GENERIC TRANSFER FOR GENERAL SPIN GROUPS

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Abstract
We prove Langlands functoriality for the generic spectrum of general spin groups (both odd and even). Contrary to other recent instances of functoriality, our resulting automorphic representations on the general linear group are not self-dual. Together with cases of classical groups, this completes the list of cases of split reductive groups whose \( L \)-groups have classical derived groups. The important transfer from \( \text{GSp}_4 \) to \( \text{GL}_4 \) follows from our result as a special case.

Contents
1. Introduction ................................................................. 137
2. Structure theory .......................................................... 141
3. Analytic properties of local \( L \)-functions .......................... 150
4. Stability of \( \gamma \)-factors .............................................. 154
5. Analytic properties of global \( L \)-functions ......................... 175
6. Proof of the main theorem .............................................. 176
7. Complements ............................................................... 183
References ................................................................. 188

1. Introduction
Let \( G \) be a connected reductive group over a number field \( k \). Let \( G = G(\mathbb{A}) \), where \( \mathbb{A} \) is the ring of adèles of \( k \). Let \( L^G \) denote the \( L \)-group of \( G \), and fix an embedding

\[
\iota : L^G \hookrightarrow \text{GL}_N(\mathbb{C}) \rtimes \text{W}(k/k),
\]

where \( \text{W}(k/k) \) is the Weil group of \( k \). Without loss of generality, we may assume that \( N \) is minimal. Let \( \pi = \bigotimes_v \pi_v \) be an automorphic representation of \( G \). Then for almost all \( v \), the local representation \( \pi_v \) is an unramified representation, and its class is determined by a semisimple conjugacy class \([t_v]\) in \( L^G \). Here \( v \) is a finite place of
Let $\Pi_v$ be the unramified representation of $\text{GL}_N(k_v)$ determined by the conjugacy class $[\iota(t_v)]$ generated by $\iota(t_v)$. Langlands’s functoriality conjecture then demands the existence of an automorphic representation $\Pi' = \bigotimes_v \Pi'_v$ of $\text{GL}_N(\mathbb{A})$ such that $\Pi'_v \simeq \Pi_v$ for all the unramified places $v$.

In this article we prove functoriality in the cases where $G$ is not classical but the derived group $L^0$ of the connected component of its $L$-group is. (We follow the convention that a classical group is the stabilizer of a symplectic, orthogonal, or Hermitian nondegenerate bilinear form. Hence, e.g., spin groups are not considered classical.) These groups do not have a useful matrix representation. This fact creates a major difficulty in proving stability of the corresponding root numbers, forcing us to use rather complicated abstract structure theory.

We are mainly concerned with quasi-split groups and those automorphic representations that are induced from generic cuspidal ones. The theory of Eisenstein series reduces our problem to establishing functoriality for generic cuspidal representations of $G = G(\mathbb{A})$.

The cases when $G$ is a quasi-split classical group were addressed in [7], [8], and [23], unless $G$ is a quasi-split special orthogonal group, which should be taken up by the authors of [8].

In this article we establish the functorial transfer of generic cuspidal representations when $G = \text{GSpin}_{m}$, the split general spin group of semisimple rank $n = [m/2]$. These groups are split reductive linear algebraic groups of type $B_n$ or $D_n$ whose derived groups are double coverings of split special orthogonal groups. Moreover, the connected component of their Langlands dual groups are $L^0 = \text{GSp}_{2n}(\mathbb{C})$ or $\text{GSO}_{2n}(\mathbb{C})$, respectively. Then $L^0 = \text{GSO}_{2n}(\mathbb{C}) \times W(\overline{k}/k)$ or $\text{GSp}_{2n}(\mathbb{C}) \times W(\overline{k}/k)$, according to whether $m$ is even or odd. The map $\iota$ is the natural embedding. Observe that $L^0$ is now a classical group and that these groups are precisely the ones for which $G^0_D$ is not classical, but $L^0_D$ is. The transfer is to the space of automorphic representations of $\text{GL}_{2n}(\mathbb{A})$.

It is predicted by the theory of (twisted) endoscopy of Kottwitz and Shelstad [28] and Langlands and Shelstad [31] that the representations of $\text{GL}_{2n}(\mathbb{A})$ which are in the image of this transfer must satisfy

$$\Pi = \tilde{\Pi} \otimes \omega$$

for some Hecke character $\omega$, where $\tilde{\Pi}$ denotes the contragredient of $\Pi$. If $\omega_\Pi$ is the central character of $\Pi$, this implies that $\omega_\Pi/\omega^n$ must be a quadratic character $\mu$ of $k^\times \backslash \mathbb{A}^\times$. Each $\mu$ then determines a quadratic extension of $k$ via class field theory, and the group $G$ which has transfers to automorphic representations of the type just mentioned is the quasi-split form $\text{GSpin}^*_n$ of $\text{GSpin}_{2n}$ associated to the quadratic
extension. The split case then corresponds to \( \mu \equiv 1 \), which is the content of the present article.

If a representation \( \pi \) of \( \text{GSpin}^*_n(\mathbb{A}) \) with central character \( \omega_\pi \) transfers to \( \Pi \) on \( \text{GL}_{2n}(\mathbb{A}) \) satisfying \( \Pi \cong \tilde{\Pi} \otimes \omega \) for some Hecke character \( \omega \), then \( \omega = \omega_\pi \) and \( \omega_\Pi = \omega_{\pi}^n \mu \), where \( \mu \) is the quadratic Hecke character associated with the quasi-split \( \text{GSpin}^*_n \). While we are not able to show that every \( \Pi \) satisfying (1) is the transfer of an automorphic representation \( \pi \), we do show that our transfers satisfy (1). (In fact, we prove that \( \Pi \) is nearly equivalent to \( \tilde{\Pi} \otimes \omega \) for now; see Theorem 1.1.)

We should note here that if \( \Pi \) is an automorphic transfer to \( \text{GL}_{2n+1}(\mathbb{A}) \) satisfying (1), then \( \omega = \theta^2 \) for some \( \theta \), and \( \Pi \otimes \theta^{-1} \) is then self-dual. Therefore, this is already subsumed in the self-dual case, which is a case of standard twisted endoscopy. On the other hand, the case of \( \text{GL}_{2n}(\mathbb{A}) \) discussed above is an example of the most general form of transfer that twisted endoscopy can handle.

As explained earlier, in this article we prove Langlands’s functoriality conjecture in the form discussed for all generic cuspidal representations of split \( \text{GSpin}_m(\mathbb{A}) \). In other words, we establish generic transfer from \( \text{GSpin}_m(\mathbb{A}) \) to \( \text{GL}_{2n}(\mathbb{A}) \). Extension of this transfer to the nongeneric case requires either the use of models other than Whittaker models or that of Arthur’s twisted trace formula. As far as we know, new models for these groups have not been developed, and the fact that these groups are not classical may make matters complicated. On the other hand, the use of Arthur’s twisted trace formula depends at present on the validity of the fundamental lemmas that are not available for these groups. We refer to [2] for information on the case of \( \text{GSp}_4 \).

To state our main theorem, fix a Borel subgroup \( B \) in \( G \) with a maximal (split) torus \( T \), and denote the unipotent radical of \( B \) by \( U \). Let \( \psi \) be a nontrivial continuous additive character of \( k \backslash \mathbb{A} \). As usual, we use \( \psi \) and a fixed splitting (i.e., the choice of Borel subgroup above along with a collection of root vectors, one for each simple root of \( T \); e.g., see [28, page 13]) to define a nondegenerate additive character of \( U(k) \backslash U(\mathbb{A}) \), again denoted by \( \psi \) (see also [41, Section 2]).

Let \( (\pi, V_\pi) \) be an irreducible cuspidal automorphic representation of \( G(\mathbb{A}) \). The representation \( \pi \) is said to be globally generic if there exists a cusp form \( \phi \in V_\pi \) such that

\[
\int_{U(k) \backslash U(\mathbb{A})} \phi(ng)\psi^{-1}(n) \, dn \neq 0. \tag{2}
\]

Note that cuspidal automorphic representations of general linear groups are always globally generic. Two irreducible automorphic representations \( \Pi \) and \( \Pi' \) of \( \text{GL}_N(\mathbb{A}) \) are said to be nearly equivalent if there is a finite set of places \( T \) of \( k \) such that \( \Pi_v \cong \Pi'_v \) for all \( v \not\in T \). Our main result is the following.
THEOREM 1.1
Let \( k \) be a number field, and let \( \pi = \bigotimes \pi_v \) be an irreducible globally generic cuspidal automorphic representation of either \( \text{GSpin}_{2n+1}(\mathbb{A}) \) or \( \text{GSpin}_{2n}(\mathbb{A}) \). Let \( S \) be a nonempty finite set of non-Archimedean places \( v \) such that for \( v \not\in S \), we have that \( \pi_v \) and \( \psi_v \) are unramified. Then there exists an automorphic representation \( \Pi = \bigotimes \Pi_v \) of \( \text{GL}_{2n}(\mathbb{A}) \) such that for all places \( v \not\in S \), the homomorphism parameterizing the local representation \( \Pi_v \) is given by

\[
\Phi_v = \iota \circ \phi_v : W_v \longrightarrow \text{GL}_{2n}(\mathbb{C}),
\]

where \( W_v \) denotes the local Weil group of \( k_v \) and \( \phi_v : W_v \longrightarrow L^0 \) is the homomorphism parameterizing \( \pi_v \). Moreover, if \( \omega_{\Pi} \) and \( \omega_{\pi} \) denote the central characters of \( \Pi \) and \( \pi \), respectively, then \( \omega_{\Pi} = \omega^n_{\pi} \). Furthermore, if \( v \) is an Archimedean place or a non-Archimedean place with \( v \not\in S \), then \( \Pi_v \simeq \widetilde{\Pi_v} \otimes \omega_{\pi_v} \). In particular, the representations \( \Pi \) and \( \widetilde{\Pi} \otimes \omega_{\pi} \) are nearly equivalent.

At the non-Archimedean places \( v \) where \( \pi_v \) is unramified with the semisimple conjugacy class \([t_v]\) in \( L^0 \) as its Frobenius-Hecke (or Satake) parameter, this amounts to the fact that the local representation \( \Pi_v \) is the unramified irreducible admissible representation determined by the conjugacy class generated by \( \iota(t_v) \) in \( \text{GL}_{2n}(\mathbb{C}) \). At the Archimedean places, the existence of \( \phi_v \) is contained in [30].

Our method of proof is that of applying an appropriate version of the Converse Theorems (see [9], [11]) to a family of \( L \)-functions whose required properties, except for one, are proved in [39], [15], [24], and [20]. The exception, that is, the main stumbling block for applying the converse theorem, is that of stability of certain root numbers under highly ramified twists. In [41] the root numbers, or more precisely the inverses of the local coefficients, were expressed as a Mellin transform of certain Bessel functions. Applying this to our case requires a good amount of development and calculations. This is particularly important since the necessary Bruhat decompositions for these groups are more complicated than for the classical groups. For that we have to resort to using the abstract theory of roots which is harder since no reasonable matrix representation is available for these groups. Moreover, the main theorem in [41] is based on certain assumptions whose verification requires our calculations.

The fact that \( \text{GSpin}_{2n} \) has a disconnected center makes matters even more complicated. This leads us to use an extended group \( \text{GSpin}_{2n} \) of \( \text{GSpin}_{2n} \), so that our proof of stability proceeds smoothly.

There are two important transfers that are special cases of this theorem. The first is the generic transfer from \( \text{GSp}_4 = \text{GSpin}_5 \) to \( \text{GL}_4 \) whose proof, as far as we know, has never been published before. We should point out that even the unpublished proofs of this result are based on methods that are fairly disjoint from ours. We finally remark

\[
\text{GSpin}_{2n+1}(\mathbb{A}) \quad \text{or} \quad \text{GSpin}_{2n}(\mathbb{A})\]
that our result in this case also gives an immediate proof of the holomorphy of spinor $L$-functions for generic cusp forms on $\text{GSp}_4$ (see Remark 7.9).

The second special case is when $G = \text{GSpin}_6$. In this case our transfer gives the exterior square transfer from $\text{GL}_4$ to $\text{GL}_6$ due to H. H. Kim [22] which, when composed with the symmetric cube of a cuspidal representation on $\text{GL}_2(\mathbb{A})$, leads to its symmetric fourth.

The issue of whether the local components of the transfer at places in $S$ are the “correct” ones (strong transfer) has been dealt with for classical groups thanks to the existence of the theory of descent from $\text{GL}_n$ to classical groups (see [16], [42]). The analogous results for our cases have to wait until the descent or other techniques are established for representations of $\text{GL}_{2n}(\mathbb{A})$ which satisfy (1). We should note here that, contrary to the case of general linear groups, the local Langlands conjecture is not yet available for the groups we deal with in the present article. What we mean by the correct local component is that for places $v \in S$ we do get the local transfer, which can be defined, at least for generic local representations, using $\gamma$-factors of representations of these groups twisted by those of general linear groups (see [8, Section 7]).

Further applications such as global estimates toward the Ramanujan conjecture as well as some of the other applications established in [7] and [8] will be addressed in future articles. As mentioned earlier, the cases of quasi-split $\text{GSpin}$ groups will be the subject matter of our next article.

Here is an outline of the contents of each section. In Section 2 we review the structure theory of the groups involved in this article. In particular, we give a detailed description of the root data for $\text{GSpin}$ groups and their extensions. We then prove the necessary analytic properties of local $L$-functions in Section 3. In particular, we discuss the Standard Module Conjecture, which is another local ingredient. In Section 4 we go on to prove the most crucial local result, stability of $\gamma$-factors under twists by highly ramified characters. This is where we do the calculations with root data mentioned above and use the extended group. We then prove the necessary analytic properties of the global $L$-functions in Section 5 which prepares us to apply the converse theorem in Section 6. In Section 7 we include the special cases mentioned above along with some other local and global consequences of the main theorem.

2. Structure theory
We review the structure theory for the families of algebraic groups relevant to the current work, namely, $\text{GSpin}_{2n+1}$ and $\text{GSpin}_{2n}$, as well as their duals $\text{GSp}_{2n}$ and $\text{GSO}_{2n}$. We also introduce the group $\widetilde{\text{GSpin}}_{2n}$, which is closely related to $\text{GSpin}_{2n}$. It shares the same derived group with $\text{GSpin}_{2n}$. However, contrary to $\text{GSpin}_{2n}$, which has disconnected center, the center of $\widetilde{\text{GSpin}}_{2n}$ is connected. We need this group for our purposes, as we explain later.
2.1. Root data for GSpin groups
We first describe the algebraic group $G = \text{GSpin}_m$, $m = 2n + 1$ or $2n$, and its standard Levi subgroups in terms of their root data. We rely heavily on these descriptions in the computations of Section 4.

The group $\text{GSpin}_m$ is the quotient of $\text{GL}_1 \times \text{Spin}_m$ by a central subgroup of order 2 (see Proposition 2.2).

PROPOSITION 2.1
The root datum of $\text{GSpin}_m$ can be described as the following. Let

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n,$$

$$X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$

and let $\langle \, , \rangle$ be the standard $\mathbb{Z}$-pairing on $X \times X^\vee$. The root datum for $\text{GSpin}_m$ is $(X, R, X^\vee, R^\vee)$ with $R$ and $R^\vee$ generated, respectively, by

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\},$$

$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = 2e_n^* - e_0^*\},$$

if $m = 2n + 1$ and by

$$\Delta = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\},$$

$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_{n-1}^* + e_n^* - e_0^*\},$$

if $m = 2n$.

Proof
See [3, Section 2].

In the odd case, $G$ has a Dynkin diagram of type $B_n$:

$$\begin{array}{ccccccccc}
\alpha_1 & \alpha_2 & \cdots & \cdots & \cdots & \alpha_{n-2} & \alpha_{n-1} & \alpha_n \\
\end{array}$$

In the even case, it has a Dynkin diagram of type $D_n$:
PROPOSITION 2.2
The derived group of $G$ is isomorphic to $\text{Spin}_{2n+1}$ or $\text{Spin}_{2n}$, the double coverings, as algebraic groups, of special orthogonal groups. In fact, $G$ is isomorphic to
\[(GL_1 \times \text{Spin}_m)\big/\{(1, 1), (-1, c)\},\]
where $c = \alpha_n^\vee(-1)$ if $m = 2n + 1$ and $c = \alpha_{n-1}^\vee(-1)\alpha_n^\vee(-1)$ if $m = 2n$. The dual of $G$ is $\text{GSp}_{2n}$ if $m = 2n + 1$ and $\text{GSO}_{2n}$ if $m = 2n$.

Moreover, if $M$ is the Levi component of a maximal standard parabolic subgroup of $G$, then $M$ is isomorphic to $GL_k \times \text{GSpin}_{m-2k}$ with $k = 1, 2, \ldots, n$ if $m = 2n + 1$ and $k = 1, 2, \ldots, n - 2, n$ if $m = 2n$.

Proof
See [3, Section 2].

We can also describe the Levi subgroup $M$ in terms of its root datum. Without loss of generality, we may assume that $M$ is maximal. Obviously, $M$ has the same character and cocharacter lattices as those of $G$. Denote the set of roots of $M$ by $\Delta_M$, and denote its coroots by $\Delta_M^\vee$. They are generated by $\Delta - \{\alpha\}$ and $\Delta^\vee - \{\alpha^\vee\}$, respectively, where $\alpha = \alpha_k$ unless $m = 2n$ and $k = n$, in which case $\alpha$ can be of either $\alpha_n$ or $\alpha_{n-1}$ (resulting in two nonconjugate isomorphic Levi components). In the sequel, the case of $m = 2n$ and $k = n - 1$ is therefore always ruled out, and we do not repeat this again.

PROPOSITION 2.3
(a) The center of $G$ is given by
\[Z_G = \begin{cases} A_0 & \text{if } m = 2n + 1, \\ A_0 \cup (\zeta_0 A_0) & \text{if } m = 2n, \end{cases}\]
where
\[A_0 = \{e_0^*(\lambda) : \lambda \in GL_1\}\]
and $\zeta_0 = e_1^*(-1)e_2^*(-1)\cdots e_n^*(-1)$. 

(b) The center of \( M \) is given by

\[
Z_M = \begin{cases} 
A_k & \text{if } m = 2n + 1, \\
A_k \cup (\zeta_k A_k) & \text{if } m = 2n,
\end{cases}
\]

where

\[
A_k = \left\{ e_0^*(\lambda)e_1^*(\mu)e_2^*(\mu)\cdots e_k^*(\mu) : \lambda, \mu \in \text{GL}_1 \right\}
\]

and \( \zeta_k = e_{k+1}^*(-1)e_{k+2}^*(-1)\cdots e_n^*(-1) \).

