LECTURE 4

Autonomous Systems and Stability

An autonomous system is a system of ordinary differential equations of the form

\[
\frac{dx_1}{dt} = F_1 (x_1, \ldots, x_n) \\
\frac{dx_2}{dt} = F_2 (x_1, \ldots, x_n) \\
\vdots \\
\frac{dx_n}{dt} = F_n (x_1, \ldots, x_n)
\]

or, in vector notation,

\[ x' = F(x) \]

That is to say, an autonomous system is a system of ODEs in which the underlying variable \( t \) does not appear explicitly in the defining equations.

For example, homogeneous linear systems of the form

\[ x' = Ax \]

where \( A \) is a constant matrix are autonomous (with \( F(x) = Ax \)).

What’s especially nice about autonomous linear systems is that the associated direction field plots are independent of \( t \), and so one can get a fairly good idea of what the solutions look like by staring the direction field associated with the vector-valued function \( F(x) \).

In this lecture we shall focus on the behavior of solutions near critical points.

DEFINITION 4.1. A point \( x_0 \) is a critical point for an autonomous system

\[ x' = F(x) \tag{1} \]

if \( F(x_0) = 0 \).

The first thing to point out is that if \( x_0 \) is a critical point of (1) then the constant function

\[ x(t) = x_0 \]

is always a solution of the differential equation: for

\[ 0 = \frac{dx_0}{dt} = \frac{d}{dt} x(t) = F(x(t)) = F(x_0) = 0 \]

DEFINITION 4.2. A critical point \( x_0 \) of an autonomous system is said to be stable if the following condition holds:

- For any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that if \( \Phi(t) \) is a solution of \( \dot{x} = F(x) \) satisfying

\[ \| \Phi(0) - x_0 \| < \delta \]
then for all $t > 0$, $\Phi(t)$ exists (as a solution) and satisfies
\[ \| \Phi(t) - x_0 \| < \varepsilon. \]

That is, is to say that if $x_0$ is a critical point, then if a solution “starts off close to $x_0$”, then it stays close to $x_0$ for all positive $t$.\(^1\)

A related but distinct concept is that of an asymptotically stable critical point:

**Definition 4.3.** A critical point $x_0$ of an autonomous system is said to be **asymptotically stable** if the following condition holds:

- There exists a $\delta_0 > 0$ such that if $\Phi(t)$ is a solution of $x' = F(x)$ satisfying
  \[ \| \Phi(0) - x_0 \| < \delta_0 \]
  then
  \[ \lim_{t \to \infty} \Phi(t) = x_0. \]

Asymptotic stability is a bit stronger than mere stability; because the stability just requires that solutions that start near a critical point never stray far from that critical point, asymptotic stability, on the other hand, requires that solutions starting near the critical point $x_0$ have to eventually (well, in the limit $t = \infty$ anyway) settle in at $x_0$.

Here is a simple physical situation that distinguishes between asymptotically stable and stable critical points. Consider a pendulum.

![Diagram of a pendulum](image)

One possible motion is the pendulum just resting at $\theta = 0$. This corresponds to the solution at a critical point. If you displace the pendulum by a small angle $\delta$ and then let go, there are two possibilities.

In the frictionless case, the pendulum just rocks back and forth indefinitely between $\theta = -\delta$ and $\theta = \delta$, but always with $|\theta(t) - 0| \leq \delta$. In this situation $\theta = 0$ is a stable critical point but not an asymptotically stable critical point (as the pendulum continues to rock back and forth forever).

In the (more realistic) case where there is friction at play, then eventually the rocking motions die down to the steady rest position. So for a damped pendulum (meaning a pendulum with friction acting) $\theta = 0$ is an asymptotically stable critical point.

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\(^1\)This is the geometric interpretation given in the text; but it’s not quite accurate. What the $\varepsilon, \delta$ test really says is that if $x_0$ is a critical point, by forcing solutions to pass through a small enough $\delta$-neighborhood around $x_0$, you can always ensure that they never stray outside an $\varepsilon$-neighborhood of $x_0$.\]
Let’s now look at the example of a pendulum a bit more quantitatively. The rotational motion analog of Newton’s second law is

\[ \text{torque} = (\text{rotational inertia}) (\text{angular acceleration}) \]

or

\[ \frac{d^2 \theta}{dt^2} = \frac{1}{mL^2} \left[ -mgL \sin \theta - cL \frac{d\theta}{dt} \right] \]

Here \( mL^2 \) is the moment of inertia of a mass \( m \) displaced from the pivot point by a distance \( L \) (the length of the pendulum string). The term \(-mgL \sin \theta = r \times F_g\) (or at least the relevant component on the right hand side). The term \(-cL \frac{d\theta}{dt}\) represents the force of friction. We can rewrite this equation as

\[ \frac{d^2 \theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin \theta = 0 \]

where

\[ \gamma = \frac{c}{mL} \]
\[ \omega = \sqrt{\frac{g}{L}} \]

