F)$ as above. Choose W_π^0 and f_s^0 as before. With the Haar measures normalized as in (8.1), we have*

$$\xi(W_\pi^0, f_s^0) = \begin{cases} \dfrac{L(s, \pi\times\tau)}{L\left(2s, \tau, \wedge^2\otimes\omega\right)}, & \textit{(case A), odd,} \\[2ex] \dfrac{L(s, \pi\times\tau)}{L\left(2s, \tau, \mathrm{Sym}^2\otimes\omega\right)}, & \textit{(case A), even, split,} \\[2ex] \dfrac{L(s, \pi\times\tau)}{L\left(2s, \tau, \wedge^2\otimes\omega\right)}, & \textit{(case A), even, quasi-split.} \end{cases} \tag{8.19}$$

(Refer to the table in Section 7 for the details of the cases.)

Proof. Recall that we have the embeddings

$$\begin{aligned} G = \mathrm{GSpin}_{2n} &\hookrightarrow H = \mathrm{GSpin}_{2m+1}, \quad n\le m, \text{(case A–odd)}, \\ G = \mathrm{GSpin}_{2n+1} &\hookrightarrow H = \begin{cases} \mathrm{GSpin}_{2m}, & \text{split}, \\ \mathrm{GSpin}_{2m}^a, & \text{quasi-split non-split}, \end{cases} \quad n < m, \text{(case A–even)}. \end{aligned}$$

The above embeddings induce embeddings at the level of F-points. By the Iwasawa decomposition, we have

$$G(F) = N_G(F)T_G(F)K_G = N_G(F)Z_G(F)T_1(F)K_G(F), \tag{8.20}$$

where K_G is the maximal compact subgroup of $G(F)$, $Z_G(F)$ denotes the connected component of the center of $G(F)$, and

$$T_1 = \left\{ t = e_1^*(t_1) \cdots e_n^*(t_n) : t_i \in F^\times \right\}. \tag{8.21}$$

Below we will also employ t to denote the image of t under the embedding of $G(F)$ into $H(F)$ as well as the element $\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \in \mathrm{GL}_n(F)$, i.e., we are using t to denote the related elements in $\mathrm{GL}_n(F)$, $G(F)$ and $H(F)$. Given these identifications, we may write

$$f_s^0(t; I_n) = \delta_{P_n}^{1/2}(t) \mid \det t \mid^s W_\tau^0(t) \tag{8.22}$$

where $W_\tau^0(t)$ is the normalized spherical Whittaker function in the ψ^{-1}-Whittaker model of τ (a representation of $\mathrm{GL}_n(F)$).

The integrand on the right hand side of (8.18) is invariant under multiplying g on the right by an element in K_G and a central element on the left. Given our normalization of the Haar measures, this means that the integral reduces to

$$\int_{T_1(F)} W_\tau^0(t) \mid t_1 t_2 \dots t_n \mid^{s+u} \delta_G^{-1}(t) \int_{\mathcal{X}(F)} W_\pi^0(\lambda \delta_\ell t) \, d\lambda \, dt, \tag{8.23}$$

where

$$u = \begin{cases} \frac{n-2}{2}, & \text{odd case,} \\ \\ \frac{n-1}{2}, & \text{even case,} \end{cases} \tag{8.24}$$

and δ_G denotes the modulus function of the Borel subgroup of $G(F)$ (restricted to T_1).

Next, we dispose of the integration over $\mathcal{X}(F)$. Here, we can argue as in [20, p. 176] or [34, p. 98]. For each t we have $W_\pi^0(\lambda\delta_\ell t) = 0$ unless $\lambda \in \mathcal{X}(O_F) \subset K_H$. This follows from the facts that $\delta_\ell \in K_H$, that W_π^0 is right K_H invariant, and that $\delta_\ell t$ normalizes $\mathcal{X}(F)$, leading, by (5.2), to a change of variables $d\lambda \to |\det t|^{-\ell} d\lambda$. where, as in Section 5,

$$\ell = \begin{cases} m-n, & \text{odd case,} \\ \\ m-n-1, & \text{even case.} \end{cases} \tag{8.25}$$

Therefore, our integral reduces to

$$\zeta(s, W_\pi^0, W_\tau^0) = \int_{T_1(F)} W_\pi^0(t) W_\tau^0(t) |t_1 t_2 \dots t_n|^{s+u-\ell} \, \delta_G^{-1}(t) \, dt. \tag{8.26}$$

