Preliminary Version

SOME TWO-ADIC DOUBLE COSETS

MAHDI ASGARI AND DAN BARBASCH

The *Iwahori-Hecke algebras* are important tools in the study of representations of p-adic groups among other things (cf. [Bu]). One often considers the algebra of complex valued functions that are bi-invariant under an *Iwahori* subgroup. However, it also desirable to consider such algebras for "deeper" compact open subgroups. This is particularly interesting, and often overlooked, for the dyadic fields ($p = 2$). The dyadic case is particularly important for Representation Theory of non-linear covers of linear algebraic groups (cf. [AB]). In this short note we consider a related combinatorial question (cf. Questions 1 and 2 below) for the special linear group.

Let $\mathbb{F} = \mathbb{Q}_2$ denote the two-adic local field and let $\mathcal{O} = \mathbb{Z}_2$ be its ring of integers, containing the unique maximal ideal $\mathcal{P} = \varpi\mathcal{O}$, generated by the fixed uniformizer $\varpi = 2$.

Let $G = \mathrm{SL}_n(\mathbb{F})$, the special linear group of $n \times n$ matrices with entries in $\mathbb{F}$ and determinant 1. As usual, we let T denote the diagonal matrices, U^+ the unipotent upper triangular matrices, and U^- the unipotent lower triangular matrices in SL_n. The roots of G (in the usual Bourbaki notation) will be denoted by $\Phi = \Phi^+ \sqcup \Phi^-$, with Φ^+ the positive roots and $\Phi^- = -\Phi^+$ the negative roots. Also, let $\mathcal{W} \cong \mathcal{S}_n$ denote the Weyl group of G.

For an intger $m \geq 0$, let $U^+_m = U^+(\mathcal{P}^m)$ and $U^-_m = U^-(\mathcal{P}^m)$. Moreover, let $T_0 = T(\mathcal{O}^\times)$. The standard *Iwahori subgroup* of G will be denoted by $\mathcal{I}$. In other words, $\mathcal{I}$ is the subgroup of $\mathrm{SL}_n(\mathcal{O})$ generated by T_0, U^+_0 and U^-_1. Also, let $\mathcal{J}$ denote the subgroup generated by T_0, U^+_0 and U^-_2. In other words, $\mathcal{I}$, resp. $\mathcal{J}$, is the subgroup of matrices in $\mathrm{SL}_n(\mathcal{O})$ that are unipotent upper triangular when reduced modulo $\mathcal{P}$, resp., $\mathcal{P}^2$.

In this short note, we would like to consider the following.

Question 1. *How many distinct $\mathcal{J}$-double cosets are there in $\mathcal{I}$?*

And more generally:

Question 2. *For $w \in \mathcal{W}$ how many distinct $\mathcal{J}$-double cosets are there in $\mathcal{I}w\mathcal{I}$?*

In other words, we would like to compute the cardinality of $\mathcal{J}\backslash\mathcal{I}w\mathcal{I}/\mathcal{J}$. As is well known, these types of questions are related to the study of dyadic Iwahori-Hecke algebras for "deeper" subgroups of the Iwahori, such as the group $\mathcal{J}$. The answer turns out to be a bit more straightforward to state for for $w = 1$ (cf. Theorem 3) than it is for a general Weyl element w (cf. Theorem 13).

We consider the case of $w = 1$ first.

Theorem 3. *Let $\mathbb{F} = \mathbb{Q}_2 \supset \mathcal{O} = \mathbb{Z}_2 \supset \mathcal{P} = \varpi\mathcal{O}$ and let $G = \mathrm{SL}_n(\mathbb{F}) \supset \mathcal{I} \supset \mathcal{J}$, as above. Then $\mathcal{I}$ is a disjoint union of $B(n)$ distinct $\mathcal{J}$-double cosets. Here, $B(n)$ denotes the n-th Bell number, i.e., the number of partitions of the set $\{1, 2, \ldots, n\}$. (See Remark 4 for more details about $B(n)$.)*

Remark 4 (Bell and Stirling Numbers). For an integer $n \geq 0$, the n-th Bell number $B(n)$ (named after Eric T. Bell) is defined to be the number of all possible partitions of the set $[n] = \{1, 2, \ldots, n\}$, or equivalently, the number of equivalence relations on $[n]$. For example, $B(0) = B(1) = 1$, $B(2) = 2$, $B(3) = 5$, $B(4) = 15$, $B(5) = 52$, etc. In fact, an elementary counting argument shows that the Bell numbers satisfy the recurrence relation

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k). \tag{5}$$

We also recall the *Stirling number of the second kind* $S(n,k)$, which denotes the number of partitions of $[n]$ into k non-empty parts. For $n \geq 1$ we clearly have

$$B(n) = \sum_{k=1}^{n} S(n,k). \tag{6}$$

See [St, (1.94a)–(1.94f)] for further basic formulas for the Bell and Stirling numbers.