Proof

The maximal torus \( T \) of \( G \) (or \( M \)) consists of elements of the form

\[
t = \prod_{j=0}^{n} e_j^*(t_j)
\]

with \( t_j \in \text{GL}_1 \). Now \( t \) is in the center of \( G \), respectively, \( M \), if and only if it belongs to the kernel of all simple roots of \( G \), respectively, \( M \). For \( G \), this leads to

\[
\frac{t_1}{t_2} = \frac{t_2}{t_3} = \cdots = \frac{t_{n-1}}{t_n} = t_n = 1
\]

if \( m = 2n + 1 \) and

\[
\frac{t_1}{t_2} = \frac{t_2}{t_3} = \cdots = \frac{t_{n-1}}{t_n} = t_{n-1}t_n = 1
\]

if \( m = 2n \). For \( M \), we get

\[
\frac{t_1}{t_2} = \frac{t_2}{t_3} = \cdots = \frac{t_{k-1}}{t_k} = \frac{t_{k+1}}{t_{k+2}} = \cdots = \frac{t_{n-1}}{t_n} = t_n = 1
\]

if \( m = 2n + 1 \) and

\[
\frac{t_1}{t_2} = \frac{t_2}{t_3} = \cdots = \frac{t_{k-1}}{t_k} = \frac{t_{k+1}}{t_{k+2}} = \cdots = \frac{t_{n-1}}{t_n} = t_{n-1}t_n = 1
\]

if \( m = 2n \). These relations prove the proposition. \( \square \)

Remark 2.4

When \( m = 2n \), the nonidentity component of \( Z_G \) can also be written as \( z'A_0 \), where \( z' \) is a nontrivial element in the center of \( \text{Spin}_{2n} \), the derived group of \( G \). We now specify this element explicitly in terms of the central element \( z \) of [3, Proposition 2.2]. There
is a typographical error in the description of $z$ in that article which we correct here:

$$z = \begin{cases} \prod_{j=1}^{n-2} \alpha_j \cdot \alpha_{n-1}((1-j) \cdot \alpha_n) & \text{if } n \text{ is even}, \\ \prod_{j=1}^{n-2} \alpha_j \cdot \alpha_{n-1}(-\sqrt{-1}) \alpha_n & \text{if } n \text{ is odd}. \end{cases}$$

To compute $z'$, note that with $m = 2n$ we have

$$e_1^* + \cdots + e_{n-1}^* + e_n^* = \sum_{j=1}^{n-2} j \alpha_j + \left( \frac{n}{2} - 1 \right) \alpha_{n-1} + \frac{n}{2} \alpha_n + \frac{n}{2} e_0^*,$$

which, when evaluated as a character at $(-1)$, yields

$$\zeta_0 = \begin{cases} z & \text{if } n = 4p, \\ ze_0^*(\sqrt{-1}) & \text{if } n = 4p + 1, \\ cze_0^*(-1) & \text{if } n = 4p + 2, \\ cze_0^*(-\sqrt{-1}) & \text{if } n = 4p + 3. \end{cases}$$

Therefore, $\zeta_0 A_0 = z'A_0$, where $z'$ is an element in the center of Spin$_{2n}$ given by

$$z' = \begin{cases} z & \text{if } n \equiv 0, 1 \pmod{4}, \\ cz & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

### 2.2. Root data for $\text{GSpin}_{\sim}^2$ groups

We describe the structure theory for the group $\text{GSpin}_{\sim}^2_n$ as well as its standard Levi subgroups in this section. For our future discussion on stability of $\gamma$-factors in Section 4, we need to work with a group with a connected center. The center of GSpin$_{2n}$ is not connected, as we observed in Proposition 2.3. To remedy this, we define a new group that is just GSpin$_{2n}$ extended by a one-dimensional torus. This group has a connected center, while its derived group remains the same as that of GSpin$_{2n}$, that is, Spin$_{2n}$. This allows us to work with GSpin$_{\sim}^2_n$, as we explain in Section 4.

**Definition 2.5**

Let GSpin$_{\sim}^2_n$ be the group

$$(\text{GL}_1 \times \text{GSpin}_{2n})/\{(1,1),(-1,\zeta_0)\},$$

where $\zeta_0$ is as in Proposition 2.3. Note that the derived group of GSpin$_{\sim}^2_n$ is again Spin$_{2n}$. 

PROPOSITION 2.6

Let

\[ X = \mathbb{Z}E_{-1} \oplus \mathbb{Z}E_0 \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_n, \]
\[ X^\vee = \mathbb{Z}E_{-1}^* \oplus \mathbb{Z}E_0^* \oplus \mathbb{Z}E_1^* \oplus \cdots \oplus \mathbb{Z}E_n^*, \]

and let \( \langle \cdot, \cdot \rangle \) be the standard \( \mathbb{Z} \)-pairing on \( X \times X^\vee \). Then \((X, R, X^\vee, R^\vee)\) is the root datum for \( \text{GSpin}_{2n} \) with \( R \) and \( R^\vee \) generated, respectively, by

\[ \Delta = \{ \alpha_1 = E_1 - E_2, \ldots, \alpha_{n-1} = E_{n-1} - E_n, \alpha_n = E_{n-1} + E_n - E_1 \} \]

and

\[ \Delta^\vee = \{ \alpha_1^\vee = E_1^* - E_2^*, \ldots, \alpha_{n-1}^\vee = E_{n-1}^* - E_n^*, \alpha_n^\vee = E_{n-1}^* + E_n^* - E_0^* \}. \]

**Proof**

Our proof is similar to the proof of [3, Proposition 2.4]. We compute the root datum of \( \text{GSpin}_{2n} \) using that of \( \text{GSpin}_{2n} \) described earlier, and we verify that it can be written as above.

Start with the character lattice of \( \text{GL}_1 \times \text{GSpin}_{2n} \) which can be written as the \( \mathbb{Z} \)-span of \( e_0, e_1, \ldots, e_n, e_{-1} \). Here, \( e_{-1} \) is a generator for the character lattice of the \( \text{GL}_1 \) factor. Now, characters of \( \text{GSpin}_{2n} \) are those characters of \( \text{GL}_1 \times \text{GSpin}_{2n} \) which are trivial when evaluated at the element \((-1, \zeta_0)\). Note that \( e_i(\zeta_0) = -1 \) for \( 1 \leq i \leq n \), \( e_0(\zeta_0) = 1 \), and \( e_{-1}(-1) = -1 \). This implies that the character lattice of \( \text{GSpin}_{2n} \) can be written as the \( \mathbb{Z} \)-span of \( 2e_{-1}, e_0, e_1 + e_{-1}, \ldots, e_n + e_{-1} \). Now, set \( E_{-1} = 2e_{-1}, E_0 = e_0, \) and \( E_i = e_{-1} + e_i \) for \( 1 \leq i \leq n \). We can compute a basis for the cocharacter lattice using the \( \mathbb{Z} \)-pairing of the root datum. It turns out to consist of \( E_{-1}^* = e_{-1}^*/2 - (e_1^* + \cdots + e_n^*)/2, E_0^* = e_0^*, \) and \( E_i^* = e_i^* \) for \( 1 \leq i \leq n \). Writing the simple roots and coroots in terms of the new bases finishes the proof. For example,

\[ \alpha_n = e_{n-1} + e_n = (e_{n-1} + e_{-1}) + (e_n + e_{-1}) - 2e_{-1} = E_{n-1} + E_n - E_1 \]

and

\[ \alpha_n^\vee = e_{n-1}^* + e_n^* - e_0^* = E_{n-1}^* + E_n^* - E_0^*. \]

We can also describe the root datum of any standard Levi subgroup \( M \) in \( \text{GSpin}_{2n} \). Again, without loss of generality, we may assume that \( M \) is maximal. Similar to the case of \( \text{GSpin}_{2n} \), the roots and coroots of \( M \) are, respectively, generated by \( \Delta - \{ \alpha_k \} \) and \( \Delta^\vee - \{ \alpha_k^\vee \} \) for some \( k \). The character and cocharacter lattices are the same as those of \( \text{GSpin}_{2n} \).
PROPOSITION 2.7
(a) The center of $\text{GSpin}_{2n}$ is given by
\[
\left\{ E_0^*(\mu)E_1^*(\lambda)E_2^*(\lambda) \cdots E_n^*(\lambda)E_{n-1}^*(\lambda^2) : \lambda, \mu \in \text{GL}_1 \right\},
\]
and it is hence connected.
(b) The center of $\text{M}$ is given by
\[
\left\{ E_0^*(\mu)E_1^*(\nu) \cdots E_k^*(\nu)E_{k+1}^*(\lambda) \cdots E_n^*(\lambda)E_{n-1}^*(\lambda^2) : \lambda, \mu, \nu \in \text{GL}_1 \right\},
\]
and it is hence connected.

Proof
The maximal torus of $\text{GSpin}_{2n}$ (or $\text{M}$) consists of elements of the form
\[
t = \prod_{j=-1}^{n} E_j^*(t_j)
\]
with $t_j \in \text{GL}_1$. Now $t$ is in the center of $\text{G}$, respectively, $\text{M}$, if and only if it belongs to the kernel of all simple roots of $\text{G}$, respectively, $\text{M}$. For $\text{G}$, this leads to
\[
\frac{t_1}{t_2} = \frac{t_2}{t_3} = \cdots = \frac{t_{n-1}}{t_n} = \frac{t_{n-1}t_n}{t_{-1}} = 1.
\]
For $\text{M}$, we get
\[
\frac{t_1}{t_2} = \frac{t_2}{t_3} = \cdots = \frac{t_{k-1}}{t_k}, \quad \frac{t_{k+1}}{t_{k+2}} = \cdots = \frac{t_{n-1}}{t_n} = \frac{t_{n-1}t_n}{t_{-1}} = 1.
\]
These relations prove the proposition. \qed

We also describe the structure of standard Levi subgroups in $\text{GSpin}_{2n}$.

PROPOSITION 2.8
The standard Levi subgroups of $\text{GSpin}_{2n}$ are isomorphic to
\[
\text{GL}_{k_1} \times \cdots \times \text{GL}_{k_r} \times \text{GSpin}_{2l},
\]
where $k_1 + \cdots + k_r + l = n$.

Proof
Without loss of generality, we may assume that $\text{M}$ is maximal. The character and cocharacter lattices of $\text{M}$, which are the same as those of $\text{G}$, were described in
Proposition 2.6 and can be written as
\[(\mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_k) \oplus (\mathbb{Z}E_{-1} \oplus \mathbb{Z}E_0 \oplus \mathbb{Z}E_{k+1} \oplus \cdots \oplus \mathbb{Z}E_n)\]
and
\[(\mathbb{Z}E^*_1 \oplus \cdots \oplus \mathbb{Z}E^*_k) \oplus (\mathbb{Z}E^*_{-1} \oplus \mathbb{Z}E^*_0 \oplus \mathbb{Z}E^*_{k+1} \oplus \cdots \oplus \mathbb{Z}E^*_n),\]
respectively. This, along with the description of roots and coroots of \(M\) given above, implies that the root datum of \(M\) can be written as a direct sum of two root data. The first one is now the well-known root datum of \(GL_k\), and the second is just our earlier description of the root datum of \(GSpin_{2(n-k)}\). Therefore, \(M\) is isomorphic to \(GL_k \times GSpin_{2(n-k)}\).

2.3. Root data for \(GSp_{2n}\) and \(GSO_{2n}\)
We describe the root data for the two groups \(GSp_{2n}\) and \(GSO_{2n}\) in detail. Since these two groups are usually introduced as matrix groups, we also describe the root data in terms of their usual matrix representations. It is evident from this description that the two groups \(GSpin_{2n+1}\) and \(GSp_{2n}\), as well as \(GSpin_{2n}\) and \(GSO_{2n}\), are pairs of connected reductive algebraic groups with dual root data.

Consider the group defined as
\[\{ g \in GL_{2n} : ^tJg = \mu(g)J \},\]
where the \((2n \times 2n)\)-matrix \(J\) is defined via
\[J = \begin{pmatrix} 1 & \cdots & 1 \\ \cdots & -1 & 1 \\ -1 & \cdots & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & \cdots & 1 \\ \cdots & 1 & \cdots \\ 1 & \cdots & 1 \end{pmatrix},\]
respectively. The former is the connected reductive algebraic group \(GSp_{2n}\). However, the latter is not connected as an algebraic group. This group is sometimes denoted by \(GO_{2n}\) (see [34, Section 2]). Its connected component is the group \(GSO_{2n}\) (also denoted by \(SGO_{2n}\)).

Proposition 2.9
The root datum of the groups \(GSp_{2n}\) and \(GSO_{2n}\) can be described as follows. Let
\[X = Ze_0 \oplus Ze_1 \oplus \cdots \oplus Ze_n,\]
\[X^\vee = Ze^*_0 \oplus Ze^*_1 \oplus \cdots \oplus Ze^*_n,\]
and let $(, )$ be the standard $\mathbb{Z}$-pairing on $X \times X^\vee$. Then $(X, R, X^\vee, R^\vee)$ is the root datum for the connected reductive algebraic group $\text{GSp}_{2n}$ or $\text{GSO}_{2n}$ with $R$ and $R^\vee$ generated, respectively, by

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n - e_0\},$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_n^*\},$$

for $\text{GSp}_{2n}$ and

$$\Delta = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n - e_0\},$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_n^* + e_n^*\},$$

for $\text{GSO}_{2n}$.

**Proof**

See [45, pages 133 – 136] for $\text{GSp}_{2n}$. Similar computations work for $\text{GSO}_{2n}$. \qed

The Dynkin diagrams are of type $C_n$ and $D_n$, respectively. A computation similar to the proof of Proposition 2.3 proves the following.

**PROPOSITION 2.10**

*Let $G$ be either $\text{GSp}_{2n}$ or $\text{GSO}_{2n}$. Then the center of $G$ is given by*

$$Z = \{e_0^*(\lambda^2) e_1^*(\lambda) \cdots e_n^*(\lambda) : \lambda \in \text{GL}_1\}.$$ 

*The maximal split torus in both $\text{GSp}_{2n}$ and $\text{GSO}_{2n}$ can be described as*

$$\hat{T} = \left\{ t(a_1, \ldots, a_n, b_n, \ldots, b_1) = \begin{pmatrix} a_1 & \cdots & a_n \\ & \ddots & \\ a_n & b_n & \cdots \\ & & b_1 \end{pmatrix} : a_i b_i = \mu \right\}.$$ 

(3)

*We can now describe $e_i$ and $e_i^*$ in terms of matrices. In either case, we have*

$$e_0(t) = \mu, e_0^*(\lambda) = t(1, \ldots, 1, \lambda, \ldots, \lambda),$$
$$e_i(t) = a_i, e_i^*(\lambda) = t(1, \ldots, 1, \lambda, 1, \ldots, 1, \lambda^{-1}, 1, \ldots, 1), \quad 1 \leq i \leq n.$$ 

(4)
3. Analytic properties of local $L$-functions

Let $F$ denote a local field of characteristic zero, either Archimedean or non-Archimedean. Let $G_n$ denote the algebraic group $\mathrm{GSpin}_{2n+1}$ (resp., $\mathrm{GSpin}_{2n}$), and let $\sigma$ be an irreducible admissible generic representation of $M = M(F)$ in $G = G_{r+n}(F)$, where $M \simeq \mathrm{GL}_r \times G_n$ is the Levi subgroup of a standard parabolic subgroup $P$ in $G_{r+n}$. Let $\tilde{M} \simeq \mathrm{GL}_r(\mathbb{C}) \times GSp_{2n}(\mathbb{C})$ (resp., $\tilde{M} \simeq \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GSO}_{2n}(\mathbb{C})$) be the Levi component of the corresponding standard parabolic subgroup $\tilde{P}$ in the dual group $\hat{G} = L G^{0} = \mathrm{GSp}_{2(n+r)}(\mathbb{C})$ (resp., $\hat{G} = \mathrm{GSO}_{2(n+r)}(\mathbb{C})$). Let $r$ denote the adjoint action of $\tilde{M}$ on the Lie algebra of the unipotent radical of $\hat{P}$. Then by [3, Proposition 5.6], $r = r_1 \oplus r_2$ if $n \geq 1$ (resp., $n \geq 2$) with $r_1 = \rho_r \otimes \tilde{R}$ and $r_2 = \mathrm{Sym}^2 \rho_r \otimes \mu^{-1}$ (resp., $r_2 = \wedge^2 \rho_r \otimes \mu^{-1}$). Here, $\rho_r$ denotes the standard representation of $\mathrm{GL}_r(\mathbb{C})$, $\tilde{R}$ denotes the contragredient of the standard representation of $\mathrm{GSp}_{2n}(\mathbb{C})$ (resp., $\mathrm{GSO}_{2n}(\mathbb{C})$), and $\mu$ denotes the similitude character of $\mathrm{GSp}_{2n}(\mathbb{C})$ (resp., $\mathrm{GSO}_{2n}(\mathbb{C})$). If $n = 0$, then $r = r_1$ with $r_1 = \mathrm{Sym}^2 \rho_r \otimes \mu^{-1}$ (resp., $r_1 = \wedge^2 \rho_r \otimes \mu^{-1}$). Recall that we have excluded $n = 1$ in the even case. The Langlands-Shahidi method defines the $L$-functions $L(s, \sigma, r_i)$ and $\epsilon$-factors $\epsilon(s, \sigma, r_i, \psi)$ for $1 \leq i \leq 2$, where $\psi$ is a nontrivial additive character of $F$. (In the global setting, it is the local component of our fixed global additive character $\psi$ of Section 1.) If $\pi$ denotes a representation of $G_n(F)$ and $\tau$ denotes one of $\mathrm{GL}_r(F)$, then we sometimes employ the following notation for these $L$-functions as well as their global analogues:

$$L(s, \pi \times \tau) := L(s, \tau \otimes \tilde{\pi}, \rho_r \otimes \tilde{R}) = L(s, \tau \otimes \tilde{\pi}, r_1),$$ (5)

$$\epsilon(s, \pi \times \tau, \psi) := \epsilon(s, \tau \otimes \tilde{\pi}, \rho_r \otimes \tilde{R}, \psi) = \epsilon(s, \tau \otimes \tilde{\pi}, r_1, \psi).$$ (6)

**PROPOSITION 3.1**

Assume that $\sigma$ is tempered. Then the local $L$-function $L(s, \sigma, r_i)$ is holomorphic for $\Re(s) > 0$ for $1 \leq i \leq 2$.

**Proof**

The result is known more generally for Archimedean $F$ (see [36] or [1]). For non-Archimedean $F$, this is [3, Theorem 5.7]. Here, $i = 1, 2$, and the first $L$-function gives the Rankin-Selberg product while the second is either the twisted symmetric or the twisted exterior square $L$-function. When $n = 0$, we get only the second $L$-function (see [3, Proposition 5.6]).

The following proposition is due to W. Kim (in his Ph.D. dissertation [25]) when $F$ is non-Archimedean and, in more generality, due to Kostant [27] and Vogan [48] when $F$ is Archimedean.

**PROPOSITION 3.2** (The Standard Module Conjecture for $G_n$)

Let $\sigma$ be an irreducible admissible generic representation of $M(F)$, where $M$ is a standard maximal Levi subgroup as before, and let $v$ be an element in the positive
Weyl chamber. Let $I(\nu, \sigma)$ be the representation unitarily induced from $\nu$ and $\sigma$ (the standard module), and denote by $J(\nu, \sigma)$ its unique Langlands quotient. Assume that $J(\nu, \sigma)$ is generic. Then $J(\nu, \sigma) = I(\nu, \sigma)$. In particular, $I(\nu, \sigma)$ is irreducible.

A similar result also holds for general linear groups (see [49]).

**Remark 3.3**

For small values of $n$, we need not rely on [25]. We can obtain the result for small $n$ from published ones as we now explain. The group $GSpin_5$ is isomorphic to $GSp_4$, whose derived group is $Sp_4$. G. Muć has proved the Standard Module Conjecture for (quasi-split) classical groups (see [33, Theorem 1.1]). The result for $GSpin_5$ now follows from Corollary 3.5.