We are reduced to proving that

$$L(2s, \tau, \wedge^2 \otimes \omega) \cdot \zeta(s, W_\pi^0, W_\tau^0) = L(s, \pi \times \tau) \tag{8.27}$$

in the odd case,

$$L(2s, \tau, \mathrm{Sym}^2 \otimes \omega) \cdot \zeta(s, W_\pi^0, W_\tau^0) = L(s, \pi \times \tau) \tag{8.28}$$

in the even split case, and

$$L(2s, \tau, \wedge^2 \otimes \omega) \cdot \zeta(s, W_\pi^0, W_\tau^0) = L(s, \pi \times \tau) \tag{8.29}$$

in the even quasi-split case.

We prove the two sides of (8.27), resp., (8.28) and (8.29), have equal coefficients when expanded as power series in q^{-s}. We start by expanding $\zeta(s, W_\pi^0, W_\tau^0)$.

By [14, Lemma 5.1] we have $W_\pi^0(t) \equiv 0$ unless

$$|\alpha(t)| \leq 1 \tag{8.30}$$

for all simple roots α of H, and $W_\tau^0(t) \equiv 0$ unless

$$|\alpha(t)| \leq 1 \tag{8.31}$$

for all simple roots α of G appearing in the Levi M_n. Therefore, (8.30) implies that

$$\operatorname{ord}(t_1) \geq \cdots \geq \operatorname{ord}(t_n) \geq 0 \tag{8.32}$$

while (8.31) implies that

$$\operatorname{ord}(t_1) \geq \cdots \geq \operatorname{ord}(t_n). \tag{8.33}$$

Notice the crucial fact that the last inequality in (8.32) holds because of the structure of the root system of H (both odd and even case), and the fact that $n \leq m$ in the odd case and $n < m$ in the even case (and that $t_{n+1} = \cdots = t_m = 1$, i.e., they do not appear).

For δ as in (8.8) and $\bar{\delta}$ as in (8.9), set

$$\varpi^\delta = e_1^*(\varpi^{k_1}) \cdots e_n^*(\varpi^{k_n}) \in T_1(F) \subset G(F) \tag{8.34}$$

and write $\varpi^{\bar{\delta}}$ for its image in the maximal torus of $H(F)$ under the embedding. We will also use ϖ^δ to denote the diagonal element in $\mathrm{GL}_n(F)$. Also, write

$$\operatorname{tr}\delta = \operatorname{tr}\bar{\delta} = k_1 + k_2 + \cdots + k_m. \tag{8.35}$$

Our integral then reduces to

$$\zeta(s, W_\pi^0, W_\tau^0) = \sum_{\substack{\delta=(k_1,\cdots,k_n)\\ \text{dominant}}} W_\pi^0(\varpi^{\bar{\delta}}) W_\tau^0(\varpi^\delta) \delta_G^{-1}(\varpi^\delta) q^{-(s+u-\ell)\operatorname{tr}\delta}, \tag{8.36}$$

with u as in (8.24) and ℓ as in (8.25). Using the Casselman-Shalika formulas (8.13) in the split case and (8.16) in the quasi-split case, we have

$$W_\tau^0(\varpi^\delta) = \delta_{\mathrm{GL}_n}^{1/2}(\varpi^\delta) \chi_\delta^{\widehat{\mathrm{GL}}_n}(t_\tau) \tag{8.37}$$

and

$$W_\pi^0(\varpi^{\bar{\delta}}) = \delta_H^{1/2}(\varpi^{\bar{\delta}}) \cdot \begin{cases} \chi_{(\bar{\delta};\operatorname{tr}\delta)}^{\mathrm{GSp}_{2m}}(t_\pi), & \text{odd, split} \\ \\ \chi_{(\bar{\delta};\operatorname{tr}\delta)}^{\mathrm{GSO}_{2m}}(t_\pi), & \text{even, split} \\ \\ \chi_{(\bar{\delta};\operatorname{tr}\delta)}^{\mathrm{GSp}_{2(m-1)}}(t'_\pi), & \text{even, quasi-split} \end{cases} \tag{8.38}$$

with t'_π as in (8.15). Also, by (8.11), the right hand sides of (8.38) are equal to

$$\delta_H^{1/2}(\varpi^{\bar{\delta}}) \cdot \begin{cases} \mu(t_\pi)^{\operatorname{tr}\delta} \chi_{\bar{\delta}}^{\mathrm{Sp}_{2m}}(\bar{t}_\pi), & \text{odd, split,} \\ \\ \mu(t_\pi)^{\operatorname{tr}\delta} \chi_{\bar{\delta}}^{\mathrm{SO}_{2m}}(\bar{t}_\pi), & \text{even, split,} \\ \\ \mu(t'_\pi)^{\operatorname{tr}\delta} \chi_{\bar{\delta}}^{\mathrm{Sp}_{2(m-1)}}(\bar{t}'_\pi), & \text{even, quasi-split.} \end{cases} \tag{8.39}$$

