Sculptures in S^3

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The 3-dimensional sphere

The 2-dimensional sphere, S^2, is the usual sphere in $\mathbb{R}^3$:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

By analogy, the 1-dimensional sphere is the circle:

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

and this is the 3-dimensional sphere:

$$S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$$

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Stereographic projection

S^3 is hard to visualise, because it lives in $\mathbb{R}^4$. However, we can use stereographic projection, which maps from S^n to $\mathbb{R}^n$, to reduce the dimension by one.

For $n = 1$, we define $\rho : S^1 \to \mathbb{R}^1$ by $\rho(x, y) = \frac{x}{1-y}$.

This is a cross-section of stereographic projection for $n > 1$.

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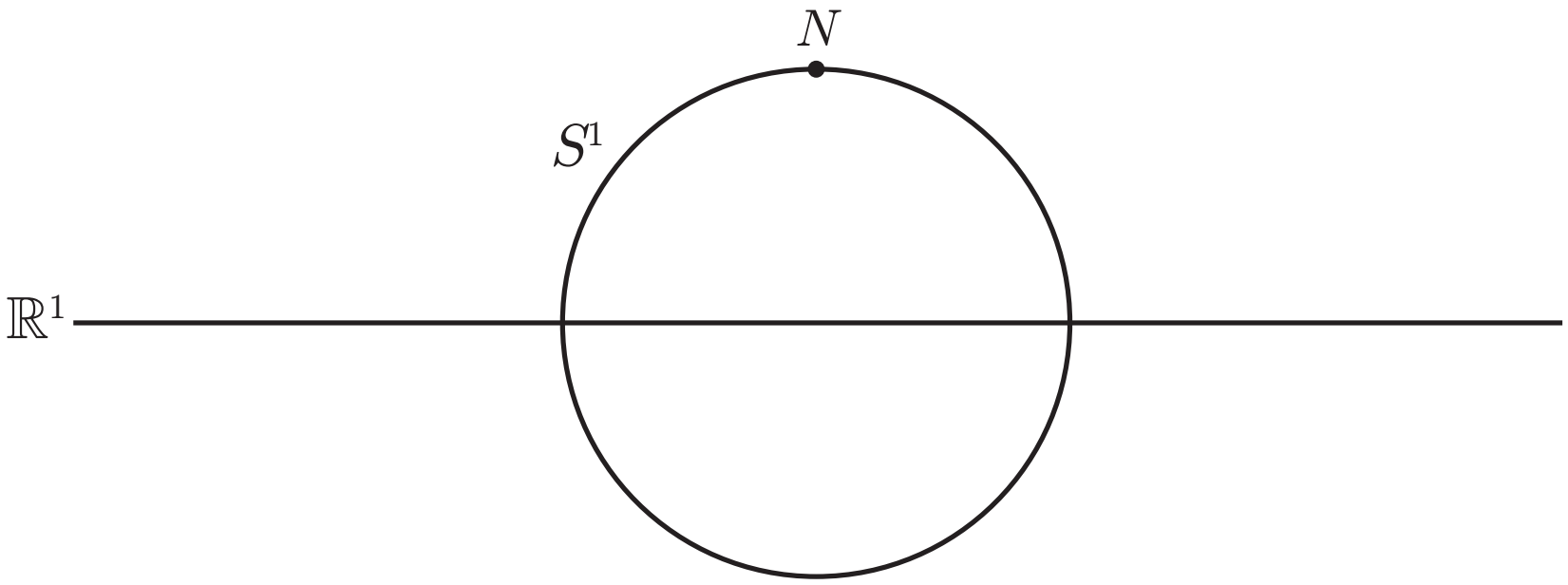