Similarly, note that the derived group of $GSpin_6$ is isomorphic to $Spin_6 \simeq SL_4$ and hence equal to the derived group of $GL_4$. Therefore, again by Corollary 3.5, the result for $GSpin_6$ follows from the Standard Module Conjecture for $GL_6$.

**Proposition 3.4**

Let $G \subset \tilde{G}$ be two connected reductive groups whose derived groups are equal. Let $\tilde{\mathcal{P}} = \tilde{M}\tilde{N}$ be a maximal standard parabolic subgroup of $\tilde{G}$, and let $\mathcal{P} = MN$ be the corresponding one in $G$ with $M = \tilde{M} \cap G$. Also, let $\tilde{T} \subset \tilde{M}$ and $T = \tilde{T} \cap G \subset M$ be maximal tori in $\tilde{G}$ and $G$, respectively. Let $\tilde{\sigma}$ be a quasi-tempered representation of $\tilde{M} = \tilde{M}(F)$, and denote by $\sigma$ its restriction to $M = M(F)$. Write $\sigma = \bigoplus \sigma_i$ with $\sigma_i$ irreducible representations of $M$. Let $I(\tilde{\sigma})$ denote the induced representation $\text{Ind}_{\tilde{M}N \uparrow \tilde{G}} 1$ of $\tilde{G} = \tilde{G}(F)$, and let $I(\sigma_i)$ denote $\text{Ind}_{MN \uparrow G} \sigma_i \otimes 1$, a representation of $G = G(F)$. Then the standard module $I(\tilde{\sigma})$ is irreducible if and only if each standard module $I(\sigma_i)$ is irreducible.

**Proof**

By the irreducibility of $\tilde{\sigma}$ and the fact that $\tilde{M} = \tilde{T}M$, choose

$$\{t_1 = 1, t_2, \ldots, t_k : t_i \in \tilde{T} = \tilde{T}(F)\}$$

such that $\sigma_i(m) = \sigma_i(t_i^{-1}m t_i)$. Observe that

$$I(\tilde{\sigma})|_G = \bigoplus I(\sigma_i).$$

In fact, for $f_1 \in V(\sigma_1)$, define $f_i(g) = f_1(t_i^{-1}g t_i)$. Then $f_i \in V(\sigma_i)$, the space of $I(\sigma_i)$, and the representation $I(\sigma_i)(t_i^{-1}g t_i)$ on $V(\sigma_1)$ is isomorphic to $I(\sigma_i)$ since

$$\left(I(\sigma_i)(t_i^{-1}g t_i)f_i\right)_i = I(\sigma_i)(g)f_i$$
for all \( g \in G \). In particular, \( I(\sigma_i) \) is irreducible if and only if \( I(\sigma_1) \) is. The assumption of the equality of the derived groups implies that \( \widetilde{G} = \widetilde{T}G \), which, in turn, implies that \( \widetilde{T} \) acts transitively on the set of \( I(\sigma_i) \).

If each \( I(\sigma_i) \) is irreducible, then \( I(\tilde{\sigma}) \) has to be irreducible. In fact, if \((\widetilde{I}_1, \widetilde{V}_1)\) is an irreducible subrepresentation of \( I(\tilde{\sigma}) \), then

\[
\widetilde{I}_1|G = \bigoplus_j I_j, \quad I_j \neq \{0\},
\]

and given \( j \), there exists \( i \) such that \( I_j \subset I(\sigma_i) \). Conversely, for each \( i \) there exists \( j \) such that \( 0 \neq I_j \subset I(\sigma_i) \). Fix \( i \) such that \( V(\sigma_i) \cap \widetilde{V}_1 \neq 0 \). Since \( \widetilde{V}_1 \) is invariant under \( \tilde{T} \), applying \( I(\tilde{\sigma})(\tilde{T}) \) to this intersection, one concludes that \( 0 \neq V(\sigma_i) \cap \widetilde{V}_1 \subset V(\sigma_i) \) for all \( i \). Consequently, \( 0 \neq I_j \subset I(\sigma_i) \), which is a contradiction.

Conversely, suppose that \( I(\tilde{\sigma}) \) is irreducible but \( I(\sigma_i) \)'s are (all) reducible. Let \( V_i \) be an irreducible \( G \)-subspace of \( V(\sigma_i) \). Then

\[
\bigoplus_i I(\tilde{\sigma})(t_i)V_i
\]

is a \( \widetilde{G} \)-invariant subspace of \( V(\tilde{\sigma}) \) which is strictly smaller than \( V(\tilde{\sigma}) \), a contradiction. \( \square \)

**Corollary 3.5**

Suppose that \( G \) and \( G' \) are two connected reductive groups having the same derived group. Then the Standard Module Conjecture is valid for \( G \) if and only if it is valid for \( G' \).

**Proof**

Let \( H \) be the common derived group. Apply Proposition 3.4 once to \( H \subset G \) and again to \( H \subset G' \). \( \square \)

The Langlands-Shahidi method defines the local \( L \)-functions via the theory of intertwining operators. With notation as above, let the standard maximal Levi subgroup \( M \) in \( G \) correspond to the subset \( \theta \) of the set of simple roots \( \Delta \) of \( G \). Then \( \theta = \Delta - \{ \alpha \} \) for a simple root \( \alpha \in \Delta \). We denote by \( w \) the longest element in the Weyl group of \( G \) modulo that of \( M \). Then \( w \) is the unique element with \( w(\theta) \subset \Delta \) and \( w(\alpha) < 0 \). Let \( A(s, \sigma, w) \) denote the intertwining operator as in \([39, (1.1), page 278]\), and let \( N(s, \sigma, w) \) be defined via

\[
A(s, \sigma, w) = r(s, \sigma, w)N(s, \sigma, w),
\]

\[
r(s, \sigma, w) = \frac{L(s, \sigma, \tilde{r}_1)L(2s, \sigma, \tilde{r}_2)}{L(1 + s, \sigma, \tilde{r}_1)\epsilon(s, \sigma, \tilde{r}_1, \psi)L(1 + 2s, \sigma, \tilde{r}_2)\epsilon(2s, \sigma, \tilde{r}_2, \psi)}.
\]

\( \square \)
In fact, the Langlands-Shahidi method inductively defines the $\gamma$-factors using the theory of local intertwining operators out of which the $L$- and $\epsilon$-factors are defined via the relation

$$
\gamma(s, \sigma, r_i, \psi) = \epsilon(s, \sigma, r_i, \psi) \frac{L(1 - s, \sigma, \tilde{r}_i)}{L(s, \sigma, r_i)}.
$$

The following proposition about analytic properties of local $L$-functions is the main result of this section. We use it to prove the necessary global analytic properties.

**PROPOSITION 3.6**

Let $\sigma$ be a local component of a globally generic cuspidal automorphic representation of $M(\mathbb{A})$. Then the normalized local intertwining operator $N(s, \sigma, w)$ is holomorphic and nonzero for $\Re(s) \geq 1/2$.

**Proof**

First, assume that $\sigma$ is tempered. Then $A(s, \sigma, w)$ is holomorphic for $\Re(s) > 0$ by a result of Harish-Chandra. Moreover, for $\Re(s) > 0$ we have that $r(s, \sigma, w)$ is nonzero by definition and holomorphic by Proposition 3.1. This implies that $N(s, \sigma, w)$ is also holomorphic for $\Re(s) > 0$.

Next, assume that $\sigma$ is not tempered but still unitary. Write $\sigma = \tau \otimes \tilde{\pi}$, where $\tau$ is a representation of $\text{GL}_r(F)$ and $\tilde{\pi}$ is one of $\text{G}_n(F)$. (We use $\tilde{\pi}$ in order to get the usual Rankin-Selberg factors for pairs of general linear groups below.) By Proposition 3.2 and the classification of irreducible unitary representations of general linear groups, we can write $\tau$ and $\tilde{\pi}$ as

$$
\tau = \text{Ind}(\nu^{\alpha_1} \tau_1 \otimes \cdots \otimes \nu^{\alpha_p} \tau_p \otimes \tau_{p+1} \otimes \nu^{-\alpha_p} \tau_p \otimes \cdots \otimes \nu^{-\alpha_1} \tau_1)
$$

and

$$
\tilde{\pi} = \text{Ind}(\nu^{\beta_1} \pi_1 \otimes \cdots \otimes \nu^{\beta_q} \pi_q \otimes \pi_0)
$$

with $0 = \alpha_{p+1} < \alpha_p < \cdots < \alpha_1 < 1/2$ and $0 < \beta_q < \cdots < \beta_1$, where $\tau_1, \ldots, \tau_{p+1}$ and $\pi_1, \ldots, \pi_q$ are tempered representations of the corresponding general linear groups and $\pi_0$ is a generic tempered representation of $\text{G}_t(F)$ for some $t$. Here, $\nu(r)$ denotes $|\det(r)|_F$. Since we are assuming that $\sigma$ is a component of a global cuspidal representation (see Remark 3.7), it follows exactly as in [21, Lemma 3.3] that $\beta_1 < 1$. However, note that one should use our $\tilde{\pi}$ in the argument.

Now $N(s, \sigma, w)$ is equal to a product of rank-one operators for either $\text{GL}_k \times \text{GL}_l \subset \text{GL}_{k+l}$ (Rankin-Selberg products) or $\text{GL}_k \times G_l \subset G_{k+l}$. Lemma 2.10 of [21] implies that the former rank-one operators are holomorphic since $\Re(s - \alpha_1 - \beta_1) > -1$ for
\[ \Re(s) \geq 1/2. \] The latter rank-one operators are also holomorphic for \( \Re(s) \geq 1/2 \) by the tempered case at the beginning of this proof since \( \alpha_1 < 1/2 \).

The fact that \( N(s, \sigma, w) \) is a nonvanishing operator now follows from applying a result of Y. Zhang to our case (see [50, pages 393–394]). Note that because of Proposition 3.1, no assumptions are needed in applying [50].

**Remark 3.7**

Note that the proof of [21, Lemma 3.3] does depend on the fact that \( \sigma \) is assumed to be a local component of a global cuspidal representation. To be more precise, the proof uses [21, Theorem 3.2(3)], and it refers to [21, Proposition 1.8], which is a global result.

### 4. Stability of \( \gamma \)-factors

We continue to denote by \( G_n \) either of the groups \( \text{GSpin}_{2n+1} \) or \( \text{GSpin}_{2n} \) in this section. In Sections 4.1 and 4.2 we denote by \( \tilde{G}_n \) the groups \( \text{GSpin}_{2n+1} \) in the odd case and \( \text{GSpin}_{2n} \) in the even case (see Remark 4.2), and \( G \) denotes \( \tilde{G}_{n+1} \) in either case.

In this section we prove a key local fact, called the *stability of \( \gamma \)-factors*, which is what allows us to connect the Langlands-Shahidi \( L \)- and \( \epsilon \)-factors to those in the converse theorem. Similar results have been proved for the group \( \text{SO}_{2n+1} \) in [10] and [7] and for other classical groups in [8], which we have followed. A more general result appears in [12] and [13].

Let \( F \) denote a non-Archimedean local field of characteristic zero. Thus, \( F \) could be one of the \( k_v \)'s, where \( v \) is a finite place. Composing a fixed splitting of \( G_n \) with \( \psi \) as in [41] defines a generic character of \( U \) as well as \( U_M \) which we still denote by \( \psi \). Let \( \pi \) be an irreducible admissible \( \psi \)-generic representation of \( \text{GSpin}_{2n+1}(F) \) or \( \text{GSpin}_{2n}(F) \), and let \( \eta \) be a continuous character of \( F^\times \). The associated \( \gamma \)-factors of the Langlands-Shahidi method defined in [39, Theorem 3.5] are denoted by \( \gamma(s, \eta \times \pi, \psi) \). They are associated to the pair \( (\text{GSpin}_{m+2}, \text{GL}_1 \times \text{GSpin}_m) \) of the maximal Levi subgroup \( M = \text{GL}_1 \times \text{GSpin}_m \) in the connected reductive group \( \text{GSpin}_{m+2} \), where \( m = 2n + 1 \) or \( 2n \). Recall that the \( \gamma \)-factor is related to the \( L \)- and \( \epsilon \)-factors by

\[
\gamma(s, \eta \times \pi, \psi) = \epsilon(s, \eta \times \pi, \psi) \frac{L(1-s, \eta^{-1} \times \tilde{\pi})}{L(s, \eta \times \pi)}.
\] (14)

The main result of this section is the following.

**THEOREM 4.1**

Let \( \pi_1 \) and \( \pi_2 \) be irreducible admissible generic representations of \( \text{GSpin}_m(F) \) with equal central characters \( \omega_{\pi_1} = \omega_{\pi_2} \). Then for a sufficiently ramified character \( \eta \) of
\( F^\times \) we have

\[
\gamma(s, \eta \times \pi_1, \psi) = \gamma(s, \eta \times \pi_2, \psi).
\]

The proof of this theorem is the subject matter of this section, including a review of some facts about partial Bessel functions.

4.1. \( \gamma(s, \eta \times \pi, \psi) \) as the Mellin transform of a Bessel function

Recall that \( G_n^- \) denotes either \( \mathrm{GSpin}_{2n+1} \) or \( \mathrm{GSpin}_{2n}^- \). Then \( \text{GL}_1 \times G_n^- \) is a maximal Levi subgroup in \( G = G_n^{+1} \). We refer to \( G = \mathrm{GSpin}_{2n+3} \) as odd and \( G = \mathrm{GSpin}_{2n+2}^- \) as even in the rest of this section. Therefore, in the odd case, \( G_n^- \) and \( G_n \) are the same, but they are not in the even case.

**Remark 4.2**

We need to assume that the center of our group \( G \) is connected for the proof of Proposition 4.16. This is not true if \( G \) is taken to be \( \mathrm{GSpin}_{2n+2} \), as we pointed out in Proposition 2.3. To remedy this, we can alternatively work with the group \( \mathrm{GSpin}_{2n+2}^- \) of Section 2.2. Since \( \mathrm{GSpin}_{2n+2}^- \) has the same derived group as \( \mathrm{GSpin}_{2n+2} \), its corresponding \( \gamma \)-factors are the same as those of \( \mathrm{GSpin}_{2n+2} \) since they (and, in fact, the local coefficients via which they are defined) depend only on the derived group of our group. This has no effect on the arguments of the next few sections, as all of our crucial computations take place inside the derived group.

Let \( G \) be as above with a fixed Borel subgroup \( B = TU \) as before. We continue to denote its root data by \((X, R, X^\vee, R^\vee)\), which we have described in detail in Section 2. Consider the maximal parabolic subgroup \( P = MN \) in \( G \), where \( N \subset U \) and the Levi component, \( M \), is isomorphic to \( \text{GL}_1 \times G_n^- \). The standard Levi subgroup \( M \supset T \) corresponds to the subset \( \theta = \Delta - \{\alpha_1\} \) of the set of simple roots \( \Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}\} \) of \((G, T)\) with notation as in Section 2. Let \( \bar{w}_0 \) denote the unique element of the Weyl group of \( G \) such that \( \bar{w}_0(\theta) \subset \Delta \) and \( \bar{w}_0(\alpha_1) < 0 \). Notice that the parabolic subgroup \( P \) is self-associate; that is, \( \bar{w}_0(\theta) = \theta \). We denote by \( G, P, M, N, B \), and so on, the groups \( G(F), P(F), M(F), N(F), B(F) \), and so on, in what follows. Also, denote the opposite parabolic subgroup to \( P \) by \( \overline{P} = MN \).

Let \( Z = Z_G \) and \( Z_M \) be the centers of \( G \) and \( M \), respectively. The following is [41, Assumption 5.1] for our cases. We need this when dealing with Bessel functions.

**Proposition 4.3**

There exists an injection \( e^* : F^\times \rightarrow Z_G \backslash Z_M \) such that for all \( t \in F^\times \) we have \( \alpha_1(e^*(t)) = t \).
Proof

We define \( e^*(t) \) to be the image in \( Z_G \setminus Z_M \) of \( e^*_1(t) \) in the odd case and that of \( E^*_1(t) \) in the even case. The proposition is now clear from our explicit descriptions in Section 2.

Denote the image of \( e^* \) by \( Z^0_M \) as in [41]. (Note that [41] uses the notation \( \alpha^\vee \) for \( e^* \).)

We now review some standard facts about the reductive group \( G \) whose proofs could be found in either [43] or [44], for example. For \( \alpha \in R \), let \( u_\alpha : F \to G \) be the root group homomorphism determined by the equation

\[
tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x), \quad t \in T, x \in F.
\] (15)

Moreover, define \( w_\alpha : F^\times \to G \) by

\[
w_\alpha(\lambda) = u_\alpha(\lambda)u_{-\alpha}(-\lambda^{-1})u_\alpha(\lambda).
\] (16)

Also, set \( w_\alpha := w_\alpha(1) \). Then

\[
w_\alpha(\lambda) = \alpha^\vee(\lambda)w_\alpha = w_\alpha \alpha^\vee(\lambda^{-1}),
\] (17)

where \( \alpha^\vee \) is the coroot corresponding to \( \alpha \). The element \( w_\alpha \) normalizes \( T \), and we denote its image in the Weyl group by \( \tilde{w}_\alpha \).

Remark 4.4

Our choice of \( w_\alpha \) is indeed the same as \( n_\alpha \) in [43, page 133]. This choice differs up to a sign from those made in [41, (4.43), (4.19), or (4.56)], requiring \( w_\alpha \) to be the image of \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) under the homomorphism from \( SL_2 \) into \( G \) determined by \( \alpha \). The latter choice introduces an occasional negative sign in some of the equations, for example, (17). Of course, this choice is irrelevant to the final results, and we have chosen Springer’s since we are using some detailed information on structure constants from [43] in what follows (see, e.g., Lemma 4.10).

Recall that

\[
w_{-\alpha}(\lambda) = w_\alpha(\frac{-1}{\lambda}),
\] (18)

\[
w_\alpha^2 = \alpha^\vee(-1),
\] (19)

\[
w_{-\alpha} = w_\alpha^{-1}.
\] (20)
For any two linearly independent roots $\alpha$ and $\beta$ in $R$ and a total order on $R$, which we now fix, we have

$$u_\alpha(x)u_\beta(y)u_\alpha(-x) = u_\beta(y) \prod_{i,j \geq 0, i\alpha + j\beta \in R} u_{i\alpha + j\beta}(c_{ij}x^j y^i)$$

(21)

for certain structure constants $c_{ij} = c_{\alpha,\beta;i,j}$ (which may depend on the total order $R$). In particular, if there are no roots of the form $i\alpha + j\beta$ with $i, j > 0$, then

$$u_\alpha(x)u_\beta(y) = u_\beta(y)u_\alpha(x).$$

We recall the following result.