(Note that by (8.15) we know that $\mu(t_\pi) = \mu(t'_\pi)$ in the even, quasi-split case.)

For $t \in T_1(F)$ as in (8.21) (where $t_{n+1} = \cdots = t_m = 1$) we see from the root data that

$$\delta_G(t) = \begin{cases} \left|t_1^{2n-2} t_2^{2n-4} \cdots t_n^0\right|, & \text{odd case,} \\ \\ \left|t_1^{2n-1} t_2^{2n-3} \cdots t_n^1\right|, & \text{even case,} \end{cases}$$

$$\delta_H(t) = \begin{cases} \left|t_1^{2m-1} t_2^{2m-3} \cdots t_n^{2m-2n+1}\right|, & \text{odd case,} \\ \\ \left|t_1^{2m-2} t_2^{2m-4} \cdots t_n^{2m-2n}\right|, & \text{even case,} \end{cases}$$

and

$$\delta_{\mathrm{GL}_n}(t) = \left|t_1^{n-1} t_2^{n-3} \cdots t_n^{1-n}\right|.$$

Therefore, for t as in (8.21)

$$\delta_G(t) = \delta_H^{1/2}(t) \cdot \delta_{\mathrm{GL}_n}^{1/2}(t) \cdot \begin{cases} |t_1 \cdots t_n|^{\frac{3n-2m-2}{2}}, & \text{odd case,} \\ \\ |t_1 \cdots t_n|^{\frac{3n-2m+1}{2}}, & \text{even case.} \end{cases} \tag{8.40}$$

Substituting in (8.36) we get

$$\zeta(s, W_\pi^0, W_\tau^0) = \begin{cases} \sum\limits_{\substack{\delta=(k_1,\cdots,k_n) \\ \text{dominant}}} \mu(t_\pi)^{\operatorname{tr}\delta} \chi_{\bar\delta}^{\mathrm{Sp}_{2m}(\mathbb{C})}(\bar t_\pi) \chi_\delta^{\widehat{\mathrm{GL}}_n}(t_\tau) q^{-s \operatorname{tr}\delta}, & \text{odd, split,} \\ \\ \sum\limits_{\substack{\delta=(k_1,\cdots,k_n) \\ \text{dominant}}} \mu(t_\pi)^{\operatorname{tr}\delta} \chi_{\bar\delta}^{\mathrm{SO}_{2m}(\mathbb{C})}(\bar t_\pi) \chi_\delta^{\widehat{\mathrm{GL}}_n}(t_\tau) q^{-s \operatorname{tr}\delta}, & \text{even, split,} \\ \\ \sum\limits_{\substack{\delta=(k_1,\cdots,k_n) \\ \text{dominant}}} \mu(t'_\pi)^{\operatorname{tr}\delta} \chi_{\bar\delta}^{\mathrm{Sp}_{2(m-1)}(\mathbb{C})}(\bar t'_\pi) \chi_\delta^{\widehat{\mathrm{GL}}_n}(t_\tau) q^{-s \operatorname{tr}\delta}, & \text{even, quasi-split.} \end{cases} \tag{8.41}$$

Next, recall the well-known identity

$$\det(I - AX)^{-1} = \sum_{r=0}^\infty \operatorname{tr}\left(\mathrm{Sym}^r(A)\right) X^r, \tag{8.42}$$

where A is an arbitrary square complex matrix and X is a sufficiently small complex variable. Applying this identity to $A = \left(\omega(\varpi) \wedge^2 (t_\tau)\right)$, resp., $A = \left(\omega(\varpi)\mathrm{Sym}^2(t_\tau)\right)$, we obtain

$$\begin{aligned} L(2s, \tau, R \otimes \omega) &= \det\left(I - \omega(\varpi)(Rt_\tau) q^{-2s}\right)^{-1} \\ &= \sum_{i=0}^\infty \operatorname{tr}\left(\mathrm{Sym}^i\left(\omega(\varpi) R t_\tau\right)\right) q^{-2is} \\ &= \sum_{i=0}^\infty \omega(\varpi)^i \operatorname{tr}\mathrm{Sym}^i\left(R t_\tau\right) q^{-2is} \end{aligned} \tag{8.43}$$

where

$$R = \begin{cases} \wedge^2 & \text{odd case,} \\ \mathrm{Sym}^2 & \text{even, split case,} \\ \wedge^2 & \text{even, quasi-split case.} \end{cases} \tag{8.44}$$

