LECTURE 12

Integrating Factors

Recall that a differential equation of the form
\[ M(x, y) + N(x, y)y' = 0 \]  
(12.1)

is said to be exact if
\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \]  
(12.2)

and that in such a case, we could always find an implicit solution of the form
\[ \psi(x, y) = C \]  
(12.3)

with
\[ \frac{\partial \psi}{\partial x} = M(x, y), \quad \frac{\partial \psi}{\partial y} = N(x, y). \]  
(12.4)

Even if (12.1) is not exact, it is sometimes possible multiply it by another function of \( x \) and/or \( y \) to obtain an equivalent equation which is exact. That is, one can sometimes find a function \( \mu(x, y) \) such that
\[ \mu(x, y)M(x, y) + \mu(x, y)N(x, y)y' = 0 \]  
(12.5)

is exact. Such a function \( \mu(x, y) \) is called an integrating factor. If an integrating factor can be found, then the original differential equation (12.1) can be solved by simply constructing a solution to the equivalent exact differential equation (12.5).

**Example 12.1.** Consider the differential equation
\[ x^2y^3 + x(1 + y^2) \frac{dy}{dx} = 0. \]

This equation is not exact; for
\[ \frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left( x^2y^3 \right) = 3x^2, \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left( x(1 + y^2) \right) = 1 + y^2. \]  
(12.6)

and so
\[ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}. \]

However, if we multiply both sides of the differential equation by
\[ \mu(x, y) = \frac{1}{xy^3} \]

we get
\[ x + \frac{1 + y^2}{y^3} \frac{dy}{dx} = 0 \]

which is not only exact, it is also separable. The general solution is thus obtained by calculating
$$H_1(x) = \int x \, dx = \frac{1}{2}x^2$$
$$H_2(y) = \int \frac{1+y^2}{y^2} \, dy = \frac{1}{2y^2} + \ln |y|$$

and then demanding that $y$ is related to $x$ by

$$H_1(x) + H_2(y) = C$$
or

$$\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln |y| = C.$$  

Now, in general, the problem of finding an integrating factor $\mu(x,y)$ for a given differential equation is very difficult. In certain cases, it is rather easy to find an integrating factor.

**0.1. Equations with Integrating Factors that depend only on $x$.** Consider a general first order differential equation

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0 \quad (12.7)$$

We shall suppose that there exists an integrating factor for this equation that depends only on $x$:

$$\mu = \mu(x) \quad (12.8)$$

If $\mu$ is to really be an integrating factor, then

$$\mu(x)M(x,y) + \mu(x)N(x,y) \frac{dy}{dx} \quad (12.9)$$
must be exact; i.e.,

$$\frac{\partial}{\partial y} (\mu(x)M(x,y)) = \frac{\partial}{\partial x} (\mu(x)N(x,y)) \quad (12.10)$$

Carrying out the differentiations (using the product rule, and the fact that $\mu(x)$ depends only on $x$), we get

$$\mu \frac{\partial M}{\partial y} = \frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x}$$
or

$$\frac{d\mu}{dx} = \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \mu \quad (12.11)$$

Now if $\mu$ is depends only on $x$ (and not on $y$), then necessarily $\frac{d\mu}{dx}$ depends only on $x$. Thus, the self-consistency of equations (12.8) and (12.11) requires the right hand side of (12.11) to be a function of $x$ alone. We presume this to be the case and set

$$p(x) = -\frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

so that we can rewrite (12.11) as

$$\frac{d\mu}{dx} + p(x)\mu = 0 \quad (12.12)$$

This is a first order linear differential equation for $\mu$ that we can solve! According to the formula developed in Section 2.1, the general solution of (12.12) is

$$\mu(x) = A \exp \left[ \int -p(x) \, dx \right] = A \exp \left[ \int \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \, dx \right] \quad (12.13)$$

The formula (12.13) thus gives us an integrating factor for (12.7) so long as

$$\frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$
depends only on $x$.  