**PROPOSITION 4.5**

Let $\alpha$ and $\beta$ be two arbitrary linearly independent roots, and let $(\beta - \alpha, \ldots, \beta + b\alpha)$ be the $\alpha$-string through $\beta$. Then

$$w_\alpha w_\beta(x) w_\alpha^{-1} = \tilde{w}_\alpha(\beta)(d_{\alpha,\beta} x),$$

(22)

where

$$d_{\alpha,\beta} = \sum_{i = \max(0,c-b)}^c (-1)^i c_{-\alpha,\beta;i,1} c_{\alpha,\beta-i\alpha;i+b-c,1}.$$  

(23)

Moreover,

$$d_{-\alpha,\beta} = (-1)^{\langle \beta,\alpha \rangle} d_{\alpha,\beta},$$

(24)

and

$$d_{\alpha,\beta} d_{\alpha,\tilde{w}_\alpha(\beta)} = (-1)^{\langle \beta,\alpha \rangle}.$$  

(25)

**Proof**

This is [43, Lemma 9.2.2]. Note that Springer defines $d_{\alpha,\beta}$ via

$$w_\alpha u_\beta(x) w_\alpha^{-1} = u_{\tilde{w}_\alpha(\beta)}(d_{\alpha,\beta} x),$$

which, using (16), immediately implies (22). \qed

Denote the image of $u_\alpha$ in $G$ by $U_\alpha$. Notice that $M$ is generated by $T$ and $U_\alpha$’s with $\alpha$ ranging over $\Sigma(\theta)$, the set of all (positive and negative) roots spanned by $\alpha_2, \alpha_3, \ldots, \alpha_n, \alpha_{n+1}$, while $N$ is generated by $U_\alpha$’s, where $\alpha$ ranges over $R(N) = R^+ - \Sigma(\theta)$, the set of positive roots of $G$ not in $M$ (i.e., involving a positive coefficient of $\alpha_1$ when written as a sum of simple roots with nonnegative coefficients), and $\overline{N}$ is generated by $U_\alpha$’s, where $\alpha$ ranges over $R(\overline{N}) = R^- - \Sigma(\theta)$, the set of negative roots.
of $G$ not in $M$ (i.e., involving a negative coefficient of $\alpha_1$ when written as a sum of simple roots with nonpositive coefficients). Let $U_M = U \cap M$. Then $U_M$ is generated by $U_\alpha$’s with $\alpha \in \Sigma(\theta)^+ = \Sigma(\theta) \cap R^+$.

The group $M$ acts via the adjoint action on $N$; in particular, both $U_M$ and $Z_M^0$ act on $N$. We are interested in the orbits of $N$ under conjugation by $Z_M^0 U_M$.

**Lemma 4.6**

*Up to a subset of measure zero of $N$, the following is a complete set of representatives for the orbits of $N$ under conjugation by $U_M$:*

$$U_M \backslash N \simeq \{ u_{\alpha_1}(a)u_\gamma(x) : a \in F^\times, x \in F \},$$

where

$$\gamma = \begin{cases} 
\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n+1} & \text{if } G \text{ is odd}, \\
\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1} & \text{if } G \text{ is even}
\end{cases}$$

is the longest positive root in $G$.

**Proof**

Using the same Bourbaki notation as in Section 2, $R(N)$ is given by (see [6])

$$\left\{ \begin{array}{l}
\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}, \alpha_1 + \alpha_2 + \cdots + \alpha_n + 2\alpha_{n+1}, \\
\alpha_1 + 2\alpha_2 + \cdots + \alpha_{n-1} + 2\alpha_n + 2\alpha_{n+1}, \ldots, \\
\gamma = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n + 2\alpha_{n+1}
\end{array} \right\}$$

(26)

for the odd case and by

$$\left\{ \begin{array}{l}
\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n + \alpha_{n+1}, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + 2\alpha_n - 2 + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}, \\
\alpha_{n+1}, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n + \alpha_{n+1}, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + 2\alpha_n - 2 + \alpha_{n-1} + \alpha_n + \alpha_{n+1}, \\
\ldots, \\
\gamma = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}
\end{array} \right\}$$

(27)

for the even case.

An arbitrary element $n \in N$ is of the form

$$n = \prod_{\alpha \in R(N)} u_\alpha(x_\alpha)$$

(28)

with $x_\alpha \in F$. The ordering in the product can be arbitrarily chosen since any linear combination with positive integer coefficients of two roots in $R(N)$ has $\alpha_1$ with an integer coefficient of at least two which cannot be a root; hence by (21), any two
terms in the above product commute. We make use of this fact in the rest of the proof.

Observe that the set $R(N)$ has the property that if $\alpha$ belongs to $R(N)$, then so does $\gamma - \alpha + \alpha_1$. Notice that if $\alpha' \in R(N) - \{\alpha_1, \gamma\}$, then $\beta = \alpha' - \alpha_1 \in \Sigma(\theta)$ and $\beta > 0$; hence $g = u_\beta(x_\beta) \in U_M$ for any $x_\beta \in F$. The observation means that $\gamma - \beta = \gamma - \alpha' + \alpha_1 \in R(N)$. Fix one such $\beta$, and consider the adjoint action of $g$ on $n$:

$$gng^{-1} = \prod_{\alpha \in R(N)} g\alpha(x_\alpha)g^{-1} = \prod_{\alpha \in R(N)} u_\beta(x_\beta)u_\alpha(x_\alpha)u_\beta(-x_\beta).$$

We now look at each term in this product. If $i\beta + j\alpha \notin R$ for positive $i$ and $j$, then by (21), the term is equal to $u_\alpha(x_\alpha)$. This is the case most of the time. The only roots of the form $i\beta + j\alpha$ with positive $i$ and $j$ are $\beta + \alpha_1$ and $\beta + (\gamma - \beta) = \gamma$, except when we are in the odd case with $\beta = \alpha_2 + \cdots + \alpha_{n+1}$ and $\alpha = \alpha_1$, in which case $2\beta + \alpha_1 = \gamma$ is also a root. Therefore,

$$\prod_{\alpha \in R(N)} u_\beta(x_\beta)u_\alpha(x_\alpha)u_\beta(-x_\beta) = \prod_{\alpha \in R(N)} u_\alpha(y_\alpha),$$

where

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \neq \beta + \alpha_1, \gamma, \\ x_\alpha + Cx_\alpha_1 x_\beta & \text{if } \alpha = \beta + \alpha_1, \\ x_\alpha + C'x_\beta x_\gamma - \beta + C''x_\beta^2 x_\alpha_1 & \text{if } \alpha = \gamma, \alpha \neq \beta + \alpha_1. \end{cases}$$

Here, $C, C', C'' \in F^\times$ are the appropriate structure constants as in (21). In fact, $C''$ is nonzero only in the exceptional case mentioned above, that is, the odd case with $\beta = \alpha_2 + \cdots + \alpha_{n+1}$ and $\alpha = \alpha_1$.*

Assuming that $x_{\alpha_1} \neq 0$, which excludes only a subset of $n \in N$ of measure zero, we can choose $x_\beta \in F$ appropriately in order to have $x_\alpha + Cx_\alpha_1 x_\beta = 0$. Applying this process for all the $\beta$ in $\Sigma(\theta)$ described above, we can make all $x_\alpha$ in (28) equal to zero, except for $x_{\alpha_1}$ and $x_\gamma$. In the process, the value of $x_{\alpha_1}$ does not change, but the value of $x_\gamma$ may change. We let $a = x_{\alpha_1}$, and we let $x$ be the final value of $x_\gamma$. This proves the lemma. □

We now consider conjugation by $Z_0^0$.

**Lemma 4.7**

Let $n = u_{\alpha_1}(a)u_\gamma(x) \in N$ with $a \in F^\times$ and $x \in F$. Then $n$ and $u_{\alpha_1}(1)u_\gamma(y)$ are in the same conjugacy class of $N$ under conjugation by $Z_0^0$ for some $y \in F$.

*We thank the referee who brought the exceptional case to our attention.
Proof
For $z = e_1^*(\lambda) \in Z_M^0$, in the odd case, we have
\[
znz^{-1} = e_1^*(\lambda)u_{\alpha_1}(a)e_1^*(\lambda^{-1}) e_1^*(\lambda)u_\gamma(x)e_1^*(\lambda^{-1})
= u_{\alpha_1}(\alpha_1(e_1^*(\lambda)) a) u_\gamma(\alpha_1(e_1^*(\lambda)) x)
= u_{\alpha_1}(\lambda^{\langle \alpha_1, e_1^* \rangle} a) u_\gamma(\lambda^{\langle \gamma, e_1^* \rangle} x)
= u_{\alpha_1}(\lambda a) u_\gamma(\lambda x).
\]
In the even case, $e_1^*$ above should be replaced by $E_1^*$. Take $\lambda = 1/a$ and $y = x/a$ to finish the proof.

Lemmas 4.6 and 4.7 immediately imply the following.

COROLLARY 4.8
Up to a subset of measure zero of $N$, the following is a complete set of representatives for the orbits of $N$ under conjugation by $Z_M^0 U_M$:
\[
Z_M^0 U_M \backslash N \simeq \{ u_{\alpha_1}(1)u_\gamma(x) : x \in F \}.
\]
If we set
\[
w_0 = \begin{cases} 
w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_n} w_{\alpha_{n+1}} w_{\alpha_n} \cdots w_{\alpha_2} w_{\alpha_1} & \text{if } G \text{ is odd,} \\
w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_{n-1}} w_{\alpha} w_{\alpha_{n-1}} \cdots w_{\alpha_2} w_{\alpha_1} & \text{if } G \text{ is even,}
\end{cases}
\tag{29}
\]
where $w_{\alpha\alpha}$ is the product of the commuting elements $w_{\alpha_n} w_{\alpha_{n+1}} = w_{\alpha_{n+1}} w_{\alpha_n}$, then $w_0$ is the representative in $G$ of the unique Weyl group element $\tilde{w}_0$ introduced at the beginning of Section 4.1.

Remark 4.9
Note that in [41] the analogue of our element $w_{\alpha\alpha}$ is denoted by $w_{\alpha_{n+1}}$ as explained in [41, (4.45)]. However, our current choices, which apply equally well to SO$_{2n}$ or other cases treated in [41, Section 4], replace the two commuting matrices on the left-hand side of [41, (4.45)] with their transposes (see Remark 4.4).

As in [41] we are interested in elements $n \in N$ such that
\[
w_0^{-1} n = mn'\bar{n} \in P\bar{N}.
\tag{30}
\]
The decomposition in (30) is clearly unique, and we compute the $m$-, $n'$-, and $\bar{n}$-parts of an element $n$, as in Corollary 4.8. We do this in Proposition 4.12. First, we prove the following auxiliary lemma.
LEMMA 4.10

We can normalize the $u_{\alpha}$’s such that the element $w_{\gamma}$ satisfies

$$
\gamma^\vee(d)w_{\gamma} = w_{\gamma}(d) = \begin{cases} 
w_{\alpha_2} \cdots w_{\alpha_n} w_{\alpha_{n+1}} w_{\alpha_n} \cdots w_{\alpha_2} w_{\alpha_1} w_{\alpha_2}^{-1} \cdots w_{\alpha_n}^{-1} w_{\alpha_{n+1}}^{-1} w_{\alpha_n}^{-1} \cdots w_{\alpha_2}^{-1} & \text{if } G \text{ is odd,} \\
w_{\alpha_2} \cdots w_{\alpha_{n-1}} w_{\alpha} w_{\alpha_{n-1}} \cdots w_{\alpha_2} w_{\alpha_1} w_{\alpha_2}^{-1} \cdots w_{\alpha_n}^{-1} w_{\alpha}^{-1} w_{\alpha_{n-1}}^{-1} \cdots w_{\alpha_2}^{-1} & \text{if } G \text{ is even,}
\end{cases}
$$

where

$$d = \begin{cases} 
(-1)^n & \text{if } G \text{ is odd,} \\
(-1)^{n-1} & \text{if } G \text{ is even.}
\end{cases} \quad (31)
$$

Remark 4.11

The $d$ in the odd case is slightly different from the corresponding value for the group $SO_{2n+3}$ carried out in [8, Section 4.2.1]; that is, it differs by a factor of $-1$. The reason for this discrepancy is that the representative we have fixed for the longest element of the Weyl group would, in the case of the group $SO_3$, lead to

$$
\begin{pmatrix}
-1 \\
-1 \\
1
\end{pmatrix}.
$$

This is the correct element that should have been used in [8, Section 4.2.1] and [41] instead of

$$
\begin{pmatrix}
1 \\
-1 \\
1
\end{pmatrix}.
$$

The latter is the corresponding Weyl group representative for the group $SL_3$. However, since the group $SO_3$ does not have a Cartan of the same dimension as that of $SL_3$, there is no natural way (i.e., not requiring a choice of basis) of embedding it in $SL_3$. Therefore, there is no reason why the representative for $SO_3$ would be the same as that of $SL_3$. Of course, both of these two matrices correspond to the same Weyl group element since they differ only by a diagonal matrix in $SO_3$. In the notation of the present article, we can fix $u_{\alpha}(\cdot)$ and $u_{-\alpha}(\cdot)$ in $SO_3$ such that

$$w_{\alpha}(\lambda) = \begin{pmatrix}
-\lambda^2 \\
-1 \\
-1/\lambda^2
\end{pmatrix}.
$$

The choice of representative is now simply $w_{\alpha}(1)$. 

Proof of Lemma 4.10

We begin by noting that

\[
\gamma = \begin{cases} 
\tilde{w}_{\alpha_2} \cdots \tilde{w}_{\alpha_n} \tilde{w}_{\alpha_{n+1}} \tilde{w}_{\alpha_n} \cdots \tilde{w}_{\alpha_2} (\alpha_1) & \text{if } G \text{ is odd,} \\
\tilde{w}_{\alpha_2} \cdots \tilde{w}_{\alpha_n} \tilde{w}_{\alpha_{n+1}} \tilde{w}_{\alpha_n} \cdots \tilde{w}_{\alpha_2} (\alpha_1) & \text{if } G \text{ is even.}
\end{cases}
\]  

(32)

Let \( \beta_1 = \alpha_1 \), and denote by \( \beta_i \) the consecutive images of \( \beta_1 \) under the first \( i - 1 \) Weyl group elements above for

\[
\begin{cases} 
1 \leq i \leq 2n & \text{if } G \text{ is odd,} \\
1 \leq i \leq 2n - 1 & \text{if } G \text{ is even.}
\end{cases}
\]

In fact, the \( \beta_i \)'s are precisely the roots listed in (26) and (27) and in the same order.

Now apply (22) repeatedly to conclude that the right-hand side of the expression of the statement of the lemma is equal to \( w_{\gamma} (d) \), where

\[
d = \begin{cases} 
d_{\alpha_2, \beta_1} \cdots d_{\alpha_n, \beta_{n-1}} d_{\alpha_{n+1}, \beta_n} d_{\alpha_n, \beta_{n+1}} \cdots d_{\alpha_2, \beta_{2n-1}} & \text{if } G \text{ is odd,} \\
d_{\alpha_2, \beta_1} \cdots d_{\alpha_{n-1}, \beta_{n-2}} d_{\alpha_n, \beta_{n-1}} d_{\alpha_{n+1}, \beta_n} \cdots d_{\alpha_2, \beta_{2n-2}} & \text{if } G \text{ is even.}
\end{cases}
\]

For \( 1 \leq i \leq n - 1 \) in the odd case and for \( 1 \leq i \leq n \) in the even case, the \( \alpha_{i+1} \)-string through \( \beta_i \) is \( (\beta_i, \beta_i + \alpha_i + 1) \); that is, \( c = 0 \) and \( b = 1 \) in the notation of Proposition 4.5. In the odd case with \( i = n \), the \( \alpha_{i+1} \)-string through \( \beta_i \) is \( (\beta_i, \beta_i + \alpha_i + 1, \beta_i + 2\alpha_i + 1) \); that is, \( c = 0 \) and \( b = 2 \). Similarly, for \( 1 \leq j \leq n - 1 \) in the odd case, the \( \alpha_{n-j-1} \)-string through \( \beta_{n+j} \) is \( (\beta_{n+j}, \beta_{n+j} + \alpha_{n-j-1}) \); that is, \( c = 0 \) and \( b = 1 \). Also, for \( 1 \leq j \leq n - 2 \) in the even case, the \( \alpha_{n-j} \)-string through \( \beta_{n+j} \) is \( (\beta_{n+j}, \beta_{n+j} + \alpha_{n-j}) \); that is, \( c = 0 \) and \( b = 1 \). Putting all these together and using (23), we can write

\[
\begin{align*}
d_{\alpha_2, \beta_1} &= c_{\alpha_2, \beta_1; 1, 1} \\
d_{\alpha_{n-1}, \beta_{n-2}} &= c_{\alpha_{n-1}, \beta_{n-2}; 1, 1} \\
d_{\alpha_n, \beta_{n-1}} &= c_{\alpha_n, \beta_{n-1}; 1, 1} \\
d_{\alpha_n, \beta_n} &= c_{\alpha_n, \beta_n; 2, 1} \\
d_{\alpha_n, \beta_{n+1}} &= c_{\alpha_n, \beta_{n+1}; 1, 1} \\
d_{\alpha_{n+1}, \beta_n} &= c_{\alpha_{n+1}, \beta_n; 1, 1} \\
d_{\alpha_2, \beta_{2n-1}} &= c_{\alpha_2, \beta_{2n-1}; 1, 1} \\
d_{\alpha_2, \beta_{2n-2}} &= c_{\alpha_2, \beta_{2n-2}; 1, 1}
\end{align*}
\]

in the odd and even cases, respectively.

We can now normalize the \( u_\alpha \)'s such that we have \( c_{\alpha_i, \alpha_{i+1}; 1, 1} = 1 \) and, in the odd case, \( c_{\alpha_i, \alpha_{i+1}; 1, 2} = -1 \). These normalizations are motivated by the explicit matrix realizations of the related groups \( \text{SO}_{2n+3} \) and \( \text{SO}_{2n+2} \), such as those in [41]. The values of other structure constants are now uniquely determined by these (see [43, [43, [43].]
Lemma 9.2.3]. We then get
\[
\begin{align*}
c_{\alpha_2, \beta_1; 1, 1} &= -1 & c_{\alpha_2, \beta_1; 1, 1} &= -1 \\
& \vdots & & \vdots \\
c_{\alpha_n, \beta_{n-1}; 1, 1} &= -1 & c_{\alpha_{n-1}, \beta_{n-2}; 1, 1} &= -1 \\
c_{\alpha_{n+1}, \beta_n; 2, 1} &= -1 & c_{\alpha_n, \beta_{n-1}; 1, 1} &= -1 \\
c_{\alpha_n, \beta_{n+1}; 1, 1} &= +1 & c_{\alpha_{n+1}, \beta_n; 1, 1} &= +1 \\
& \vdots & & \vdots \\
c_{\alpha_2, \beta_{2n-1}; 1, 1} &= +1 & c_{\alpha_2, \beta_{2n-2}; 1, 1} &= +1.
\end{align*}
\]

Therefore, in the expression for \(d\), the first \(n\) terms in the odd case and the first \(n - 1\) terms in the even case are equal to \(-1\), and others are equal to \(+1\). This implies that \(d = (-1)^n\) in the odd case and \(d = (-1)^{n-1}\) in the even case.

**PROPOSITION 4.12**

Assume that \(x \in F\), and assume that \(n = u_{\alpha_1}(1)u_{\gamma}(x)\) satisfies (30). Moreover, assume that \(x\) is nonzero (which rules out only a subset of \(N\) of measure zero). Then
\[
m = w' \gamma \left( \frac{d}{x} \right),
\]
\[
n' = u_{\gamma}(-x)u_{\alpha_1}(-1),
\]
\[
\overline{n} = u_{-\gamma}\left( \frac{1}{x} \right)u_{-\alpha_1}(1),
\]

where \(d\) is as in (31) and
\[
w' = \begin{cases} 
  w_{\alpha_2}^{-1} \cdots w_{\alpha_n}^{-1} w_{\alpha_{n+1}} w_{\alpha_n}^{-1} \cdots w_{\alpha_2}^{-1} & \text{if } G \text{ is odd}, \\
  w_{\alpha_2}^{-1} \cdots w_{\alpha_{n-1}}^{-1} w_{\alpha_n}^{-1} w_{\alpha_{n-1}} w_{\alpha_{n-2}}^{-1} \cdots w_{\alpha_2}^{-1} & \text{if } G \text{ is even},
\end{cases}
\]

with \(w_{\alpha_\alpha}\) again as in (29). Moreover, we could also write \(m\) as
\[
m = \alpha_1 \gamma \left( \frac{d}{x} \right) w',
\]

which is analogous to [41, Propositions 4.4, 4.8] modulo our Remark 4.4.