Multiplying (8.41) by (8.43), we see that the left hand side of (8.27), resp., (8.28), (8.29), is equal to

$$\sum_{r=0}^{\infty}\left[\sum_{2i+j=r}\operatorname{tr}\operatorname{Sym}^i\left(\wedge^2 t_\tau\right)\omega(\varpi)^i\sum_{\substack{\operatorname{tr}\delta=j\\ \delta\text{ dominant}}}\mu(t_\pi)^j\chi_{\bar\delta}^{\mathrm{Sp}_{2m}(\mathbb{C})}(\bar t_\pi)\chi_\delta^{\widehat{\mathrm{GL}}_n}(t_\tau)\right]q^{-sr} \tag{8.45}$$

in the odd case, to

$$\sum_{r=0}^{\infty}\left[\sum_{2i+j=r}\operatorname{tr}\operatorname{Sym}^i\left(\operatorname{Sym}^2 t_\tau\right)\omega(\varpi)^i\sum_{\substack{\operatorname{tr}\delta=j\\ \delta\text{ dominant}}}\mu(t_\pi)^j\chi_{\bar\delta}^{\mathrm{SO}_{2m}(\mathbb{C})}(\bar t_\pi)\chi_\delta^{\widehat{\mathrm{GL}}_n}(t_\tau)\right]q^{-sr} \tag{8.46}$$

in the even, split case, and to

$$\sum_{r=0}^{\infty}\left[\sum_{2i+j=r}\operatorname{tr}\operatorname{Sym}^i\left(\wedge^2 t_\tau\right)\omega(\varpi)^i\sum_{\substack{\operatorname{tr}\delta=j\\ \delta\text{ dominant}}}\mu(t_\pi)^j\chi_{\bar\delta}^{\mathrm{Sp}_{2(m-1)}(\mathbb{C})}(\bar t'_\pi)\chi_\delta^{\widehat{\mathrm{GL}}_n}(t_\tau)\right]q^{-sr} \tag{8.47}$$

in the even, quasi-split case.

On the other hand, the right hand sides of (8.27), resp., (8.28), (8.28), are

$$L(s,\pi\times\tau)=\det\left(I-(t_\pi\otimes t_\tau)q^{-s}\right)^{-1}=\sum_{r=0}^{\infty}\operatorname{tr}\operatorname{Sym}^r\left(t_\pi\otimes t_\tau\right)q^{-sr} \tag{8.48}$$

and it is enough to verify that $\operatorname{tr}\operatorname{Sym}^r(t_\pi\otimes t_\tau)$ is equal to the expression in brackets in (8.45) in the odd case, in (8.46) in the even split case, and in (8.47) in the even, quasi-split case. We do this with the help of our earlier discussion of the symmetric algebra decompositions in Section 8.1. Noting that $\mu(t_\pi)=\omega_\pi(\varpi)=\omega(\varpi)$, the equation (8.12) finishes the proof. □

Next we consider (case B).

Theorem 8.2. *Let π be an irreducible, admissible, unramified, globally ψ-generic representation of $G(F)$ where $G=\mathrm{GSpin}_{2n+1}$ referred to as the (odd case), or $G=\mathrm{GSpin}_{2n}$, split or non-split quasi-split, referred to as the (even case). Assume that $m>n$ in the odd case or $m\geq n$ in the even case. Let τ be an irreducible, admissible, unramified, globally ψ^{-1}-generic representation of $GL_m(F)$ as above. Also, choose W^0 and f_s^0 as before. With the Haar measures normalized as in (8.1), we have*

$$\xi(W_\pi^0,f_s^0)=\begin{cases}\dfrac{L(s,\pi\times\tau)}{L\left(2s,\tau,\wedge^2\otimes\omega\right)}, & \textit{(case B), odd,}\\[2ex] \dfrac{L(s,\pi\times\tau)}{L\left(2s,\tau,\operatorname{Sym}^2\otimes\omega\right)}, & \textit{(case B), even, split.}\\[2ex] \dfrac{L(s,\pi\times\tau)}{L\left(2(s),\tau,\wedge^2\otimes\omega\right)}, & \textit{(case B), even, quasi-split.}\end{cases} \tag{8.49}$$

Here, $\xi(W_\pi^0,f_s^0)$ is as in (case B) of (8.18). (Again refer to the table in Section 7 for the details of the cases.)