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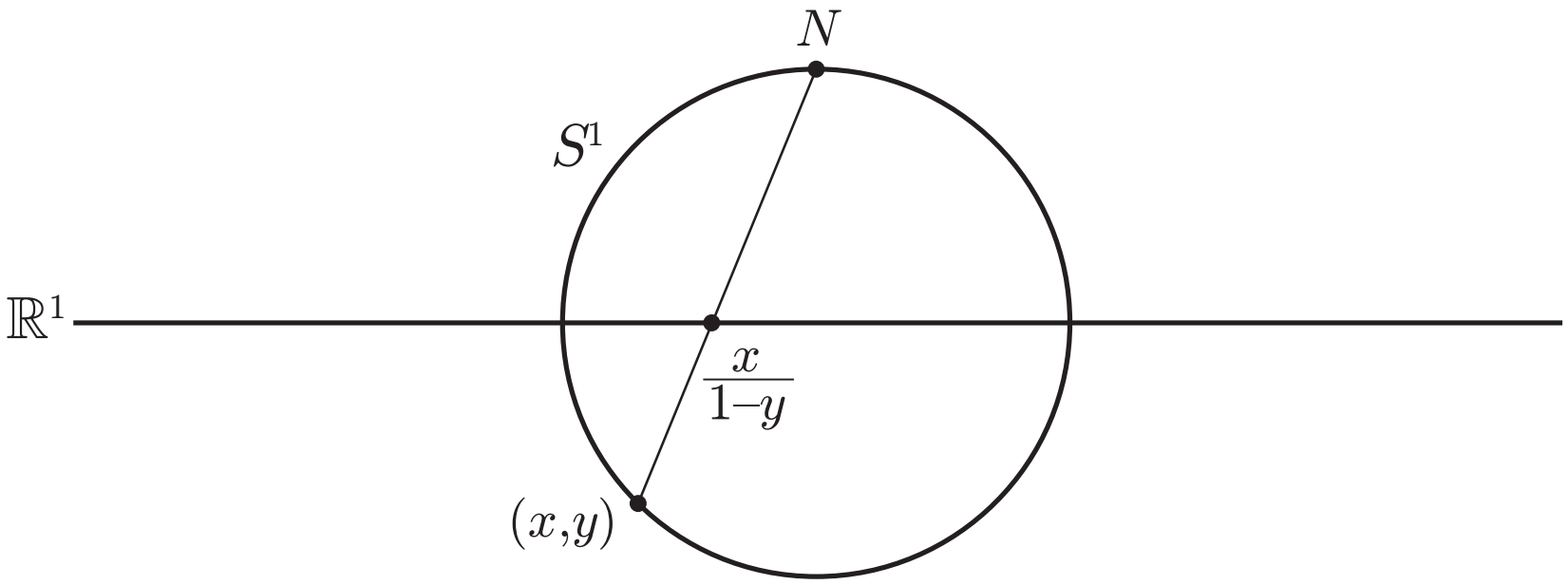