0.2. Equations with Integrating Factors that depend only on \( y \). Consider again the general first order differential equation

\[
(12.14) \quad M(x, y) + N(x, y) \frac{dy}{dx} = 0.
\]

We shall suppose that there exists an integrating factor for this equation that depends only on \( y \):

\[
(12.15) \quad \mu = \mu(y) .
\]

If \( \mu \) is to really be an integrating factor, then

\[
(12.16) \quad \mu(y)M(x, y) + \mu(y)N(x, y) \frac{dy}{dx}
\]

must be exact; i.e.,

\[
(12.17) \quad \frac{\partial}{\partial y}(\mu(y)M(x, y)) = \frac{\partial}{\partial x}(\mu(y)N(x, y)) .
\]

Carrying out the differentiations (using the product rule, and the fact that \( \mu(y) \) depends only on \( y \)), we get

\[
\frac{d\mu}{dy} + \mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x}
\]

or

\[
(12.18) \quad \frac{d\mu}{dy} = \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mu .
\]

Now since \( \mu \) is depends only on \( y \) (and not on \( x \)), then necessarily \( \frac{d\mu}{dy} \) depends only on \( y \). Thus, the self-consistency of equations (12.15) and (12.18) requires the right hand side of (12.11) to be a function of \( y \) alone. We presume this to be the case and set

\[
p(y) = -\frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)
\]

so that we can rewrite (12.11) as

\[
(12.19) \quad \frac{d\mu}{dy} + p(y)\mu = 0 .
\]

According to the formula developed in Section 2.1, the general solution of (12.19) is

\[
(12.20) \quad \mu(y) = A \exp \left[ \int p(y) \, dx \right] = A \exp \left[ \int \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \right] .
\]

The formula (12.20) thus gives us an integrating factor for (12.14) so long as

\[
\frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)
\]

depends only on \( y \).

0.3. Summary: Finding Integrating Factors. Suppose that

\[
(12.21) \quad M(x, y) + N(x, y) y' = 0
\]

is not exact.

A. If

\[
(12.22) \quad F_1 = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}
\]

depends only on \( x \) then

\[
(12.23) \quad \mu(x) = \exp \left( \int F_1(x) \, dx \right)
\]

will be an integrating factor for (12.21).
B. If

\begin{equation}
F_2 = \frac{\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}}{M}
\end{equation}

depends only on \( y \) then

\begin{equation}
\mu(y) = \exp \left( \int F_2(y) dy \right)
\end{equation}

will be an integrating factor for (12.21).

C. If neither A nor B is true, then there is little hope of constructing an integrating factor.

**Example 12.2.**

\begin{equation}
(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0
\end{equation}

Here

\[
M(x, y) = 3x^2y + 2xy + y^3 \\
N(x, y) = x^2 + y^2 .
\]

Since

\[
\frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2 \neq 2x = \frac{\partial N}{\partial x}
\]

this equation is not exact.

We seek to find a function \( \mu \) such that

\[
\mu(x, y)(3x^2y + 2xy + y^3)dx + \mu(x, y)(x^2 + y^2)dy = 0
\]

is exact. Now

\[
F_1 = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = \frac{3(x^2 + y^2)}{x^2 + y^2} = 3
\]

\[
F_2 = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{N} = \frac{2x - 3x^2 - 2x - 3y^2}{3xy + 2xy + y^3} = \frac{-3(x^2 + y^2)}{3x^2y + 2xy + y^3}
\]

Since \( F_2 \) depends on both \( x \) and \( y \), we cannot construct an integrating factor depending only on \( y \) from \( F_2 \). However, since \( F_1 \) does not depend on \( y \), we can consistently construct an integrating factor that is a function of \( x \) alone. Applying formula (12.23) we get

\[
\mu(x) = \exp \left( \int F_1(x) dx \right) = \exp \left[ \int 3dx \right] = e^{3x} .
\]

We can now employ this \( \mu(x) \) as an integrating factor to construct a general solution of

\[
e^{3x}(3x^2 + 2x + 3y^2) + e^{3x}(x^2 + y^2)y' = 0
\]

which, by construction, must be exact. So we seek a function \( \psi \) such that

\begin{equation}
(12.27) \quad \frac{\partial \psi}{\partial y} = e^{3x}(3x^2y + 2xy + y^3) \\
\frac{\partial \psi}{\partial x} = e^{3x}(x^2 + y^2) .
\end{equation}

Integrating the first equation with respect to \( x \) and the second equation with respect to \( y \) yields

\[
\psi(x, y) = x^2 ye^{3x} + \frac{1}{3}y^3 e^{3x} + h_1(y) \\
\psi(x, y) = x^2 ye^{3x} + \frac{1}{3}y^3 e^{3x} + h_2(x)
\]

Comparing these expressions for \( \psi(x, y) \) we see that we must take \( h_1(y) = h_2(x) = h(x) = C \), a constant. Thus, function \( \psi \) satisfying (12.27) must be of the form

\[
\psi(x, y) = e^{3x}x^2y + e^{3x}y^3 + C .
\]
Therefore, the general solution of (12.20) is found by solving

\[ e^{3x}x^2y + e^{3x}y^3 = C \]

for \( y \).