Proof. The proof in the non-split quasi-split case is a little more involved than the split case so we present them separately, indicating the extra issues in that case. In the split cases, we follow Soudry's technique for the odd special orthogonal groups [34, §12] and its adaptation to the even special orthogonal case by Kaplan [24, §3.2.3].

The split cases. We first deal with the odd case. The even split case will be similar, as we explain below. Since π is in general a quotient of a full parabolically induced representation from an unramified representation of the Siegel Levi, we may assume that

$$\pi = \mathrm{Ind}_{\bar{P}_n(F)}^{G(F)}(\sigma \otimes \omega)$$

with σ an unramified representation of $\mathrm{GL}_n(F)$ and $\omega = \omega_\pi$ the central character of π. Here, $\bar{P}_n = M_n \bar{N}_n$ denotes the opposite of the Siegel parabolic P_n (cf. 3.2.2).

Let ϕ be a function in V_π and assume that for $g \in G(F)$, $\phi(g)$ takes values in $\mathcal{W}(\sigma \otimes \omega, \psi)$. As a function of g it is smooth and

$$\phi(a_0 \bar{y} g) = |\det a_0|^{-n/2} \phi(g)$$

for $\bar{y} \in \bar{N}_n(F)$ and $a_0 \in \mathrm{GL}_n(F)$ considered as the factor in the Siegel Levi of $G(F)$. Consider

$$W_\phi(g) = \int\limits_{N_n(F)} \phi(yg) \psi_n^{-1}(y)\, dy, \tag{8.50}$$

with ψ_n obtained from ψ as in Section 4 (with everything local now). We formally have

$$W_\phi(ug) = \psi_n(u) W_\phi(g), \quad u \in N_n(F), g \in G(F).$$

However, the integral (8.50) may not converge absolutely. To remedy this problem, we replace σ by

$$\sigma_\zeta = \sigma \otimes |\det \cdot|^{-\zeta}$$

for $\mathrm{Re}(\zeta) \gg 0$. Replacing σ by σ_ζ and taking a holomorphic section ϕ_ζ instead of ϕ, we see just as in [34] that the integral defining W_{ϕ_ζ} converges absolutely for $\mathrm{Re}(\zeta)$ large enough and has a continuation to a holomorphic function on the whole plane. This is seen exactly as in the case of special orthogonal groups by noting that the integral (8.50) always has a principal value and if ϕ_ζ is a standard section, this principal value is a polynomial in $q^{-\zeta}$. Thus, for $\mathrm{Re}(\zeta)$ large enough, $W_{\phi_\zeta}(g)$ is a polynomial in $q^{-\zeta}$ which provides the holomorphic continuation (cf. [34, §12]).

Choose the vector $\phi^0_\zeta = \phi^0_{\sigma,\zeta}$ that gives the normalized unramified Whittaker function $W^0_\sigma \in \mathcal{W}(\sigma, \psi)$. It follows from the Casselman-Shalika formula [14] (see also [32, Remark 3.5.14]) that

$$W^0_\pi = W_{\phi^0_{\sigma,\zeta}}(I)^{-1} W_{\phi^0_{\sigma,\zeta}} = L(1 + 2\zeta, \hat{\sigma}, \mathrm{Sym}^2 \otimes \omega) W_{\phi^0_{\sigma,\zeta}}.$$

The proof now proceeds the same way as in [34, §12] and we briefly review the steps indicating the points of difference with our case, which include the appearance of the twisted versions of the symmetric and exterior square L-functions. We should also mention that the local γ- and ϵ-factors enter in the steps, which we did not define earlier. However, these factors, which are defined via applying intertwining operators to ρ_s, are defined for the GSpin groups in [28].