**Proof**

By the uniqueness of the decomposition in (30), it is enough to prove that these values do satisfy (30). This is a straightforward computation utilizing (21) multiple times.
First, observe that if $i, j > 0$ are integers, then $i\alpha_1 + j\gamma$ cannot be a root. Hence, $u_\gamma(\cdot)$ and $u_{\alpha_1}(\cdot)$ commute by (21). Also, $i\alpha_1 + j(-\gamma)$ cannot be a root, which again implies by (21) that $u_{-\gamma}(\cdot)$ and $u_{\alpha_1}(\cdot)$ commute. Similarly, $u_\gamma(\cdot)$ and $u_{-\alpha_1}(\cdot)$ commute. Moreover, by (29), we have

$$w_0 = w_{\alpha_1} w'^{-1} w_{\alpha_1}$$  \hspace{1cm} (34)$$

and, by Lemma 4.10,

$$w_\gamma(d) = w'^{-1} w_{\alpha_1} w'.$$  \hspace{1cm} (35)$$

Now,

$$w_0^{-1} n = w_{\alpha_1}^{-1} w' w_{\alpha_1}^{-1} u_{\alpha_1}(1) u_\gamma(x)$$

$$= w_{\alpha_1}^{-1} w' u_{\alpha_1}(-1) u_{-\alpha_1}(1) u_{\alpha_1}(1) u_\gamma(x)$$

$$= w_{\alpha_1}^{-1} w' u_\gamma(x) u_{\alpha_1}(-1) u_{-\alpha_1}(1)$$

$$= w_{\alpha_1}^{-1} w' w_\gamma(x) u_\gamma(-x) u_\gamma \left( \frac{1}{x} \right) u_{\alpha_1}(-1) u_{-\alpha_1}(1)$$

$$= w_{\alpha_1}^{-1} w' w_\gamma(x) \cdot u_\gamma(-x) u_{\alpha_1}(-1) \cdot u_{-\gamma} \left( \frac{1}{x} \right) u_{-\alpha_1}(1)$$

$$= w' w'^{-1} w_{\alpha_1} w_\gamma(x) \cdot n' \cdot \overline{n}$$

$$= w' w_\gamma(d)^{-1} w_\gamma(x) \cdot n' \cdot \overline{n}$$

$$= w' \gamma \left( \frac{1}{d} \right) \cdot n' \cdot \overline{n}$$

$$= w' \gamma \left( \frac{d}{x} \right) \cdot n' \cdot \overline{n}$$

$$= mn' \overline{n}.$$  

To see (33), we use (22) repeatedly to write

$$m = w_{\alpha_1}^{-1} w' w_\gamma(x)$$

$$= w_{\alpha_1}^{-1} \cdot w' w_\gamma(x) w'^{-1} \cdot w'$$

$$= w_{\alpha_1}^{-1} w_{\alpha_1}(Dx) w'$$

$$= w_{\alpha_1}^{-1} w_{\alpha_1} \gamma \left( \frac{1}{Dx} \right) w'$$

$$= \alpha_1 \gamma \left( \frac{1}{Dx} \right) w',$$

$$= \alpha_1^{-1} \gamma(x) w'.$$
where

\[
D = \begin{cases}
    d_{-\alpha_2, \beta_2} \cdots d_{-\alpha_n, \beta_n} d_{-\alpha_{n+1}, \beta_{n+1}} d_{-\alpha_n, \beta_n+2} \cdots d_{-\alpha_2, \beta_2} & \text{if } G \text{ is odd,} \\
    d_{-\alpha_2, \beta_2} \cdots d_{-\alpha_{n-1}, \beta_{n-1}} d_{\alpha_n, \beta_n} d_{-\alpha_{n+1}, \beta_{n+1}} d_{-\alpha_{n-1}, \beta_{n-1}+2} \cdots d_{-\alpha_2, \beta_2} & \text{if } G \text{ is even.}
\end{cases}
\]

(36)

Notice that conjugation by \( w' \) sends \( \gamma = \beta_{2n} \) in the odd case and \( \gamma = \beta_{2n-1} \) in the even case back to \( \alpha_1 \).

Finally, we claim that \( Dd = 1 \) in both even and odd cases. To see this, note that we can write

\[
Dd = \prod_{i=1}^{n} d_{\alpha_{i+1}, \beta_{i+1}} d_{-\alpha_i, \beta_i} d_{-\alpha_{j+1}, \beta_{j+1}} \prod_{j=1}^{n-1} d_{\alpha_{j+1}, \beta_{j+1}} d_{-\alpha_j, \beta_j} \prod_{j=1}^{n-2} d_{\alpha_{j+1}, \beta_{j+1}} d_{-\alpha_j, \beta_j}
\]

if \( G \) is odd,

\[
\prod_{i=1}^{n} d_{\alpha_{i+1}, \beta_{i+1}} d_{-\alpha_i, \beta_i} d_{-\alpha_{j+1}, \beta_{j+1}} \prod_{j=1}^{n-2} d_{\alpha_{j+1}, \beta_{j+1}} d_{-\alpha_j, \beta_j}
\]

if \( G \) is even.

Using (24) followed by (25), we can rewrite this as

\[
Dd = \begin{cases}
    \prod_{i=1}^{n} (-1)^{\langle \beta_i + \beta_{i+1}, \alpha_i \rangle} (-1)^{\langle \beta_{2n-j} + \beta_{2n-1-j}, \alpha_{j+1} \rangle} & \text{if } G \text{ is odd,} \\
    \prod_{i=1}^{n} (-1)^{\langle \beta_i + \beta_{i+1}, \alpha_i \rangle} (-1)^{\langle \beta_{2n-j} + \beta_{2n-1-j}, \alpha_{j+1} \rangle} & \text{if } G \text{ is even.}
\end{cases}
\]

Using the explicit root data that we described earlier, we can see easily that in every single term of these products the power of \((-1)\) is an even integer. In fact, \( \beta_i + \beta_{i+1} \) is equal to \( \alpha_{i+1} \) plus twice a root for all \( i \). Similarly, \( \beta_{2n-j} + \beta_{2n-1-j} \) is equal to \( \alpha_{j+1} \) plus twice a root for all \( j \) in the odd case, and \( \beta_{2n-1-j} + \beta_{2n-j} \) is equal to \( \alpha_{j+1} \) plus twice a root for all \( j \) in the even case. This completes the proof. \( \square \)

The following is [41, Assumption 4.1] for our cases.

PROPOSITION 4.13

Let \( n \in N \) satisfy (30). Then, except for a subset of measure zero of \( N \), we have

\[
U_{M,n} = U'_{M,m},
\]

(37)

where the notation is as in [41, Section 4]; that is,

\[
U_{M,n} = \{ u \in U_M : u n u^{-1} = n \},
\]

(38)

and

\[
U'_{M,m} = \{ u \in U_M : m u m^{-1} \in U_M \text{ and } \psi(m u m^{-1}) = \psi(u) \}.
\]

Note that the condition \( \psi(m u m^{-1}) = \psi(u) \) in the definition of \( U'_{M,m} \) in our case is just the compatibility of \( \psi \) with elements of the Weyl group (see [41, pages 2079 – 2080]).
Proof

By arguments such as those on [41, page 2085], if the proposition is true for \( n \in \mathbb{N} \), then it is also true for every member of the intersection of its conjugacy class under \( M \) with \( N \), provided that for the \( m \)-part we use the twisted conjugacy classes instead (see [41, (4.10)]). Hence, it is enough to verify the proposition for those \( n \), as in Corollary 4.8. Fix one such \( n = u_{\alpha_1}(1)u_\gamma(q) \) with \( q \neq 0 \) for the rest of this proof.

We can explicitly compute both sides of (37) as follows. Any \( u \in U_M \) can be written as

\[ u = \prod_{\beta \in \Sigma(\theta)^+} u_\beta(x_\beta), \tag{39} \]

where the order of the terms in the product is with respect to the total order of \( R \) we have fixed.

We have \( uu^{-1} = u\) if and only if \( uu_{\alpha_1}(1)u_\gamma(q)u^{-1} = u_{\alpha_1}(1)u_\gamma(q) \). Notice that by (21) we know that \( u_\gamma(q) \) commutes with all \( u_\beta(x_\beta) \) in the product. Hence, \( u \in U_{M,n} \) if and only if \( uu_{\alpha_1}(1)u^{-1} = u_{\alpha_1}(1) \). Among the terms \( u_\beta(x_\beta) \), the element \( u_{\alpha_1}(1) \) commutes with those with \( \beta \in \Sigma(\Omega)^+ \), where \( \Omega = \Delta - \{\alpha_1, \alpha_2\} \). Also, if \( \beta \) belongs to \( \Sigma(\theta)^+ - \Sigma(\Omega)^+ \), then so does \( \gamma - \beta - \alpha_1 \). Using (21) several times, we can now write

\[ uu_{\alpha_1}(1)u^{-1} = \prod_{\beta \in \Sigma(\theta)^+} u_\beta(x_\beta)u_{\alpha_1}(1)\left( \prod_{\beta \in \Sigma(\theta)^+} u_\beta(x_\beta) \right)^{-1} \]

\[ = u_\gamma \left( \sum c_{\alpha_1, \beta; 1, 1} c_{\alpha_1 + \beta, \delta; 1, 1} x_\delta \right) \cdot \prod_{\beta \in \Sigma(\theta)^+ - \Sigma(\Omega)^+} u_{\alpha_1 + \beta}(-c_{\alpha_1, \beta} x_\beta)u_{\alpha_1}(1), \]

where the sum in the first term is over unordered pairs \((\beta, \delta)\) of roots in \( \Sigma(\theta)^+ - \Sigma(\Omega)^+ \) such that \( \beta + \delta = \gamma - \alpha_1 \) and \( \beta \neq \delta \). Here the order of terms is prescribed by the order we fixed in (39). This implies that \( uu_{\alpha_1}(1)u^{-1} = u_{\alpha_1}(1) \) if and only if \( x_\beta = 0 \) for all \( \beta \in \Sigma(\theta)^+ - \Sigma(\Omega)^+ \). Therefore,

\[ U_{M,n} = \left\{ \prod_{\beta \in \Sigma(\Omega)^+} u_\beta(x_\beta) \right\}. \tag{40} \]

To compute \( U'_{M,m} \), note that with \( d \) as in (31) we have

\[ m u m^{-1} = w' \gamma^\vee \left( \frac{d}{q} \right) \prod_{\beta \in \Sigma(\theta)^+} u_\beta(x_\beta) \gamma^\vee \left( \frac{d}{q} \right)^{-1} w'^{-1} \]

\[ = w' \prod_{\beta \in \Sigma(\theta)^+} u_\beta \left( \beta \left( \gamma^\vee \left( \frac{d}{q} \right) \right) x_\beta \right) w'^{-1} \]

\[ = \prod_{\beta \in \Sigma(\theta)^+} w' u_\beta \left( \beta \left( \gamma^\vee \left( \frac{d}{q} \right) \right) x_\beta \right) w'^{-1}. \]
Conjugation by the element \( u' \) sends each positive root group of a root in \( \Sigma(\theta)^+ - \Sigma(\Omega)^+ \) to a root group corresponding to a negative root and sends those with roots in \( \Sigma(\Omega)^+ \) to themselves. Therefore, again,

\[
U'_{M,m} = \left\{ \prod_{\beta \in \Sigma(\Omega)^+} u_\beta(x_\beta) \right\}
\]

Now (37) follows from (40) and (41).

We wish to have an explicit identification of \( GL_1(F) \times G_n^\sim(F) \) with \( M \) as a Levi subgroup of \( G \). Going back to our descriptions of the groups \( GSpin_{2n+1} \) and \( GSpin_n^\sim \) in Section 2, note that if we consider the root datum obtained from that of \( G \) by eliminating \( e_1 \) and \( e_1^\ast \) in the odd case and \( E_1 \) and \( E_1^\ast \) in the even case as well as the root \( \alpha_1 \) and its corresponding coroot, then the remaining root datum corresponds to a subgroup of \( G \) isomorphic to \( G_n^\sim \). Denote the \( F \)-points of this subgroup by \( G_n^\sim \). Let \( k \in G_n^\sim \), and let \( a \in F^\times \). We claim that \( e_1^\ast(a) \) in the odd case (or \( E_1^\ast(a) \) in the even case) and \( k \) commute. To see this, it is enough to observe that \( e_1^\ast(a) \) (or \( E_1^\ast(a) \)) commutes with \( u_\beta(x) \) for all \( \beta \in \Sigma(\theta) \) since \( G_n^\sim \) is generated by the corresponding \( U_\beta \)’s along with a subtorus of \( T \). By (15), we have

\[
e_1^\ast(a)u_\beta(x)e_1^\ast(a)^{-1} = u_\beta(\beta(e_1^\ast(a))x) = u_\beta(a^{(\beta,e_1^\ast)})x
\]

and similarly for \( E_1^\ast(a) \). Moreover, \( (\beta, e_1^\ast) = 0 \) for all \( \beta \in \Sigma(\theta) \). Therefore, \( e_1^\ast(a) \) in the odd case (or \( E_1^\ast(a) \) in the even case) and all the \( u_\beta(x) \) commute. This implies that the maps \( (a, k) \mapsto e_1^\ast(a)k \) in the odd case and \( (a, k) \mapsto E_1^\ast(a)k \) in the even case give an isomorphism identifying \( GL_1(F) \times G_n^\sim(F) \) with \( M \). In particular, the element \( m = \alpha_1^\vee(d/x)w' \) in (33) is identified with

\[
\left( \frac{d}{x}, e_2^\ast\left( \frac{x}{d} \right)w' \right) \text{ or } \left( \frac{d}{x}, E_2^\ast\left( \frac{x}{d} \right)w' \right) \in GL_1(F) \times G_n^\sim(F)
\]

since \( \alpha_1^\vee = e_1^\ast - e_2^\ast \) in the odd case and \( \alpha_1^\vee = E_1^\ast - E_2^\ast \) in the even case (and noting \( w' \in G_n^\sim \)). Moreover, \( e_2^\ast(x/d)w' \) (or \( E_2^\ast(x/d)w' \)) is an element of a maximal Levi subgroup in \( G \) just as in the case of classical groups in [41].

We are now prepared to express the \( \gamma \)-factors as Mellin transforms.

PROPOSITION 4.14

Let \( \sigma \) be an irreducible admissible \( \psi \)-generic representation of \( G_n^\sim(F) \) (see Remark 4.2). Consider \( GL_1 \times G_n^\sim \) as a standard Levi subgroup in \( G \) as above. Let \( \eta \) be any nontrivial character of \( F^\times \) with \( \eta^2 \) ramified. Then

\[
\gamma(s, \eta \times \sigma, \psi)^{-1} = g(s, \eta) \cdot J_{F^\times} j_{\psi, N_0, \eta} (a(x)w') \eta(x) |x|^{s-n-\delta} dx^\times,
\]

(43)
where \( a(x) = e_2^s(x/d) \) or \( E_2^s(x/d) \) is as in (42) (with \( d \) as in (31)), \( \delta = 1/2 \) and \( g(s, \eta) = \eta(-1)^n \) in the odd case and \( \delta = 1 \) and \( g(s, \eta) = \eta(-1)^{n-1} \psi(2s, \eta^2, \psi)^{-1} \) in the even case. Here \( v \in V_\sigma \) and \( W_v \in \mathcal{W}(\sigma, \psi) \) with \( W_v(e) = 1 \), where \( \mathcal{W}(\sigma, \psi) \) denotes the Whittaker model of \( \sigma \). Moreover, \( \overline{N}_0 \subset \overline{N} \) is a sufficiently large compact open subgroup of the opposite unipotent subgroup \( \overline{N} \) to \( N \), where \( P = MN \subset G \) is the Levi decomposition of the corresponding standard parabolic subgroup. The function \( j_v, \overline{N}_0 \) denotes the partial Bessel function defined in [41].

**Proof**

Given that \( \eta^2 \) is ramified, this proposition is the main result of [41, Theorem 6.2, (6.39)] applied to our cases. Notice that our Propositions 4.13 and 4.3 verify the two hypotheses of that theorem (i.e., [41, Assumptions 4.1, 5.1]) for our cases.

To get from [41, (6.39)] to (43), note that we have

\[
\omega_{\sigma_s}^{-1}(\dot{x}_\sigma)(w_0 \omega_{\sigma_s})(\dot{x}_\sigma) = \eta(x)^2 |x|^{2s}.
\]

Moreover, as in [41, Section 7], we have

\[
q^{(s\bar{a} + \rho, H_M(\dot{m}))} \, d\dot{m} = |x|^{-s-n+\delta} \, dx^x
\]

and

\[
j_{v, \overline{N}_0}(\dot{m}) = \eta \left( \frac{d}{x} \right) j_{v, \overline{N}_0}(a(x)w').
\]

where \( a(x) = e_2^s(x/d) \) in the odd case and \( E_2^s(x/d) \) in the even case. □

In Section 4.2 we first rewrite (43) in terms of Bessel functions defined similarly to [10], and then we study their asymptotics.

**4.2. Bessel functions and their asymptotics**

We now briefly review some basic facts from [10]. Because of [8], and particularly [12] and [13], where these issues are studied more generally, we concentrate only on the cases at hand and leave out the details of the more general situation.

We use the same notation as in [10]. Consider the group \( G_n \) in both even and odd cases, and consider \( w' \) as an element of its Weyl group. Notice that \( w' \) supports a Bessel function and (in Bruhat order) is minimal nontrivial with respect to this property. Moreover, \( A_w = Z_{M_\Omega} \), where \( \Omega = \Delta - \{\alpha_1, \alpha_2\} \), \( M_\Omega \) is the standard Levi subgroup determined by \( \Omega \), and \( Z_{M_\Omega} \) denotes its center.

Let \( \sigma \) be an irreducible admissible \( \psi \)-generic representation of \( G_n(F) \), and take \( v \in V_\sigma \) such that the associated Whittaker function \( W_v \in \mathcal{W}(\sigma, \psi) \) satisfies
$W_v(e) = 1$. The associated Bessel function on $Z_{M_{02}}$ is defined via

$$J_{\sigma, w'}(a) = \int_{U_{w'}} W_v(aw'u)\psi^{-1}(u) \, du$$  \hspace{1cm} (44)$$

with $a \in Z_{M_{02}}$ and $U_{w'} = \prod_\alpha U_\alpha$, where the product is over all those $\alpha \in \Sigma(\theta)^+$ for which $w'(\alpha) < 0$, and $U_\alpha$ is as before.

Similar to [10] and [8], we have that $J_{\sigma, w'}$ exists and is independent of $v \in V_\sigma$, and for convergence purposes we use a slight modification of it, namely, the partial Bessel function

$$J_{\sigma, w', v, Y}(a) = \int_Y W_v(aw'y)\psi^{-1}(y) \, dy,$$  \hspace{1cm} (45)$$

where $Y \subset U_{w'}$ is a compact open subgroup.