Proof of Theorem 3. Using the usual notation, for $a \in \mathbb{F}$ and $i \neq j$, let us write $x_{ij}(a)$ for the root group associated with the root $\epsilon_i - \epsilon_j$ of SL_n. In other words, $x_{ij}(a)$ is the element of G with 1's on the diagonal, a as the (i,j)-entry, and zeros everywhere else. For $i > j$ let us write $y_{ij} = x_{ij}(\varpi)$. Also, for any subset $S \subseteq \Phi^- = \{(i,j) : n \geq i > j \geq 1\}$, where Φ^- is equipped with a fixed total order, the choice of which will not matter, write

$$y_S = \prod_{(i,j) \in S} y_{ij} \quad \text{(in the fixed order).}$$

Finally, write

$$\mathcal{Y} = \{y_S : S \subseteq \Phi^-\}. \tag{7}$$

Since we have the Iwahori factorizations

$$\mathcal{I} = U^-(\mathcal{P})T(\mathcal{O}^\times)U^+(\mathcal{O}) \quad \text{and} \quad \mathcal{J} = U^-(\mathcal{P}^2)T(\mathcal{O}^\times)U^+(\mathcal{O}),$$

for $g = u^- t u^+ \in \mathcal{I}$, we have $\mathcal{J} g \mathcal{J} = \mathcal{J} u^- \mathcal{J}$ and we clearly have a *possibly non-disjoint* union

$$\mathcal{I} = \bigcup_{y \in \mathcal{Y}} \mathcal{J} y \mathcal{J}. \tag{8}$$

We need to turn this into a disjoint union and count the number of double cosets that survive.

Multiplying $y = y_S$ on the left or the right by elements in T_0 or U_2^- does not change the double coset $\mathcal{J} y \mathcal{J}$. Next, we consider the effect of multiplication by elements in U_0^+. Notice that U_1^-/U_2^- is in bijection with the $y_S \in \mathcal{Y}$ as in (7) and, for $i > j$ and $i' > j'$, the commutators $[y_{ij}, y_{i'j'}] \in U_2^-$. Hence the order of the terms in y_S does not matter and we have $U_1^-/U_2^- \cong U^-(\mathbb{F}_2)$, where $\mathbb{F}_2 = \{0, 1\}$.

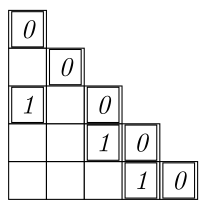
Reducing modulo $\mathcal{P}^2$ we must consider the effect of multiplying an element of $U^-(\mathbb{F}_2)$ on the left or the right with $x_{ij}(1)$ for $1 \leq i < j \leq n$. An element $u \in U^-(\mathbb{F}_2)$ can be uniquely represented by a unipotent lower triangular, equivalently strictly lower triangular, matrix with entries in $\mathbb{F}_2$ and

- multiplication of u on the left by $x_{ij}(1)$ with $i < j$ amounts to adding row j to row i (keeping the entries on or above the diagonal zero), and
- multiplication of u on the right by $x_{ij}(1)$ with $i < j$ amounts to adding column i to column j (keeping the entries on or above the diagonal zero).

The number of $\mathcal{J}$-double cosets in $\mathcal{I}$ is equal to the number of equivalence classes of the equivalence relation on $U^-(\mathbb{F}_2)$ defined by these row and and column operations. Each class may be uniquely represented by its reduced "echelon" form, which is a strictly lower triangular $n \times n$ matrix with each row and each column containing at most a single 1. In other words, we would like to find the number of *rook placements* in a strictly lower triangular $n \times n$ board.