Using the above expression for W_π^0 and proceeding as in [34, §12] we see that $\xi(W^0, f_s^0)$ is equal to

$$\frac{L(1+2\zeta, \hat{\sigma}, \mathrm{Sym}^2 \otimes \omega)}{\gamma(s-\zeta, \sigma \times \tau, \psi^{-1}) L(2s, \tau, \wedge^2 \otimes \omega)}$$

times a double integral. This double integral can be manipulated just as in [34, pp. 97–98] as the presence of the center in the GSpin case does not disturb those arguments. As a result, the double integral is seen to represent the Rankin-Selberg convolution for $\mathrm{GSpin}(2m) \times \mathrm{GL}(n)$ (as in (case A) we considered earlier), which is equal to

$$\frac{L(\zeta_1 + (s-1/2), \tau \times \hat{\sigma}\omega) L(\zeta_1 - (s-1/2), \hat{\tau} \times \hat{\sigma})}{L(2\zeta_1, \hat{\sigma}, \mathrm{Sym}^2 \otimes \omega)}.$$

with $\zeta_1 = \zeta + \frac{1}{2}$. As we pointed out above, the only new phenomenon is the appearance of the central character ω and the twisted symmetric square L-function in the above expression. Therefore,

$$\xi(W^0, f_s^0) = \frac{L(s-\zeta, \sigma \times \tau) L(s+\zeta, \hat{\sigma}\omega \times \tau)}{L(2s, \tau, \wedge^2 \otimes \omega)}.$$

Finally, note that the above equalities are all in the sense of equality of rational functions in q^{-s} and $q^{-\zeta}$ in their region of convergence and have analytic continuation to $\zeta = 0$. As such, since they are defined for $\zeta = 0$, we may indeed substitute $\zeta = 0$. Consequently we obtain

$$\xi(W^0, f_s^0) = \frac{L(s, \sigma \times \tau) L(s, \hat{\sigma}\omega \times \tau)}{L(2s, \tau, \wedge^2 \otimes \omega)} = \frac{L(s, \pi \times \tau)}{L(2s, \tau, \wedge^2 \otimes \omega)}.$$

This finishes the proof in the odd case. The even case proceeds in a similar way. The steps, for the even special orthogonal groups, are detailed in [24, §3.2.3]. As above, the same steps go through in the split general spin groups, except for the fact that again ω appears and now twisted symmetric square L-function shows up. Again, similar to the above, we will have

$$\xi(W^0, f_s^0) = \frac{L(s, \pi \times \tau)}{L\big(2s, \tau, \mathrm{Sym}^2 \otimes \omega\big)}.$$

The non-split quasi-split case. It only remains to consider the case where G is non-split, quasi-split, i.e., when $G = \mathrm{GSpin}_{2n}^* = \mathrm{GSpin}_{2n}^a$, as in Section 2.2.3, and $H = \mathrm{GSpin}_{2m+1}$ with $m \geq n$. (Recall that a quasi-split GSpin_{2n+1} is already split, so covered above.) In this case, the technique described above does not quite extend in a straightforward way due to "the presence of the compact modulo center GSpin_2^* in the middle" so we need to make some modifications in the proof. Here, we can follow the similar proof for the case of non-split quasi-split even special orthogonal groups due to E. Kaplan in his thesis [24, §3.2.3 and §7.2.3] and in [26]. Essentially, we may assume the representation π of GSpin_{2n}^* is (a quotient of) an induced representation from $\mathrm{GL}_{n-1} \times \mathrm{GSpin}_2^*$ and Kaplan's arguments go through because the presence of the center in the general spin groups does not impact those arguments. Along the way one also needs a local multiplicity at most one result [1, 30], which fortunately for the case of $\mathrm{GSpin}_{2n} \times \mathrm{GL}_m$ with $n \leq m$ is now available due to the work of E. Kaplan, J.-F. Lau and B. Liu [28, Theorem 3.3]. Similar results hold for the case of $\mathrm{GSpin}_{2n+1} \times \mathrm{GL}_m$ with $n < m$ as in [34, §8.3], where it is worked out for the odd special orthogonal groups, and similar to [28]. (There is typo in [34, §8.3] where (1.2.4) is for $\ell < n$ in the notation of that paper.) To complete the proof in the case of special orthogonal groups Kaplan uses a formal identity of power series [26, (4)] which follows from [9], particularly its Appendix, and one would use an analog of that for GSpin groups. □

9. Global L-functions

We now state the major consequence of the above discussions in the global setting, which we need. We use the notation of the earlier sections.