4.2.1. Domain of integration

We now show that the partial Bessel functions of [41] are the same as those in [10].

Recall that $M = M_0 = \text{GL}_1(F) \times G_{\sim}^n(F) \subset G = G(F)$, and consider $m \in M$ as in (42); that is, $m = (d/x, e_2^*(x/d)w')$ in the odd case and $m = (d/x, E_2^*(x/d)w')$ in the even case with $d$ as in (31).

**Lemma 4.15**

We can choose $Y$ appropriately such that with $m' = e_2^*(x/d)w'$ or $E_2^*(x/d)w'$ as above, we have

$$j_{v, N_0}(m') = \begin{cases} 
J_{\sigma, w', v, Y} \left( e_2^* \left( \frac{x}{d} \right) \right) & \text{in the odd case,} \\
J_{\sigma, w', v, Y} \left( E_2^* \left( \frac{x}{d} \right) \right) & \text{in the even case.}
\end{cases}$$

Here, $m'$ is an element of the maximal Levi subgroup $M' = \text{GL}_1(F) \times G_{n-1}(F)$ in $G_{\sim}(F)$.

**Proof**

Let us first recall [41, Theorem 6.2]. In the notation of that article, we have $j_{v, \overline{N}_0}(m') = j_{v, \overline{N}_0}(m', y_0)$ with $y_0 \in F^\times$ satisfying $\text{ord}_F(y_0) = -\text{cond}(\psi) - \text{cond}(\eta^2)$. Here, the function $j_{v, \overline{N}_0}(m', y_0)$ is given by

$$\int_{U_{M'} \setminus U_M} W_v(m'u^{-1}) \phi(u e^*(y_0)^{-1} e^*(x_{\alpha})\overline{n} e^*(x_{\alpha})^{-1} e^*(y_0) u^{-1}) \psi(u) \, du,$$  \hspace{1cm} (46)$$

where $\phi$ is the characteristic function of $\overline{N}_0, x_{\alpha} = 1/x, \overline{n}$ is as in Proposition 4.12, and $e^*$ is as in Proposition 4.3. Again as in [10] and [8], it follows from Proposition 4.3...
that we can take $U_{M,n} \setminus U_M$ to be $U_w^-$. Notice that this depends only on $w'$. On the other hand, $u \in U_w^-$ is in the domain of integration if and only if

$$ue^*(y_0)^{-1}e^*(x_\alpha)\bar{n}e^*(x_\alpha)^{-1}e^*(y_0)u^{-1} \in \bar{N}_0.$$ 

This condition is equivalent to $ue^*(x_\alpha)\bar{n}e^*(x_\alpha)^{-1}u^{-1} \in e^*(y_0)\bar{N}_0e^*(y_0)^{-1}$, and $e^*(y_0)\bar{N}_0e^*(y_0)^{-1}$ is another compact open subgroup of the same type as $\bar{N}_0$, which we may replace it with.

Recall that an arbitrary element of $\bar{N}_0$ is given by

$$\bar{n}(y) = \prod_{\alpha \in R(\bar{N})} u_\alpha(y_\alpha),$$

(47)

where $y = (y_\alpha)_{\alpha \in R(\bar{N})}$ and $y_\alpha \in F$. Also recall that $\bar{n}$ in (46) was given by $\bar{n} = u_{-\gamma}(1/x)u_{-\alpha_1}(1)$. Moreover, note that $x_\alpha = 1/x$. Hence, $e^*(x_\alpha)\bar{n}e^*(x_\alpha)^{-1} = \bar{n}(y')$, where $y'_\gamma = 1$, $y'_{-\alpha_1} = x$, and all other coordinates of $y'$ are zero (see (15)). Of course, throughout we have a fixed ordering of the roots in the products similar to that of (28). Hence, the domain of integration is determined by $u\bar{n}(y')u^{-1} \in \bar{N}_0$.

We may take $\bar{N}_0 = \{\bar{n}(y) : y_\alpha \in p^{M_\alpha}\}$ for all $\alpha \in R(\bar{N})$ for a sufficiently large integer vector $M = (M_\alpha)_{\alpha \in R(\bar{N})}$. As the $M_\alpha$’s increase, $\bar{N}_0$ exhausts $\bar{N}$.

On the other hand, any $u \in U_w^-$ is given by

$$u = u(b) = \prod_{\alpha \in \Sigma^+(\Omega)} u_\alpha(b_\alpha)$$

(48)

for $b = (b_\alpha)_{\alpha \in \Sigma^+(\Omega)}$ and $b_\alpha \in F$. Now, $u\bar{n}(y')u^{-1} = \bar{n}(y'')$, where $y''$ depends linearly on $b$ and $y'$. In other words, $y''$ depends upon $x$ and $b$. Of course, we could compute $y''$ explicitly in terms of $x$ and $b$ using structure constants; however, that has no bearing on what follows and is not needed. Now, choose $Y = \{u = u(b) : y'' \geq M\}$. This defines the domain of integration. Enlarging $\bar{N}_0$ if need be would then imply that the domain $Y$ does not depend on $m$, and we conclude the lemma.

Therefore, we can rewrite (43) as

$$\gamma(s, \eta \times \sigma, \psi)^{-1} = g(s, \eta) \cdot \int_{F^x} J_{\sigma,w',v}(a(x))\eta(x)|x|^{s-n-\delta} dx$$

(49)

with $a(x) = e^*_2(x/d)$ or $E^*_2(x/d)$.

4.2.2. Asymptotics of Bessel functions

We now study the asymptotics of our Bessel functions near zero and infinity. This allows us to prove our result on stability given that the $\gamma$-factors are already written as the Mellin transform of Bessel functions.
Our starting point is the following analogue of [10, Proposition 5.1] for our groups. Note that the proposition was proved only for the group $SO_{2n+1}$. However, as was pointed out in [8], the methods used to prove it are quite general. This was pointed out for classical groups (with finite center) in [8]. But the same also holds for GSpin or GSpin~ groups (see [12]), which are of interest to us since the only difference is the infinite center that is already contained in the fixed Borel subgroup we are dividing out with.

**Proposition 4.16**

There exist a vector $v'_\sigma \in V_\sigma$ and a compact neighborhood $B K_1$ of the identity in $B \setminus G_n^\sim$ such that if $\chi_1$ is the characteristic function of $B K_1$, then for all sufficiently large compact open sets $Y \subset U_{\overline{w}}$ we have

$$J_{\sigma,w,v,Y}(a) = \int_Y W_v(awy) \chi_1(awy) \psi^{-1}(y) dy + W_{v'_\sigma}(a). \quad (50)$$

Notice that $w$ here would be our earlier $w'$ if we wanted to consider the group $G_n^\sim$ as part of the Levi subgroup $M$ in $G$, as in (42).

We now rewrite this in a way that depends only on the central character of $\sigma$. To this end, we argue similarly to [8], making some necessary modifications along the way. For any positive integer $M$, set

$$U(M) = \{ u_\alpha(t) : \alpha \in \Delta, |t| \leq q^M \}.$$  

These are compact open subgroups of $U$, and as the integer $M$ grows, they exhaust $U$. For any $v \in V_\sigma$, we define

$$v_M = \frac{1}{\text{Vol}(U(M))} \int_{U(M)} \psi^{-1}(u) \sigma(u) v du.$$  

The smoothness of $\sigma$ implies that this is a finite sum and $v_M \in V_\sigma$. Then just as in [10], if $Y$ is sufficiently large relative to $M$, we may choose $v'_\sigma$ and $K_1$ such that $K_1 \subset \text{Stab}(v_M)$, and we have

$$\int_Y W_v(awy) \chi_1(awy) \psi^{-1}(y) dy = \int_Y W_{v_M}(awy) \chi_1(awy) \psi^{-1}(y) dy. \quad (51)$$

Write $awy = utk_1$ with $u \in U$, $t \in T$, and $k_1 \in K_1$. Then $K_1 \subset \text{Stab}(v_M)$ implies that $W_{v_M}(awy) = \psi(u) W_{v_M}(t)$. As in [10], the support of $W_{v_M}$ is contained in

$$T_M = \{ t \in T : \alpha(t) \in 1 + p^M \text{ for all simple } \alpha \}. \quad (52)$$
At this point we assume that the center $Z$ is connected, which we can do (see Remark 4.2). This is why we chose to work with the group $\text{GSpin}^\sim$ in the even case in this section. By the connectedness of $Z$, and since the groups are split, we have the following exact sequence of $F$-points of tori:

$$0 \rightarrow Z \rightarrow T \rightarrow T_{ad} \rightarrow 0,$$

(53)

which splits (see [12], [13]). Recall that $Z = Z(F)$ and so on. Identify $T_{ad}$ with $(F \times)^n$ through values of roots, and let $T_M^1 \subset T$ be the image of

$$(1 + p^M)^n \subset T_{ad},$$

under the splitting map. Here, the rank of $T_{ad}$ is $n$ and $T_M = Z T_M^1$.

Now if $t \in T$, then we can write $t = z t^1$ with $z \in Z$ and $t \in T_M^1$. Also, we have $W_v(t) = W_{v_M}(t)$, and if we choose $M$ large enough, so that $T_M^1 \subset T \cap \text{Stab}(v)$, then

$$W_{v_M}(t) = W_{v_M}(zt^1) = \omega_{\sigma}(z) W_v(t^1) = \omega_{\sigma}(z).$$

Next, we note that in our integral, $W_{v_M}(aw y) \chi_1(aw y) \neq 0$ if and only if $aw y \in UT_M K_1$ or $y \in (aw)^{-1}UT_M K_1$. Writing $aw y = ut k_1 = u(aw)z(aw)t^1 k_1$ then implies that

$$\int_{Y} W_v(aw y) \chi_1(aw y) \psi^{-1}(y) dy = \int_{Y \cap (aw)^{-1}UT_M K_1} \psi(u(aw)) \psi^{-1}(y) \omega_{\sigma}(z(aw)) dy.$$

(54)

Therefore, we can rewrite Proposition 4.16 as follows.

PROPOSITION 4.17
Let $v \in V_{\sigma}$ with $W_v(e) = 1$, and choose $M$ sufficiently large, so that $T_M^1 \subset T \cap \text{Stab}(v)$. There exist a vector $v'_\sigma \in V_{\sigma}$ and a compact open subgroup $K_1$ such that for $Y$ sufficiently large we have

$$J_{\sigma,w,v,Y}(a) = \int_{Y \cap (aw)^{-1}UT_M K_1} \psi(u(aw)) \psi^{-1}(y) \omega_{\sigma}(z(aw)) dy + W_{v'_\sigma}(a).$$

4.3. Proof of Theorem 4.1
Let $\sigma_i = \pi_i$, $i = 1, 2$, in the odd case. In the even case, choose a character $\mu$ of the center $Z^\sim$ of $\text{GSpin}^\sim_{2n}(F)$ (which contains the center of $\text{GSpin}_{2n}(F)$) such that $\mu$ agrees with the central characters $\omega_{\pi_1} = \omega_{\pi_2}$ on the center of $\text{GSpin}_{2n}(F)$. Consider the representation of $\text{GSpin}^\sim_{2n}(F)$ induced from the representation $\mu \otimes \pi_i$ on $Z^\sim \cdot \text{GSpin}_{2n}(F)$ (which is of finite index in $\text{GSpin}^\sim_{2n}(F)$), and let $\sigma_i$ be an irreducible constituent of this induced representation (see [45]). Note that the choice of $\sigma_i$ is
irrelevant. Then
\[ \gamma(s, \eta \times \sigma_i, \psi) = \gamma(s, \eta \times \pi_i, \psi), \]
by Remark 4.2. Also, the assumption \( \omega_{\pi_1} = \omega_{\pi_2} \) implies \( \omega_{\sigma_1} = \omega_{\sigma_2} \).

Choose \( v_i \in V_{\sigma_i}, i = 1, 2, \) with \( W_{v_i}(e) = 1 \), and let \( M \) be a large-enough integer, so that \( T_M \subset T \cap \text{Stab}(v_i) \). Choose a compact open subgroup \( K_0 \subset \text{Stab}(v_1) \cap \text{Stab}(v_2) \). Then in Proposition 4.17 we may take
\[ K_1 = \bigcap_{u \in U(M)} u^{-1}K_0u; \]
that is, we can take the same \( K_1 \) for both \( \sigma_1 \) and \( \sigma_2 \). Consequently, by Proposition 4.17, there exist \( v'_{\sigma_i} \in V_{\sigma_i} \) such that
\[ J_{\sigma_i,w,v,Y}(a) = \int_{Y \cap (aw)^{-1}UTMK_1} \psi(u(awy))\psi^{-1}(y)\omega_{\sigma_i}(z(awy))dy + W_{v'_{\sigma_i}}(a). \quad (55) \]

Now \( \omega_{\sigma_1} = \omega_{\sigma_2} \) implies that
\[ J_{\sigma_1,w,v,Y}(a) - J_{\sigma_2,w,v,Y}(a) = W_{v'_{\sigma_1}}(a) - W_{v'_{\sigma_2}}(a). \quad (56) \]

Now taking \( a = a(x) \) to be \( e_2^*(x/d) \) or \( E_2^*(x/d) \) and \( w \) to be the \( w' \) described before, we apply (49) to conclude that
\[ \gamma(s, \eta \times \sigma_1, \psi)^{-1} - \gamma(s, \eta \times \sigma_2, \psi)^{-1} = g(s, \eta) \int_{F^\times} \left( J_{\sigma_1,w,v,Y}(a(x)) - J_{\sigma_2,w,v,Y}(a(x)) \right) \eta(x)|x|^{s-n+\delta}d^\times x \]
\[ = g(s, \eta) \int_{F^\times} \left( W_{v'_{\sigma_1}}(a(x)) - W_{v'_{\sigma_2}}(a(x)) \right) \eta(x)|x|^{s-n+\delta}d^\times x. \]

However, note that Whittaker functions are smooth, and for \( \Re(s) \gg 0 \) and \( \eta \) sufficiently ramified, we have
\[ \int_{F^\times} W_{v'_{\sigma_1}}(a(x))\eta(x)|x|^{s-n+\delta}d^\times x \equiv 0. \]

Hence, for \( \Re(s) \gg 0 \), we have \( \gamma(s, \eta \times \sigma_1, \psi)^{-1} - \gamma(s, \eta \times \sigma_2, \psi)^{-1} \equiv 0 \), which then implies \( \gamma(s, \eta \times \sigma_1, \psi) = \gamma(s, \eta \times \sigma_2, \psi) \) for all \( s \) by analytic continuation. Therefore, \( \gamma(s, \eta \times \pi_1, \psi) = \gamma(s, \eta \times \pi_2, \psi) \).

4.4. Stable form of \( \gamma(s, \eta \times \pi, \psi) \)
We now prove some consequences of Theorem 4.1 that are needed later.
First, let us compute the stable form of Theorem 4.1 by taking \( \pi_2 \) to be an appropriate principal series representation and computing its right-hand side explicitly.

**Proposition 4.18**

Let \( \pi \) be an irreducible generic representation of \( G_n(F) \) with central character \( \omega = \omega_\pi \). Let \( \mu_1, \ldots, \mu_n \) be \( n \) characters of \( F^\times \). Then for every sufficiently ramified character \( \eta \) of \( F^\times \) we have

\[
\gamma(s, \eta \times \pi, \psi) = \prod_{i=1}^{n} \gamma(s, \eta \mu_i, \psi) \gamma(s, \eta \omega \mu_i^{-1}, \psi).
\]

**Proof**

Set \( \mu_0 = \omega \), and consider the character

\[
\mu = (\mu_0 \circ e_0) \otimes (\mu_1 \circ e_1) \otimes \cdots \otimes (\mu_n \circ e_n)
\]

of \( T(F) \) with \( e_i \)'s as in Section 2.1. Proposition 2.3 implies that the restriction of the character \( \mu \) to the center of \( G_n(F) \) is \( \mu_0 = \omega \). Consider the induced representation \( \text{Ind}(\mu) \) from the Borel subgroup to \( G_n(F) \). Reordering the \( \mu_i \) if necessary, we may assume that it has an irreducible admissible generic subrepresentation \( \pi_2 \) (see Proposition 3.2). Since \( \omega_{\pi_2} = \mu_0 = \omega = \omega_\pi \), we can apply Theorem 4.1 to get

\[
\gamma(s, \eta \times \pi_2, \psi) = \gamma(s, \eta \times \pi_2, \psi).
\]

The multiplicativity of \( \gamma \)-factors can now be used to compute the right-hand side to get

\[
\gamma(s, \eta \times \pi, \psi) = \prod_{i=1}^{n} \gamma(s, \eta \mu_i, \psi) \gamma(s, \eta \omega \mu_i^{-1}, \psi).
\]  

(57)

This completes the proof. \( \square \)

**Corollary 4.19**

Let \( \pi \) be an irreducible generic representation of \( G_n(F) \) with central character \( \omega = \omega_\pi \). Let \( \mu_1, \ldots, \mu_n \) be \( n \) characters of \( F^\times \) as in Proposition 4.18. Then for every sufficiently ramified character \( \eta \) of \( F^\times \) we have

\[
L(s, \eta \times \pi) \equiv 1
\]

and

\[
\epsilon(s, \eta \times \pi, \psi) = \prod_{i=1}^{n} \epsilon(s, \eta \mu_i, \psi) \epsilon(s, \eta \omega \mu_i^{-1}, \psi).
\]
Proof
If \( \eta \) is sufficiently ramified, then by [40] we have
\[
L(s, \eta \times \pi) \equiv 1.
\]
This implies that \( \epsilon(s, \eta \times \pi, \psi) = \gamma(s, \eta \times \pi, \psi) \). Moreover, since \( \eta \) is highly ramified, so is each \( \eta \mu_i \) and \( \eta \omega \mu_i^{-1} \). This implies that \( L(s, \eta \mu_i) \equiv 1 \) and \( L(s, \eta \omega \mu_i^{-1}) \equiv 1 \). Therefore, \( \epsilon(s, \eta \mu_i, \psi) = \gamma(s, \eta \mu_i, \psi) \) and \( \epsilon(s, \eta \omega \mu_i^{-1}, \psi) = \gamma(s, \eta \omega \mu_i^{-1}, \psi) \). The second statement of the corollary follows from Proposition 4.18. \( \square \)

5. Analytic properties of global \( L \)-functions
In this section we prove the properties of global \( L \)-functions which we need in order to apply the Converse Theorems.

We again let \( G_n \) denote either the group \( \text{GSpin}_{2n+1} \) or \( \text{GSpin}_{2n} \) as in Section 2.1. Let \( k \) be a number field, and let \( \mathbb{A} \) be its ring of adeles. Let \( S \) be a finite set of finite places of \( k \). Let \( \mathcal{F}(S) \) denote the set of irreducible cuspidal automorphic representations \( \tau \) of \( \text{GL}_r(\mathbb{A}) \) for \( 1 \leq r \leq N-1 \) such that \( \tau_v \) is unramified for all \( v \in S \). If \( \eta \) is a continuous complex character of \( k^\times \backslash \mathbb{A}^\times \), then we let \( \mathcal{T}(S; \eta) = \{ \tau = \tau' \otimes \eta : \tau' \in \mathcal{F}(S) \} \).