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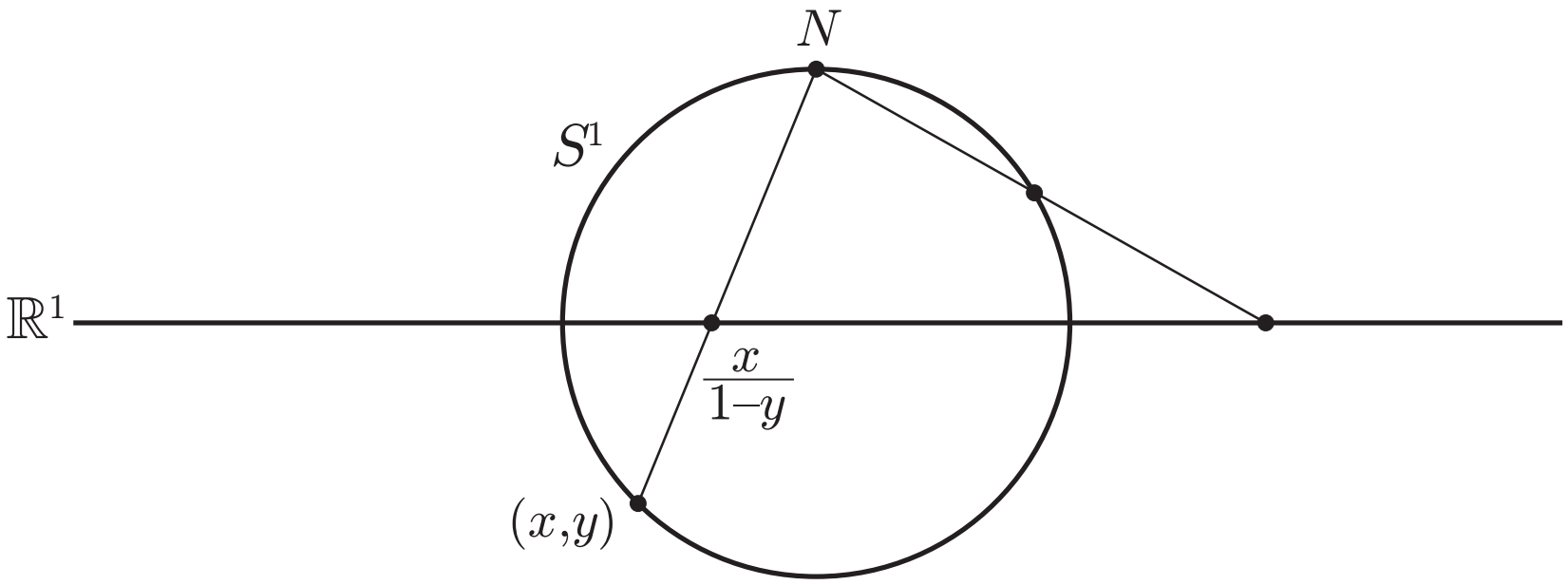

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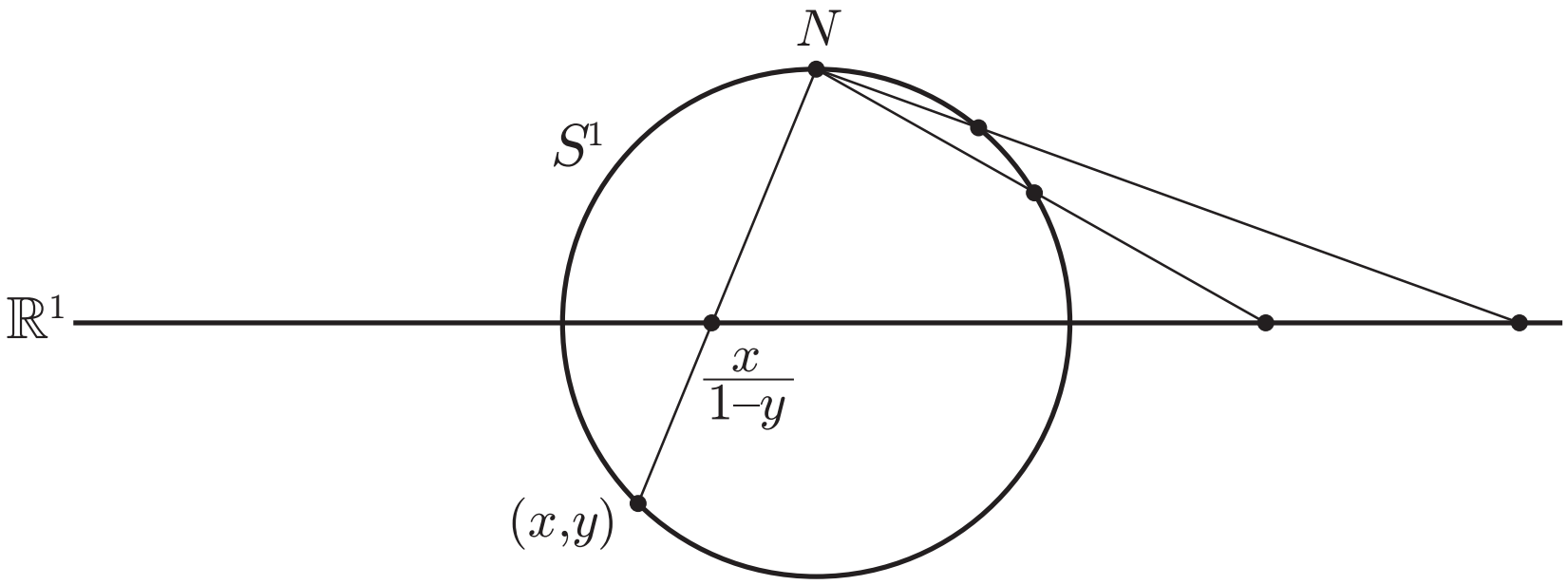