Let G and H be as in (case A), resp. (case B), of the Table in Section 7. In (case A) consider $H \times \mathrm{GL}_n$ as the Levi of a maximal parabolic inside a larger GSpin group of the same type and similarly in (case B) consider $G \times \mathrm{GL}_m$ as the Levi of a maximal parabolic inside the larger GSpin group. This is the setup for the Langlands-Shahidi method and in the cases we are considering the adjoint action of the dual of the Levi on the unipotent radical of the dual parabolic decomposes into two irreducible components. The first component is the tensor product representation, leading to the Rankin-Selberg L-functions. The second component, ϱ, will be either the twisted symmetric square or the twisted exterior square representation of the complex $\mathrm{GL}_n \times \mathrm{GL}_1$, resp. $\mathrm{GL}_m \times \mathrm{GL}_1$. To be more precise, we have

$$\varrho = \begin{cases} \wedge^2 \otimes \omega, & \text{(case A) or (case B) , odd,} \\ \mathrm{Sym}^2 \otimes \omega, & \text{(case A) or (case B), even, split,} \\ \wedge^2 \otimes \omega, & \text{(case A) or (case B), even, quasi-split,} \end{cases} \tag{9.1}$$

where ω denotes the character on GL_1.

Theorem 9.1. *Let π be a unitary, cuspidal, globally generic, automorphic representation of $H(\mathbb{A})$ in (case A), resp. of $G(\mathbb{A})$ in (case B), and let τ be a unitary, cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$, resp. of $\mathrm{GL}_m(\mathbb{A})$. Let ω be as in* (4.1) *(essentially the central character ω_π of π). Let S be a sufficiently large finite set of places, including all the archimedean places, such that for $v \notin S$ all data are unramified. Then we have*

$$\mathcal{L}(\varphi, f_s) = \frac{L^S(s, \pi \times \tau)}{L(2s, \tau, \varrho)} \cdot R(s), \tag{9.2}$$

where, $R(s)$ is a meromorphic function, which can be made holomorphic and nonzero in a neighborhood of any given $s = s_0$ for an appropriate choice of φ and f_s as in (4.2) *or* (5.1). *Here, $\mathcal{L}(\varphi, f_s)$ is as in Theorem 4.2 or Theorem 5.1, as appropriate, and ϱ is as in* (9.1)

Proof. The theorem follows from Theorems 8.1 and 8.2 if we take $R(s)$ to be equal to the product of the local zeta integrals (7.3) over $v \in S$. The fact that $R(s)$ is meromorphic is clear. To show that it can be made holomorphic in the neighborhood of any point $s = s_0$ the argument in [34, §§6-7] applies. The presence of the nontrivial center in the GSpin case does not have an impact on those arguments. □

As a corollary we obtain the following result, which is the precise statement we already used in an earlier work as [7, Prop. 4.9].

Proposition 9.2. *Let π be globally generic unitary cuspidal automorphic representations of $\mathrm{GSpin}_{2n+1}(\mathbb{A})$ or $\mathrm{GSpin}_{2n}(\mathbb{A})$ and let τ be a unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, as above. If $L^S(s, \pi \times \tau)$ has a pole at s_0 with $\mathrm{Re}(s_0) \geq 1$, then for a choice of f_s the Eisenstein series $E(h, f_s)$ has a pole at $s = s_0$.*

Proof. This statement follows immediately from Theorem 9.1. Recall that any pole of $\mathcal{L}(\varphi, f_s)$ must come from a pole of the Eisenstein series that is used to define it. Moreover, the twisted symmetric and twisted exterior square L-functions appearing on the right hand sides in Theorem 9.1 are holomorphic in $\mathrm{Re}(s) \geq 1$ so they can not cancel a possible pole of $L(s, \pi \times \tau)$.

We note that it follows from [15, Appendix] that the above argument holds for generic representations with respect to an arbitrary ψ and any particular "standard" one that we may have fixed. This issue is relevant when the some of our groups are not of adjoint type, as here. See [15, Appendix] for more details. □

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Department of Mathematics, Oklahoma State University, Stillwater, OK 74078–1058, USA

Email address: `asgari@math.okstate.edu`

Department of Mathematics, Ohio State University, Columbus, OH 43210, USA

Email address: `cogdell@math.osu.edu`

Mathematics Department, Purdue University, West Lafayette, IN 47907, USA

Email address: `shahidi@math.purdue.edu`