If \( \pi \) is a globally generic cuspidal representation of \( G_n(\mathbb{A}) \) and \( \tau \) is a cuspidal representation of \( \text{GL}_r(\mathbb{A}) \) in \( \mathcal{F}(S; \eta) \), then \( \sigma = \tau \otimes \tilde{\pi} \) is a (unitary) cuspidal globally generic representation of \( \text{M}(\mathbb{A}) \), where \( \text{M} = \text{GL}_r \times G_n \) is a Levi subgroup of a standard parabolic subgroup in \( G_{r+n} \). The machinery of the Langlands-Shahidi method as mentioned in Section 3 now applies (see [37], [39]). Recall that
\[
L(s, \pi \times \tau) = \prod_v L(s, \pi_v \times \tau_v), \tag{58}
\]
\[
\epsilon(s, \pi \times \tau) = \prod_v \epsilon(s, \pi_v \times \tau_v, \psi_v), \tag{59}
\]
where the local factors are as in (5) and (6).

PROPOSITION 5.1
Let \( S \) be a nonempty set of finite places of \( k \), and let \( \eta \) be a character of \( k^\times \backslash \mathbb{A}^\times \) such that \( \eta_v \) is highly ramified for \( v \in S \). Then for all \( \tau \in \mathcal{F}(S; \eta) \), the \( L \)-function \( L(s, \pi \times \tau) \) is entire.

Proof
These \( L \)-functions are defined via the Langlands-Shahidi method, as we outlined in Section 3. The proposition is a special case of a more general result, [24, Theorem 2.1] (see [20] for the original idea). Note that we have proved the necessary assumption of that theorem, [24, Assumption 1.1], for our cases in Proposition 3.6. \( \square \)
The following lemma is an immediate consequence of Proposition 3.6.

LEMMA 5.2
The global normalized intertwining operator $N(s, \sigma, w)$ is a holomorphic and nonzero operator for $\Re(s) \geq 1/2$.

PROPOSITION 5.3
For any cuspidal automorphic representation $\tau$ of $\text{GL}_r(\mathbb{A}_F)$, $1 \leq r \leq 2n - 1$, the $L$-function $L(s, \pi \times \tau)$ is bounded in vertical strips.

Proof
This follows as a consequence of [15, Theorem 4.1] along the lines of [15, Corollary 4.5], given the fact that we have proved [15, Assumption 2.1] in our Lemma 5.2 for our cases.

PROPOSITION 5.4
For any cuspidal automorphic representation $\tau$ of $\text{GL}_r(\mathbb{A}_F)$, $1 \leq r \leq 2n$, we have the functional equation

$$L(s, \pi \times \tau) = \epsilon(s, \pi \times \tau)L(1 - s, \tilde{\pi} \times \tilde{\tau}).$$

Proof
This is a special case of [39, Theorem 7.7].

6. Proof of the main theorem
As mentioned before, we use the following variant of converse theorems of Cogdell and Piatetski-Shapiro. This version of the Converse Theorems appeared in [7, Section 2].

THEOREM 6.1
Let $\Pi = \bigotimes \Pi_v$ be an irreducible admissible representation of $\text{GL}_N(\mathbb{A})$ whose central character $\omega_{\Pi}$ is invariant under $k^\times$ and whose $L$-function $L(s, \Pi) = \prod_v L(s, \Pi_v)$ is absolutely convergent in some right half-plane. With notation as in Section 5, suppose that for every $\tau \in \mathcal{F}(S; \eta)$, we have that:

1. $L(s, \Pi \times \tau)$ and $L(s, \tilde{\Pi} \times \tilde{\tau})$ extend to entire functions of $s \in \mathbb{C}$;
2. $L(s, \Pi \times \tau)$ and $L(s, \tilde{\Pi} \times \tilde{\tau})$ are bounded in vertical strips; and
3. $L(s, \Pi \times \tau) = \epsilon(s, \Pi \times \tau)L(1 - s, \tilde{\Pi} \times \tilde{\tau})$.

Then there exists an automorphic representation $\Pi'$ of $\text{GL}_N(\mathbb{A})$ such that $\Pi_v \simeq \Pi'_v$ for all $v \notin S$. 
Here, the twisted $L$- and $\epsilon$-factors are defined via

$$L(s, \Pi \times \tau) = \prod_v L(s, \Pi_v \times \tau_v), \quad \epsilon(s, \Pi \times \tau) = \prod_v \epsilon(s, \Pi_v \times \tau_v, \psi_v)$$

with local factors as in [9].

We can now prove Theorem 1.1.

**Proof of Theorem 1.1**

We apply Theorem 6.1 with $N = 2n$. We continue to denote by $G_n$ either $GSpin_{2n+1}$ or $GSpin_{2n}$. First, we introduce a candidate for the representation $\Pi$. Consider $\pi = \bigotimes \pi_v$, and let $S$ be as in the statement of the theorem, that is, a nonempty set of non-Archimedean places $v$ such that for all finite $v \notin S$, both $\pi_v$ and $\psi_v$ are unramified.

(i): $v < \infty$ and $\pi_v$ unramified. Choose $\Pi_v$ as in the statement of the theorem via the Frobenius-Hecke (or Satake) parameter. More precisely, since $\pi_v$ is unramified, it is given by an unramified character $\chi$ of the maximal torus $T(k_v)$. This means that there are unramified characters $\chi_0, \chi_1, \ldots, \chi_n$ of $k_v^\times$ such that for $t \in T(k_v)$,

$$\chi(t) = (\chi_0 \circ e_0)(t)(\chi_1 \circ e_1)(t) \cdots (\chi_n \circ e_n)(t),$$

(60)

where $e_i$’s form the basis of the rational characters of the maximal torus of $G$ as in Section 2.1. The character $\chi$ corresponds to an element $\hat{t}$ in $\hat{T}$, the maximal torus of (the connected component) of the Langlands dual group which is $GSp_{2n}(\mathbb{C})$ or $GSO_{2n}(\mathbb{C})$, uniquely determined by the equation

$$\chi(\phi(\sigma)) = \phi(\hat{t}),$$

(61)

where $\sigma$ is a uniformizer of our local field $k_v$ and $\phi \in X_*(T) = X^*(\hat{T})$ (see [14, (1.2.3.3), page 26]). We make this identification explicit via the correspondence $e_i^* \leftrightarrow e_i$ for $i = 0, \ldots, n$ as in Section 2.1, which gave the duality of $GSpin_{2n+1} \leftrightarrow GSp_{2n}$ and $GSpin_{2n} \leftrightarrow GSO_{2n}$. Applying (61) with the $\phi$ on the left-hand side replaced with $e_i^*$ and the one on the right-hand side replaced with $e_i$ for $i = 0, 1, \ldots, n$ yields

$$\chi_i(\sigma) = \chi(e_i^*(\sigma)) = e_i(\hat{t}), \quad i = 0, 1, \ldots, n.$$  

(62)

We can now compute the Satake parameter explicitly as an element $\hat{t}$ in the maximal torus $\hat{T}$ of $GSp_{2n}(\mathbb{C})$ or $GSO_{2n}(\mathbb{C})$, as described in (3). If we write our
unramified characters as \( \chi_i(\cdot) = |\cdot|_v^{s_i} \) for \( s_i \in \mathbb{C} \) and \( 0 \leq i \leq n \), then we get

\[
\hat{t} = \begin{pmatrix}
|\varpi|^{s_1} \\
\vdots \\
|\varpi|^{s_n} \\
|\varpi|^{s_0-s_n} \\
\vdots \\
|\varpi|^{s_0-s_1}
\end{pmatrix}.
\] (63)

Hence, \( \Pi_v \) is the unique unramified constituent of the representation of \( \text{GL}_{2n}(k_v) \) induced from the character

\[
\chi_1 \otimes \cdots \otimes \chi_n \otimes \chi_0 \chi_n^{-1} \otimes \cdots \otimes \chi_0 \chi_1^{-1}
\] (64)
of the \( k_v \)-points of the standard maximal torus in \( \text{GL}_{2n} \).

A crucial point here is what the central characters of \( \pi_v \) and \( \Pi_v \) are. It follows from Proposition 2.3 that the central character \( \omega_{\pi_v} = \chi_0 \). Moreover, the central character \( \omega_{\Pi_v} \) of \( \Pi_v \) is \( \chi_0^n \); hence, we have \( \omega_{\Pi_v} = \omega_{\pi_v}^n \).

Furthermore, note that \( \tilde{\Pi}_v \) is the unique unramified constituent of the representation induced from

\[
\chi_1^{-1} \otimes \cdots \otimes \chi_n^{-1} \otimes \chi_0^{-1} \chi_n \otimes \cdots \otimes \chi_0^{-1} \chi_1.
\]
Therefore, we have \( \tilde{\Pi}_v \simeq \chi_0^{-1} \otimes \Pi_v \). In other words, \( \Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v} \).

(ii): \( v|\infty \). Choose \( \Pi_v \) as in the statement of Theorem 1.1 (see [30]). To be more precise, Langlands associates to \( \pi_v \) a homomorphism \( \phi_v \) from the local Weil group \( W_v = W_{k_v} \) to the dual group \( \hat{G} \) which is \( \text{GSp}_{2n}(\mathbb{C}) \) or \( \text{GSO}_{2n}(\mathbb{C}) \) in our cases. Both of these groups have natural embeddings \( \iota \) into \( \text{GL}_{2n}(\mathbb{C}) \), and we take \( \Pi_v \) to be the irreducible admissible representation of \( \text{GL}_{2n}(k_v) \) associated to \( \Phi_v = \phi_v \circ \iota \) in [30].

Again, we want to show that \( \omega_{\Pi_v} = \omega_{\pi_v}^n \) and \( \Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v} \). To do this, we use some well-known facts regarding representations of \( W_v \) and local Langlands correspondence for \( \text{GL}_n(\mathbb{R}) \) and \( \text{GL}_n(\mathbb{C}) \). We refer to [26] for a nice survey of these results.

First, assume that \( k_v = \mathbb{C} \). Then \( W_v = \mathbb{C}^\times \) and any irreducible representation of \( W_v \) is one-dimensional and of the form

\[
z \mapsto [z]^\ell |z|^t, \quad \ell \in \mathbb{Z}, \ t \in \mathbb{C},
\]
where \( [z] = z/|z| \) and \( |z|_\mathbb{C} = |z|^2 \).
The 2n-dimensional representation \( \Phi_v \) of \( W_v \) can now be written as a direct sum of \( 2n \) one-dimensional representations as above. Moreover, \( \Phi_v(z) = \phi_v(z) \), considered as a diagonal matrix in \( \text{GL}_{2n}(\mathbb{C}) \), actually lies, up to conjugation, in \( \text{GSp}_{2n}(\mathbb{C}) \) or \( \text{GSO}_{2n}(\mathbb{C}) \) as in (3). Therefore, there exist one-dimensional representations \( \phi_0, \phi_1, \ldots, \phi_n \) as above such that \( \Phi_v \) is the direct sum of \( \phi_1, \ldots, \phi_n, \phi_n^{-1} \phi_0, \ldots, \phi_1^{-1} \phi_0 \). Now the central characters of \( \Pi_v \) and \( \pi_v \) can be written as \( \omega_{\Pi_v}(z) = \det(\Phi_v(z)) \) and \( \omega_{\pi_v}(z) = e_0(\phi_v(z)) \), where \( \phi_v(z) = \Phi_v(z) \) is considered as a diagonal \((2n \times 2n)\)-matrix as in (3) and \( e_0 \) is as in (4). In other words, \( \omega_{\pi_v} = \phi_0 \) and \( \omega_{\Pi_v} = \phi_0^\ell \) or \( \omega_{\Pi_v} = \omega_{\pi_v}^n \).

Moreover, \( \tilde{\Pi}_v \) corresponds to the 2n-dimensional representation of \( W_v \) which is the direct sum of \( \phi_1^{-1}, \ldots, \phi_n^{-1}, \phi_n \phi_0^{-1}, \ldots, \phi_1 \phi_0^{-1} \), implying that the two representations \( \Pi_v \) and \( \tilde{\Pi}_v \otimes \omega_{\pi_v} \) have the same parameters; that is, \( \Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v} \).

Next, assume that \( k_v = \mathbb{R} \). Then \( W_v = \mathbb{C}^\times \cup j\mathbb{C}^\times \) with \( j^2 = -1 \) and \( j \bar{z} j^{-1} = \bar{z} \) for \( z \in \mathbb{C}^\times \). Here the situation is identical, and the only difference is that \( W_v \) also has two-dimensional irreducible representations. The one-dimensional representations of \( W_v \) can be described as

\[
\begin{align*}
z &\mapsto |z|_{\mathbb{R}}, \quad j \mapsto 1, \quad t \in \mathbb{C}, \\
z &\mapsto |z|_{\mathbb{R}}, \quad j \mapsto -1, \quad t \in \mathbb{C},
\end{align*}
\]

with \( |z|_{\mathbb{R}} = |z| \), and the irreducible two-dimensional representations are of the form

\[
z = re^{i\theta} \mapsto \begin{pmatrix} r e^{i\ell \theta} & \phantom{1} \\ -\ell e^{-i\ell \theta} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & (-1)^\ell \\ 1 & 0 \end{pmatrix},
\]

where \( t \in \mathbb{C} \) and \( \ell \geq 1 \) is an integer. These correspond, respectively, to representations \( 1 \otimes | \cdot |_{\mathbb{R}} \) and \( \text{sgn} \otimes | \cdot |_{\mathbb{R}} \) of \( \text{GL}_1(\mathbb{R}) \) and \( D_\ell \otimes | \cdot |_{\mathbb{R}} \) of \( \text{GL}_2(\mathbb{R}) \). Here, \( D_\ell \) is the representation of \( \text{SL}_2^+(\mathbb{R}) \) induced from the discrete series (limit of discrete series when \( \ell = 1 \)) representation \( D_1^+ \) on the group \( \text{SL}_2(\mathbb{R}) \), the discrete series of lowest weight \( \ell + 1 \) (see [26, Section 2]).

Notice that again \( \Phi_v(z) = \phi_v(z) \) is a diagonal \((2n \times 2n)\)-matrix in \( \text{GSp}_{2n}(\mathbb{C}) \) or \( \text{GSO}_{2n}(\mathbb{C}) \), as in the previous case, while \( \Phi_v(j) \) may have \((2 \times 2)\)-blocks as well. Therefore, we still have \( \omega_{\Pi_v} = \omega_{\pi_v}^n \) and \( \Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v} \) as before.

(iii): \( v < \infty \) and \( \pi_v \) ramified. Choose \( \Pi_v \) to be an arbitrary irreducible admissible representation of \( \text{GL}_{2n}(k_v) \) with \( \omega_{\Pi_v} = \omega_{\pi_v}^n \).

Let \( \Pi = \bigotimes_v \Pi_v \). Then \( \Pi \) is an irreducible admissible representation of \( \text{GL}_{2n}(\mathbb{A}) \) whose central character \( \omega_\Pi \) is equal to \( \omega_{\pi_v}^n \) and hence is invariant under \( k^\times \). Moreover, for all \( v \not\in S \), we have that \( L(s, \pi_v) = L(s, \Pi_v) \) by construction. Hence, \( L^S(s, \Pi) = \)
\( L^S(s, \pi), \) where

\[
L^S(s, \Pi) = \prod_{v \not\in S} L(s, \Pi_v), \quad L^S(s, \pi) = \prod_{v \not\in S} L(s, \pi_v).
\]

Therefore, \( L(s, \Pi) = \prod_v L(s, \Pi_v) \) is absolutely convergent in some right half-plane.

Choose \( \eta = \bigotimes_v \eta_v \) to be a unitary character of \( k^\times \backslash \A^\times \) such that \( \eta_v \) is sufficiently ramified for \( v \in S \) in order for Theorem 4.1 to hold and such that at one place \( \eta_v^2 \) is still ramified. For \( \tau \in \mathcal{F}(S; \eta) \), we claim the following equalities (along with their analogous equalities for the contragredients):

\[
L(s, \Pi_v \times \tau_v) = L(s, \pi_v \times \tau_v), \quad (65)
\]

\[
\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \epsilon(s, \pi_v \times \tau_v, \psi_v). \quad (66)
\]

Here the \( L \)- and \( \epsilon \)-factors on the left are as in [9], and those on the right are defined via the Langlands-Shahidi method (see [39], [37]).

To see (65) and (66), we again consider different places separately.

(i): \( v < \infty \) and \( \pi_v \) unramified. Let \( \pi_v \) be again as in (60) with Satake parameter (63). Then \( \Pi_v \) is as in (64). By [17], we have

\[
L(s, \Pi_v \times \tau_v) = \prod_{i=1}^n L(s, \tau_v \otimes \chi_i) L(s, \tau_v \otimes \chi_0 \chi_i^{-1}), \quad (67)
\]

\[
L(s, \tilde{\Pi}_v \times \tilde{\tau}_v) = \prod_{i=1}^n L(s, \tilde{\tau}_v \otimes \chi_i^{-1}) L(s, \tilde{\tau}_v \otimes \chi_0 \chi_i^{-1}),
\]

and

\[
\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{i=1}^n \epsilon(s, \tau_v \otimes \chi_i, \psi_v) \epsilon(s, \tau_v \otimes \chi_0 \chi_i^{-1}, \psi_v). \quad (68)
\]

On the other hand, it follows from the inductive property of \( \gamma \)-factors in the Langlands-Shahidi method (see [39, Theorem 3.5] or [38]) that

\[
\gamma(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^n \gamma(s, \tau_v \otimes \chi_i, \psi_v) \gamma(s, \tau_v \otimes \chi_0 \chi_i^{-1}, \psi_v), \quad (69)
\]

just as in (57).

Since \( \tau_v \) is generic, it is a full-induced representation from generic essentially tempered ones. Thus, we can write

\[
\tau_v \simeq \text{Ind}(v^{b_1} \tau_{1,v} \otimes \cdots \otimes v^{b_p} \tau_{p,v}), \quad (70)
\]
where each $\tau_{j,v}$ is a tempered representation of some $\text{GL}_{r_j}(k_v)$, $\nu(\ ) = |\det(\ )|_v$ on $\text{GL}_{r_j}(k_v)$, $r_1 + \cdots + r_p = r$, and the $\tau_{j,v}$ are in the Langlands order. Moreover, recall that $\pi_v$ is the unique irreducible unramified subrepresentation of the representation of $G_n(k_v)$ induced from the character $\chi$ as in (60) after an appropriate reordering, if necessary.

Now, by the definition of $L$-functions (see [39, Section 7]) and their multiplicative property (see [38, Theorem 5.2]), we have

$$L(s, \pi_v \times \tau_v) = \prod_{j=1}^{p} L(s + b_j, \pi_v \times \tau_{j,v})$$

$$= \prod_{j=1}^{p} \prod_{i=1}^{n} L(s + b_j, \tau_{j,v} \otimes \chi_i)L(s + b_j, \tau_{j,v} \otimes \chi_0 \chi_i^{-1})$$

$$= \prod_{i=1}^{n} L(s, \tau_v \otimes \chi_i)L(s, \tau_v \otimes \chi_0 \chi_i^{-1}), \quad (71)$$

and likewise,

$$L(s, \tilde{\pi}_v \times \tilde{\tau}_v) = \prod_{i=1}^{n} L(s, \tilde{\tau}_v \otimes \chi_i^{-1})L(s, \tilde{\tau}_v \otimes \chi_0^{-1} \chi_i). \quad (72)$$

Note that [38, Conjecture 5.1], which is a hypothesis of [38, Theorem 5.2], is known in our cases by [3, Theorem 5.7].