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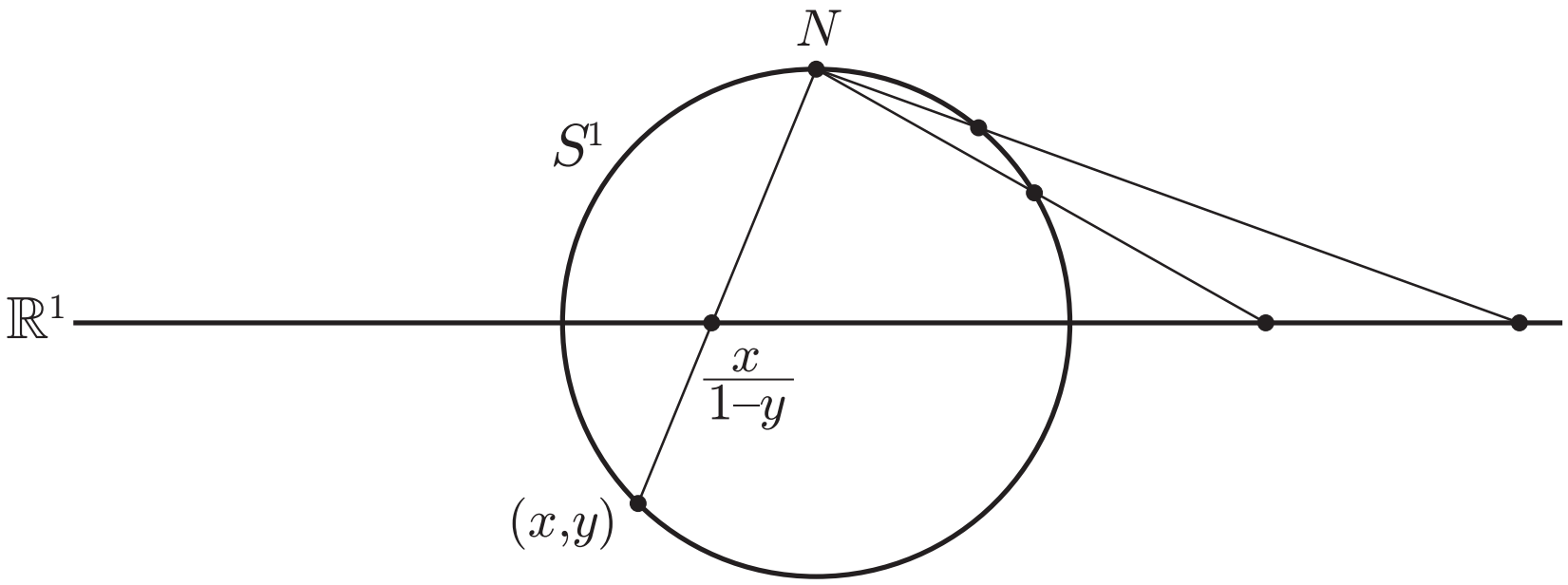

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2-dimensional stereographic projection