The number of such representatives is the same as the number of partitions of $[n]$, i.e., $B(n)$. While we prove a more general version of this below we give a direct proof for this particular case. It is enough to observe that the number of placements of k rooks in the strictly lower triangular board is equal to $S(n, n-k)$. The bijection between the rook placements and partitions is given as follows. The strictly lower triangular board has a 1 in entry (i, j) (with $i > j$) if and only if i and j appear in the same part of the corresponding partition. Now, a placement with k entries 1 gives a partition with $n - k$ parts and hence the number of such placements is equal to $S(n, n-k)$. Our claim follows from (6). □

Example 9. *Let $n = 5$ and $k = 3$. Here is an example of a "rook placement" with 1's in positions $(3,1)$, $(4,3)$, and $(5,4)$, and its corresponding 2-partition of $[5]$:*



$$\longleftrightarrow \quad [5] = \{1,3,4,5\} \sqcup \{2\}$$

The number of placements with three entries 1 is $S(5,2) = 15$, the number of partitions of $[5]$ into two parts.

Next, we consider the non-trivial Weyl elements. We first review some combinatorial background. We refer to [St, §2.3–2.4] for the details.

A *board* is a subset $B \subseteq [n] \times [n]$. For a fixed B, let r_k denote the number of k-subsets of B such that no two elements have a common coordinate. Equivalently, r_k is the number of ways to place k non-attacking rooks on B. The *rook polynomial* of B is

$$r_B(x) = \sum_k r_k x^k. \tag{10}$$

Given $0 < b_1 \leq \cdots \leq b_m$, the *Ferrers board* of shape $(b_1, \ldots, b_m)$ is the board

$$B = \{(i,j) \,:\, 1 \leq i \leq m, 1 \leq j \leq b_i\}.$$

(While $b_i = 0$ is allowed for the following theorem, we will not need it for our purposes.) Note that B is a reflection and rotation of the Young diagram of the partition $\lambda = (b_m, \ldots, b_1)$. We will use the following result from Stanley's book.

Theorem 11 ([St, 2.4.1])**.** *Let $\sum r_k x^k$ be the rook polynomial of the Ferrers board B of shape $(b_1, \ldots, b_m)$. Set $s_i = b_i - i + 1$. Then*

$$\sum_k r_k \cdot (x)_{m-k} = \prod_{i=1}^{m} (x + s_i). \tag{12}$$

Here, $(x)_\ell = x(x-1)\cdots(x-\ell+1)$ for $\ell \geq 1$, and $(x)_0 = 1$ by convention.

To each $w \in \mathcal{W}$ we associate a Ferrers board (equivalently a Young diagram) as follows. For $1 \leq j \leq n$ let $b'_j = \#\{i : i > j \text{ and } w(i) > w(j)\}$. Rearrange the non-zero b'_j's into a non-decreasing sequence $(b_1, \ldots, b_m)$ and let B_w denote the associated Ferrers board.

Theorem 13. *Let $\mathbb{F} = \mathbb{Q}_2 \supset \mathcal{O} = \mathbb{Z}_2 \supset \mathcal{P} = \varpi\mathcal{O}$ and let $G = \mathrm{SL}_n(\mathbb{F}) \supset \mathcal{I} \supset \mathcal{J}$, as before. Assume that w is an element of the Weyl group $\mathcal{W}$ and let B_w denote the Ferrers board associated with w as above. Then the number of distinct $\mathcal{J}$-double cosets in $\mathcal{I}w\mathcal{I}$ is*

$$r_{B_w}(1) = \sum_k r_k,$$

where $r_{B_w}(x)$ is the rook polynomial of B_w as in (10)*. The values of r_k may be found from* (12)*.*

In particular, when $w = 1$ the associated Ferrers board is given by $(1, 2, \ldots, n-1)$*. Then $r_k = S(n, n-k)$, $r_B(1) = B(n)$ and we recover Theorem 3.*

Proof. Let $1 \neq w \in \mathcal{W}$. Similar to (8), we have a *possibly non-disjoint* union

$$\mathcal{I}w\mathcal{I} = \bigcup_{y_L, y_R \in \mathcal{Y}} \mathcal{J}\, y_L\, w\, y_R\, \mathcal{J}, \tag{14}$$

with $\mathcal{Y}$ as in (7). Define

$$\begin{aligned} S_R(w) &= \{(i,j) : i > j \text{ and } w(i) > w(j)\}, \text{ and} \\ S_L(w) &= S_R(w^{-1}). \end{aligned}$$