To view this second order differential equation as an autonomous system of first order differential equations we set

\[ x_1 = \theta \]
\[ x_2 = \frac{d\theta}{dt} \]

to get

\[ \frac{dx_1}{dt} = x_2 \]
\[ \frac{dx_2}{dt} = -\omega^2 \sin (x_1) - \gamma x_2 \]

Thus, for this system

\[ F(x) = \begin{bmatrix} x_2 \\ -\omega^2 \sin (x_1) - \gamma x_2 \end{bmatrix} \]

The critical points are then the solution of \( F(x) = 0 \)

\[ \implies \begin{cases} x_2 = 0 \\ -\omega^2 \sin (x_1) - \gamma x_2 = 0 \end{cases} \implies \begin{cases} x_2 = 0 \\ x_1 = k\pi \end{cases}, \quad k \in \mathbb{Z} \]

Now \( x_2 = 0 \implies \frac{d\theta}{dt} = 0 \), meaning at these critical points the pendulum is at rest. The critical points where \( x_1 = 0, \pm 2\pi, \pm 4\pi, \ldots \) correspond to the pendulum resting at the bottom of its swing; these are asymptotically stable critical points (if \( c \neq 0 \)). The critical points where \( x_1 = \pm \pi, \pm 3\pi, \pm 5\pi, \ldots \) correspond to the pendulum resting at the top of its swing. These are the unstable critical points.

1. Stability of Linear Autonomous Systems

We have already discussed the solutions of autonomous systems of the form

\[ x' = Ax \]

Note that so long as \( A \) is a nonsingular matrix (i.e. so long as the only solution of \( Ax = 0 \) is \( x = 0 \)), we will have only one critical point, namely \( x_0 = 0 \).
2. Determination of Trajectories

When we were examining the solutions of linear systems of the form (2), we found were three basic cases for $2 \times 2$ systems. We’ll now look at these basic cases again to assess their stability around the critical point $x_0 = 0$.

- If $A$ has two real eigenvalues $r_1, r_2$ and two corresponding eigenvectors $\xi^{(1)}, \xi^{(2)}$, the general solution of (2) took the form
  
  $$x(t) = c_1 e^{r_1 t} \xi^{(1)} + c_2 e^{r_2 t} \xi^{(2)}$$

  Evidently, such solutions will be stable and asymptotically stable at $0$ whenever $r_1$ and $r_2$ are both negative. If, however, one or both of the eigenvalues are positive then $x(t)$ will be unstable as
  
  $$\lim_{t \to \infty} \|x(t)\| \to \infty$$

  in these cases.
- If $A$ has a pair of complex eigenvalues $r \pm i\mu_1$, then the solutions of (2) will look like
  
  $$x(t) = e^{rt} \begin{bmatrix} \cos(-\mu t + \theta_0) \\ \sin(-\mu t + \theta_0) \end{bmatrix}$$

  and so solutions will be stable and asymptotically stable if $r < 0$, stable if $r = 0$, and unstable if $r > 0$.
- If $A$ is non-diagonalizable with a single eigenvalue $r$, then the solution of (2) will look like
  
  $$x(t) = c_1 e^{rt} \xi + c_2 (t e^{rt} \xi + e^{rt} \eta)$$

  where $\xi$ is the eigenvector of $A$ corresponding to the eigenvalue $r$ and $\eta$ is the solution of $(A - r I) \eta = \xi$. Again is the exponential factor $e^{rt}$ that dictates the behavior of solutions near $0$ for positive $t$. $x_0 = 0$ will be stable and asymptotically stable if $r$ is negative, unstable if $r \geq 0$. (When $r = 0$, the factor of $t$ in the second solution will destabilize the general solution).

2. Determination of Trajectories

For two dimensional autonomous systems the trajectories of solutions can sometimes be found by eliminating the appearance of the underlying variable $t$ from the system of differential equations - via the identity

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Indeed, if

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y)$$

then $x$ and $y$ are related by the following first order differential equation

$$(3) \quad \frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$$

Below we give some examples, we find the trajectories of an autonomous system by solving (3).

**Example 4.4.** Find the critical points and trajectories of the system

$$\frac{dx}{dt} = 4 - 2y$$
$$\frac{dy}{dt} = 12 - 3x^2$$
We have
\[ 0 = F(x, y) = \begin{bmatrix} 4 - 2y \\ 12 - 3x^2 \end{bmatrix} \implies \begin{cases} 4 - 2y = 0 \\ 12 - 3x^2 = 0 \end{cases} \]
\[ \implies \begin{cases} y = 2 \\ x = \pm 2 \end{cases} \]
Thus, the critical points are \((2, 2)\) and \((-2, 2)\).

Next we have
\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12 - 3x^2}{4 - 2y} \]
or
\[ 3x^2 - 12 + (4 - 2y) \frac{dy}{dx} = 0 \]
This ODE is separable and so we can solve it like
\[ \int (3x^2 - 12) \, dx + \int (4 - 2y) \, dy = C \]
\[ \implies x^3 - 12x + 4y - y^2 = C \]
or
\[ y^2 - 4y - x^3 + 12x - C = 0 \]
\[ y = \frac{-4 \pm \sqrt{16 - 4 (-x^3 + 12x - C)}}{2} \]