Equations (69), (71), and (72) in turn imply

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{n} \epsilon(s, \tau_v \otimes \chi_i, \psi_v)\epsilon(s, \tau_v \otimes \chi_0 \chi_i^{-1}, \psi_v). \quad (73)$$

Note that the product $L$-functions for $\text{GL}_a \times \text{GL}_b$ of the Langlands-Shahidi method and the $L$-functions of [17] are known to be equal (see [35]). Hence, to see (65) and (66), all we need is to compare the right-hand sides of (67) and (68) with those of (71), (72), and (73).

(ii): $v | \infty$. By the local Langlands correspondence (see [30]), the representations $\pi_v$ and $\tau_v$ are given by admissible homomorphisms

$$\phi : W_v \longrightarrow \begin{cases} 
\text{GSp}_{2n}(\mathbb{C}) & \text{if } G_n = \text{GSpin}_{2n+1}, \\
\text{GSO}_{2n}(\mathbb{C}) & \text{if } G_n = \text{GSpin}_{2n}, 
\end{cases}$$

and

$$\phi' : W_v \longrightarrow \text{GL}_r(\mathbb{C}),$$
respectively, and the tensor product

$$(\iota \circ \phi) \otimes \phi' : W_v \longrightarrow \text{GL}_{2n_r}(\mathbb{C})$$

is again admissible. Now,

$$L(s, \Pi_v \times \tau_v) = L(s, (\iota \circ \phi) \otimes \phi') = L(s, \pi_v \times \tau_v),$$

and

$$\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \epsilon(s, (\iota \circ \phi) \otimes \phi', \psi_v) = \epsilon(s, \pi_v \times \tau_v, \psi_v),$$

where the middle factors are the local Artin-Weil factors (see [47]) and equalities hold by [36] (see also [5]).

(iii): $v < \infty$ and $\pi_v$ ramified. This is where we need the stability of $\gamma$-factors. Since $v \in S$, the representation $\tau_v$ can be written as

$$\tau_v \cong \text{Ind}(\nu_v^{b_1} \otimes \cdots \otimes \nu_v^{b_r}) \otimes \eta_v \cong \text{Ind}(\eta_v^{b_1} \nu_v^{b_1} \otimes \cdots \otimes \eta_v^{b_r} \nu_v^{b_r}),$$

(74)

where $v(x) = |x|_v$. Then

$$L(s, \pi_v \times \tau_v) = \prod_{i=1}^{r} L(s + b_i, \pi_v \times \eta_v),$$

(75)

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{r} \epsilon(s + b_i, \pi_v \times \eta_v, \psi_v).$$

(76)

However, since $\eta_v$ is sufficiently ramified (depending on $\pi_v$), Corollary 4.19 implies that

$$L(s, \pi_v \times \eta_v) \equiv 1,$$

(77)

$$\epsilon(s, \pi_v \times \eta_v) = \prod_{i=1}^{n} \epsilon(s, \eta_v \chi_i, \psi_v) \epsilon(s, \eta_v \chi_0 \mu_i^{-1}, \psi_v),$$

(78)

for $n$ arbitrary characters $\chi_1, \chi_2, \ldots, \chi_n$, and $\chi_0 = \omega_{\pi_v}$. We choose them to be as in (60).

On the other hand, by either [17] or [39], we have

$$L(s, \Pi_v \times \tau_v) = \prod_{i=1}^{r} L(s + b_i, \Pi_v \otimes \eta_v),$$

(79)

$$\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{r} \epsilon(s + b_i, \Pi_v \otimes \eta_v, \psi_v).$$

(80)
Again, since $\eta_v$ is highly ramified (depending on $\Pi v$) and $\omega_{\Pi v} = \omega_{\pi_v}^n = \chi_0^n$ is equal to the product of the $2n$ characters $\chi_1, \ldots, \chi_n, \chi_0\chi_n^{-1}, \ldots, \chi_0\chi_1^{-1},$

[19, Proposition 2.2] implies that

$$L(s, \Pi_v \otimes \eta_v) \equiv 1,$$

(81)

$$\epsilon(s, \Pi_v \otimes \eta_v) = \prod_{i=1}^n \epsilon(s, \eta_v \chi_i, \psi_v)\epsilon(s, \eta_v \chi_0\chi_i^{-1}, \psi_v).$$

(82)

Comparing equations (75)–(82) now proves (65) and (66) for a non-Archimedean place $v$ at which $\pi_v$ is ramified.

Now that we have (65) and (66) for all places $v$ of $k$, we conclude globally that

$$L(s, \Pi \times \tau) = L(s, \pi \times \tau), \quad L(s, \tilde{\Pi} \times \tilde{\tau}) = L(s, \tilde{\pi} \times \tilde{\tau}),$$

(83)

$$\epsilon(s, \Pi \times \tau) = \epsilon(s, \pi \times \tau), \quad \epsilon(s, \tilde{\Pi} \times \tilde{\tau}) = \epsilon(s, \tilde{\pi} \times \tilde{\tau}),$$

(84)

for all $\tau \in \mathcal{T}(S; \eta)$. All that remains is to verify the three conditions of Theorem 6.1, which we can now check for the factors coming from the Langlands-Shahidi method thanks to (83) and (84). Conditions (1) – (3) of Theorem 6.1 are Propositions 5.1, 5.3, and 5.4, respectively.

Therefore, there exists an automorphic representation $\Pi'$ of $GL_{2n}(A)$ such that for all $v \not\in S$, we have $\Pi_v \simeq \Pi'_v$. In particular, for all $v \not\in S$, the local representation $\Pi'_v$ is related to $\pi_v$, as prescribed in Theorem 1.1. Moreover, note that for all $v \not\in S$, we have $\omega_{\Pi'_v} = \omega_{\Pi_v} = \omega_{\pi_v}^n$. Since $\omega_{\Pi'}$ is a Hecke character that agrees with the Hecke character $\omega_{\pi}^n$ at all but possibly finitely many places, we conclude that $\omega_{\Pi'} = \omega_{\pi}^n$.

On the other hand, if $v$ is an Archimedean place or a non-Archimedean place with $v \not\in S$, then we proved earlier that

$$\Pi'_v \simeq \Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v} \simeq \tilde{\Pi}'_v \otimes \omega_{\pi_v},$$

which means, in particular, that $\Pi'$ is nearly equivalent to $\tilde{\Pi}' \otimes \omega_{\pi'}$.  

\[ \square \]

7. Complements

7.1. Local consequences

Our first local result is to show that the local transfers at the unramified places remain generic. Let us first recall a general result of Jian-Shu Li which we use. The following is a special case of [32, Theorem 2.2].
**Proposition 7.1** (J.-S. Li; see [32])

Let $G$ be a split connected reductive group over a non-Archimedean local field $F$, and let $B = TU$ be a fixed Borel subgroup, where $T$ is a maximal torus and $U$ is the unipotent radical of $B$. Let $\chi$ be an unramified character of $T(F)$, and let $\pi(\chi)$ be the unique irreducible unramified subquotient of the corresponding principal series representation. Then $\pi(\chi)$ is generic if and only if for all roots $\alpha$ of $(G, T)$, we have $\chi(\alpha^{\vee}(\varpi)) \neq |\varpi|_F$. Here $\alpha^{\vee}$ denotes the coroot associated to $\alpha$, and $\varpi$ is a uniformizer of $F$.

**Proposition 7.2**

Let $\pi = \bigotimes_v \pi_v$ be an irreducible globally generic cuspidal automorphic representation of $\text{GSpin}_m(A)$, $m = 2n + 1$ or $2n$, and let $\Pi = \bigotimes_v \pi_v$ be a transfer of $\pi$ to $GL_{2n}(A)$ (see Theorem 1.1). If $v < \infty$ is a place of $k$ with $\pi_v$ unramified, then the local representation $\Pi_v$ is irreducible and unramified, and we have $\Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v}$. Moreover, if $m = 2n + 1$ (see Remark 7.3), then $\Pi_v$ is generic (and, hence, a full-induced principal series representation).

**Proof**

The representation $\Pi_v$ is irreducible and unramified by construction (see (i) in the proof of Theorem 1.1). We also proved that $\Pi_v$ satisfies $\Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v}$ in the course of the proof of Theorem 1.1 in Section 6.

Assume that $m = 2n + 1$. We now show that $\Pi_v$ is generic. Our tool is Proposition 7.1. Let $\chi$ and $\chi_0, \ldots, \chi_n$ be as in (60). Since $\pi_v$ is generic, by Proposition 7.1 we have that $\chi(\alpha^{\vee}(\varpi)) \neq |\varpi|_F$ for all roots $\alpha$. Using the notation of Section 2, the roots in the odd case $m = 2n + 1$ are $\alpha = \pm(e_i - e_j)$, $\pm(e_i + e_j)$ with $1 \leq i < j \leq n$, and $\pm(e_i)$ with $1 \leq i \leq n$. The corresponding coroots are $\alpha^{\vee} = \pm(e_i^* - e_j^*)$, $\pm(e_i^* + e_j^* - e_0^*)$ with $1 \leq i < j \leq n$, and $\pm(2e_i^* - e_0^*)$ with $1 \leq i \leq n$, respectively. This implies that $\chi_i \chi_j^{-1} \neq |\pm1|$ for $i \neq j$ and $\chi_i \chi_j \chi_0^{-1} \neq |\pm1|$ for all $i, j$.

The representation $\Pi_v$ was chosen to be the unique irreducible unramified subquotient of the representation on $GL_{2n}(F)$ induced from the $2n$ unramified characters $\chi_1, \ldots, \chi_n, \chi_0 \chi_n^{-1}, \ldots, \chi_0 \chi_1^{-1}$ as in (64). Therefore, these relations imply that $\Pi_v$ is generic and full-induced.

**Remark 7.3**

The above argument does not quite work in the even case, and one can easily construct local examples, where the transferred local representation is the (unique) unramified subquotient of an induced representation on $GL_{2n}$ far from the generic constituent.

For example, consider $\text{GSpin}_6$ with $\chi_0 = \mu^2$, $\chi_1 = \mu(5/2)$, $\chi_2 = \mu(1/2)$, and $\chi_3 = \mu(-3/2)$, where $\mu$ is a unitary character of $F^\times$ and $\mu(r)$ means $\mu|r|$. Now $\Pi_v$ is the unique unramified constituent of the representation on $GL_6(F)$ induced from
\[ \mu(5/2), \mu(3/2), \mu(1/2), \mu(-1/2), \mu(-3/2), \text{ and } \mu(-5/2) \text{ and, in fact, is far from being generic. In this case, there is another constituent that is square-integrable and hence tempered and generic.} \]

Of course, we do expect \( \Pi_v \) in the case of \( m = 2n \) to be generic as well. However, this phenomenon is not a purely local one in the case of \( m = 2n \). In fact, it is automatic that the local transfers at the unramified places are generic once we prove that the automorphic representation \( \Pi \) is induced from unitary cuspidal representations (see Remark 7.5). As we discuss in Remark 7.5 this will follow from our future work.

### 7.2. Global consequences

In this section we make some comments about the automorphic representation \( \Pi \) which are almost immediate consequences of our main result, and we leave more detailed information about \( \Pi \) for a future article.

**Proposition 7.4**

Let \( \pi \) be a globally generic cuspidal automorphic representation of \( \text{GSpin}_m(\mathbb{A}) \), \( m = 2n + 1 \) or \( 2n \), and let \( \omega = \omega_\pi \). Then there exist a partition \( (n_1, n_2, \ldots, n_t) \) of \( 2n \) and (not necessarily unitary) cuspidal automorphic representations \( \sigma_1, \ldots, \sigma_t \) of \( \text{GL}_{n_i}(\mathbb{A}) \), \( i = 1, \ldots, t \), and a permutation \( p \) of \( \{1, \ldots, t\} \) with \( n_i = n_{p(i)} \) and \( \sigma_i \simeq \tilde{\sigma}_{p(i)} \otimes \omega \) such that any transfer \( \Pi \) of \( \pi \) as in Theorem 1.1 is a constituent of \( \Sigma_1 = \text{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_t) \), where the induction is, as usual, from the standard parabolic subgroup of \( \text{GL}_{2n} \) having Levi subgroup \( \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_t} \).

**Proof**

Let \( \Pi \) be any transfer of the globally generic cuspidal representation \( \pi \) as in Theorem 1.1. By [29, Proposition 2] there exist a partition \( p \) and \( \sigma_i \)'s, such that \( \Pi \) is a constituent of \( \Sigma \). Furthermore, for finite places \( v \), where \( \pi_v \) is unramified, we have that \( \Pi_v \) is the unique unramified constituent of \( \Sigma_v = \text{Ind}(\sigma_{1,v} \otimes \cdots \otimes \sigma_{t,v}) \). As part of Theorem 1.1, we showed that \( \Pi \) and \( \tilde{\Pi} \otimes \omega \) are nearly equivalent (see the definition prior to Theorem 1.1). Now, \( \tilde{\Pi} \otimes \omega \) is a constituent of \( \tilde{\Sigma} \otimes \omega = \text{Ind}(\tilde{\sigma}_1 \otimes \omega) \otimes \cdots \otimes (\tilde{\sigma}_t \otimes \omega) \), and by the classification theorem of Jacquet and Shalika (see [18, Theorem 4.4]), we have that there is a permutation \( p \) of \( \{1, \ldots, t\} \) such that \( n_i = n_{p(i)} \) and \( \sigma_i \simeq \tilde{\sigma}_{p(i)} \otimes \omega \).

Now let \( \Pi' \) be another transfer of \( \pi \) as in Theorem 1.1. Then \( \Pi' \) is again a constituent of some \( \Sigma' = \text{Ind}(\sigma'_1 \otimes \cdots \otimes \sigma'_t) \), where each \( \sigma'_i \) is a cuspidal automorphic representation of \( \text{GL}_{n'_i}(\mathbb{A}) \) and \( (n'_1, \ldots, n'_t) \) is a partition of \( 2n \). Moreover, for almost all finite places \( v \), we have that \( \Pi'_v \) is the unique unramified constituent of \( \Sigma'_v \). On the other hand, by construction, \( \Pi_v \simeq \Pi'_v \) for almost all \( v \), and therefore, the classification theorem of Jacquet and Shalika again implies that \( t = t' \) and, up to a permutation, \( n_i = n'_i \) and \( \sigma_i \simeq \sigma'_i \) for \( i = 1, \ldots, t \). Therefore, \( \Pi' \) is also a constituent of \( \Sigma \). \( \square \)
Remark 7.5
If we write \( \sigma_i = \tau_i \otimes |\det( )|^{r_i} \) for \( i = 1, 2, \ldots, t \), with \( \tau_i \) unitary cuspidal and \( r_i \in \mathbb{R} \), then we expect that all \( r_i = 0 \); that is, \( \Pi \) is an isobaric sum of unitary cuspidal representations. We will take up this issue, which will have important consequences, in our future work.

7.3. Exterior square transfer
In this section we show that exterior square transfer from \( \text{GL}_4 \) to \( \text{GL}_6 \) due to H. H. Kim [22] can be deduced as a special case of our main result. However, note that in this article we are proving only the weak transfer. Once we prove the strong version of the transfer from \( \text{GSpin}_{2n} \) to \( \text{GL}_{2n} \), again it will have the full content of the results of [22]. A similar remark also applies to Section 7.4.

PROPOSITION 7.6
Let \( \phi : \text{GSpin}_6 \rightarrow \text{GL}_4 \) be the (double) covering map (see Proposition 2.2), and denote by \( \hat{\phi} \) the map induced on the connected components of the \( L \)-groups:

\[
\begin{array}{c}
\text{GSO}_6(\mathbb{C}) \\
\downarrow \hat{\phi} \\
\text{GL}_{44}(\mathbb{C})
\end{array}
\]

Then \( \iota \circ \hat{\phi} = \bigwedge^2 \).

Proof
The group GSO\(_6\) is of type \( D_3 \), and we denote its simple roots by \( \alpha_1, \alpha_2, \alpha_3 \) as in Section 2. Also, GL\(_4\) is of type \( A_3 \) (or \( D_3 \)), and we denote its corresponding simple roots by \( \bar{\alpha}_2, \bar{\alpha}_1, \bar{\alpha}_3 \), respectively, and similarly for other root data (see Section 2). Let \( A = \text{diag}(a_1, a_2, a_3, a_4) \in \text{GL}_4(\mathbb{C}) \). For a fixed appropriate choice of fourth root of unity and \( \delta = (a_1a_2a_3a_4)^{1/4} \), we have

\[
\iota \circ \hat{\phi}(A) = \iota \circ \hat{\phi} \left( \delta \bar{\alpha}_2 \vee \left( \frac{a_1}{\delta} \right) \bar{\alpha}_1 \vee \left( \frac{a_1a_2}{\delta^2} \right) \bar{\alpha}_3 \vee \left( \frac{a_1a_2a_3}{\delta^3} \right) \right)
\]

\[
= \iota \left( e_0^*(\delta^4)e_1^*(\delta^2)e_2^*(\delta^2)e_3^*(\delta^2)\alpha_2 \vee \left( \frac{a_1}{\delta} \right) \alpha_1 \vee \left( \frac{a_1a_2}{\delta^2} \right) \alpha_3 \vee \left( \frac{a_1a_2a_3}{\delta^3} \right) \right)
\]

\[
= \iota \left( e_0^*(\delta^4)e_1^*(a_1a_2)e_2^*(a_1a_3)e_3^*(a_2a_3) \right)
\]

\[
= \text{diag}(a_1a_2, a_1a_3, a_2a_3, a_2a_4, a_1a_4, a_3a_4) = \bigwedge^2 A.
\]

Here the third equality follows from Proposition 2.10. \( \square \)
As a corollary, we see that our Theorem 1.1 in the special case of \( m = 2n \) with \( n = 3 \) gives Kim’s exterior square transfer.

**Proposition 7.7**

If \( \pi \) is an irreducible cuspidal automorphic representation of \( \text{GL}_4(\mathbb{A}) \) considered as a representation of \( \text{GSpin}_6(\mathbb{A}) \) via the covering map \( \phi \), then the automorphic representation \( \Pi \) of Theorem 1.1 is such that \( \Pi_v = \bigwedge^2 \pi_v \) for almost all \( v \).

7.4. Transfer from \( \text{GSp}_4 \) to \( \text{GL}_4 \)

The special case of \( m = 2n + 1 \) with \( n = 2 \) of our Theorem 1.1 gives the following.

**Proposition 7.8**

Let \( \pi \) be an irreducible globally generic cuspidal automorphic representation of \( \text{GSp}_4(\mathbb{A}) \). Then \( \pi \) can be transferred to an automorphic representation \( \Pi \) of \( \text{GL}_4(\mathbb{A}) \) associated to the embedding \( \text{GSp}_4(\mathbb{C}) \hookrightarrow \text{GL}_4(\mathbb{C}) \).

**Proof**

Notice that \( \text{GSpin}_5 \) is isomorphic, as an algebraic group, to the group \( \text{GSp}_4 \). Now the corollary is a special case of Theorem 1.1, as mentioned previously.

In fact, we can prove more in this special case. We refer to our separate work [4] for more details about this as well as its applications to the generalized Ramanujan conjecture for \( \text{GSp}_4 \).

**Remark 7.9**

Proposition 7.8, in particular, proves that the spinor \( L \)-function of \( \pi \) is entire. R. Takloo-Bighash [46] also has a proof of some cases of this result using an integral representation. His proof differs from our method in that we use the integral representations only through the Converse Theorems.

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