Observe that if $(i,j) \notin S_R(w)$, then $w y_{ij} w^{-1} \in U_1^+$ so $\mathcal{J} w y_{ij} \mathcal{J} = \mathcal{J} w \mathcal{J}$. Similarly, if $(i,j) \notin S_L(w)$, then $\mathcal{J} y_{ij} w \mathcal{J} = \mathcal{J} w \mathcal{J}$. Therefore, we may reduce (14) to

$$\mathcal{I}w\mathcal{I} = \bigcup_{\substack{y_L \in Y_L(w) \\ y_R \in Y_R(w)}} \mathcal{J}\, y_L\, w\, y_R\, \mathcal{J}, \tag{15}$$

where $Y_L(w) = \{y_S : S \subseteq S_L(w)\}$ and $Y_R(w) = \{y_S : S \subseteq S_R(w)\}$. Notice that (15) may still be a non-disjoint union.

Next, observe that any $y_L \in Y_L(w)$ may be moved across w to some $y_R \in Y_L(w)$ modulo U_2^- and vice versa. A similar argument as in the proof of Theorem 3 shows that each double coset $\mathcal{J}\, y_L\, w\, y_R\, \mathcal{J}$ is determined by strictly lower triangular matrices u_L and u_R with u_R having $\mathbb{F}_2$-entries in positions in $S_R(w)$ and u_L having $\mathbb{F}_2$-entries in positions belonging to $S_L(w)$. The "row moves" from before now modify u_L while the "column moves" modify u_R. We may also move u_L and u_R across w. After applying the row/column moves to simplify each u_L and u_R as much as possible, we may move u_L across w to arrive at a double coset $\mathcal{J} w u_R \mathcal{J}$, with u_R having entries in the positions belonging to $S_R(w)$. (We may also choose to move u_R across w and arrive at a double coset of the form $\mathcal{J} u_L w \mathcal{J}$.) Consequently, J-double coset in $\mathcal{I}w\mathcal{I}$ is determined by strictly lower triangular matrix with $\mathbb{F}_2$ entries in positions belonging to $S_R(w)$ where at most a single 1 may appear in each row and each column. This is precisely the number of "rook placements" on the Ferrers board B_w defined above. (Had we chosen to go with u_L we would have the Ferrers board $B_{w^{-1}}$ here.) The number of rook placements on B_w containing k non-zero entries is r_k and the total number of J-double cosets is therefore $\sum_k r_k$. The values of r_k, and their sum, may then be calculated using Theorem 11.

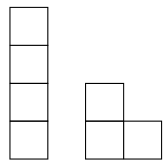
We also point out that when $w = 1$, the above argument works. We simply have

$$S_L = S_R = \{(i,j) \,:\, i > j\}$$

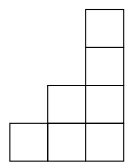
and y_L and y_R coalesce into a single term. The Ferrers board for $w = 1$ is the full strictly lower triangular board. Then $r_k = S(n, n-k)$ and $\sum_k r_k = B(n)$. □

Remark 16. We could write down a full list of representative for the $\mathcal{J}$-double cosets in $\mathcal{I}w\mathcal{I}$ using the rook placements of B_w.

Example 17. *Let $n = 5$. Take $w \in \mathcal{W}$ to be the cyle $(2543) \in \mathcal{S}_5$. Then $S_R(w)$ consists of the following strictly lower triangular entries:*



In our earlier notation, we have $(b_1', b_2', b_3', b_4') = (4, 0, 2, 1)$, $(b_1, b_2, b_3) = (1, 2, 4)$ (and $m = 3$). The Ferrers board B_w is the following.



To find the number of double cosets $\sum_k r_k$ we proceed as follows. Recall that $s_i = b_i - i + 1$ so $(s_1, s_2, s_3) = (1, 1, 2)$. Theorem 11 in this case gives

$$r_0\, x(x-1)(x-2) + r_1\, x(x-1) + r_2\, x + r_3 = (x+1)^2(x+2),$$

which implies that

$$r_0 = 1, \quad r_1 = 7, \quad r_2 = 10, \quad r_3 = 2$$

and

$$\#\left(\mathcal{J}\backslash\mathcal{I}w\mathcal{I}/\mathcal{J}\right) = 1 + 7 + 10 + 2 = 20.$$

References

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Oklahoma State University, Department of Mathematics, Stillwater, OK, USA
Email address: `mahdi.asgari@okstate.edu`

Department of Mathematics, Cornell University, Ithaca, NY, USA.
Email address: `dmb14@cornell.edu`