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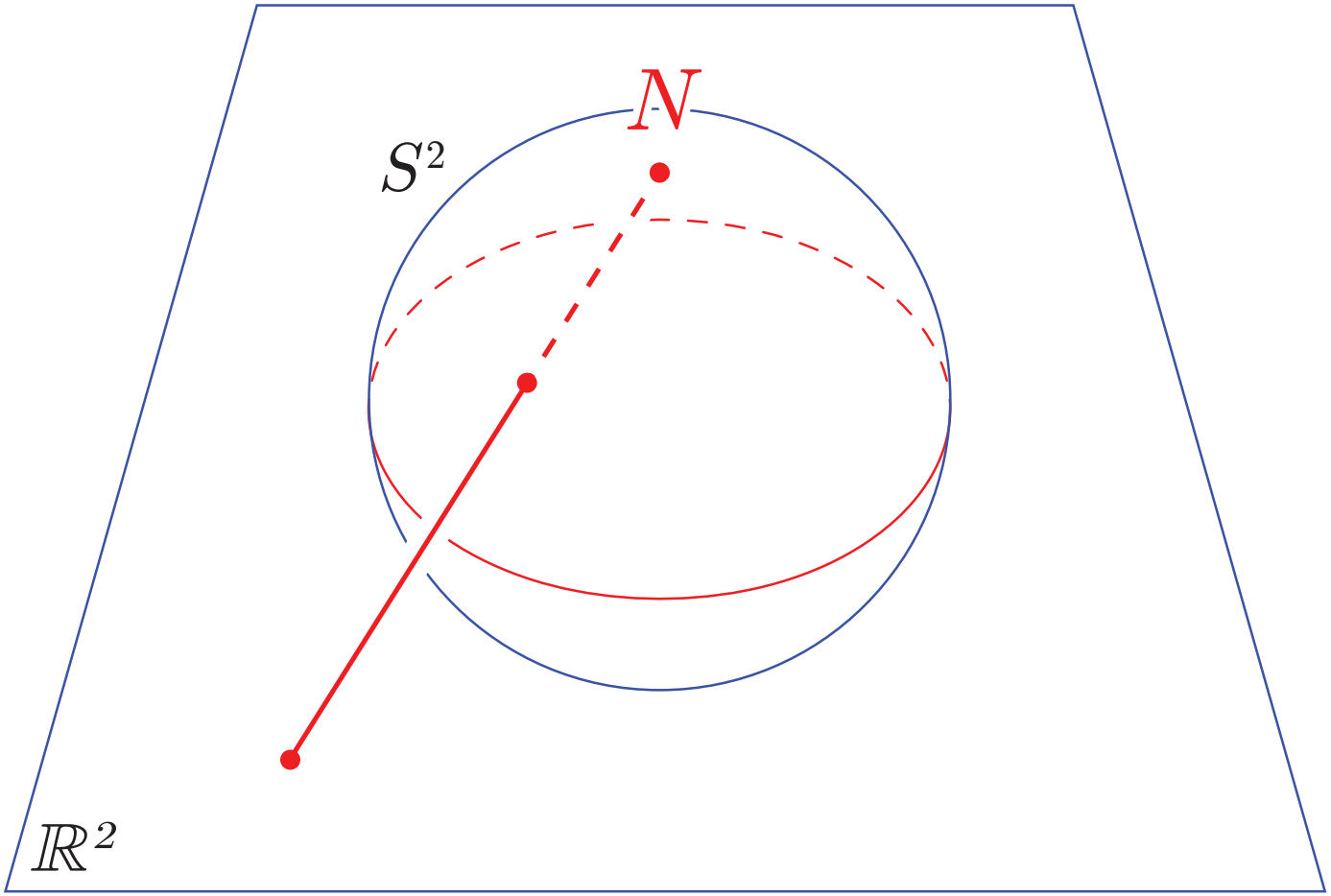

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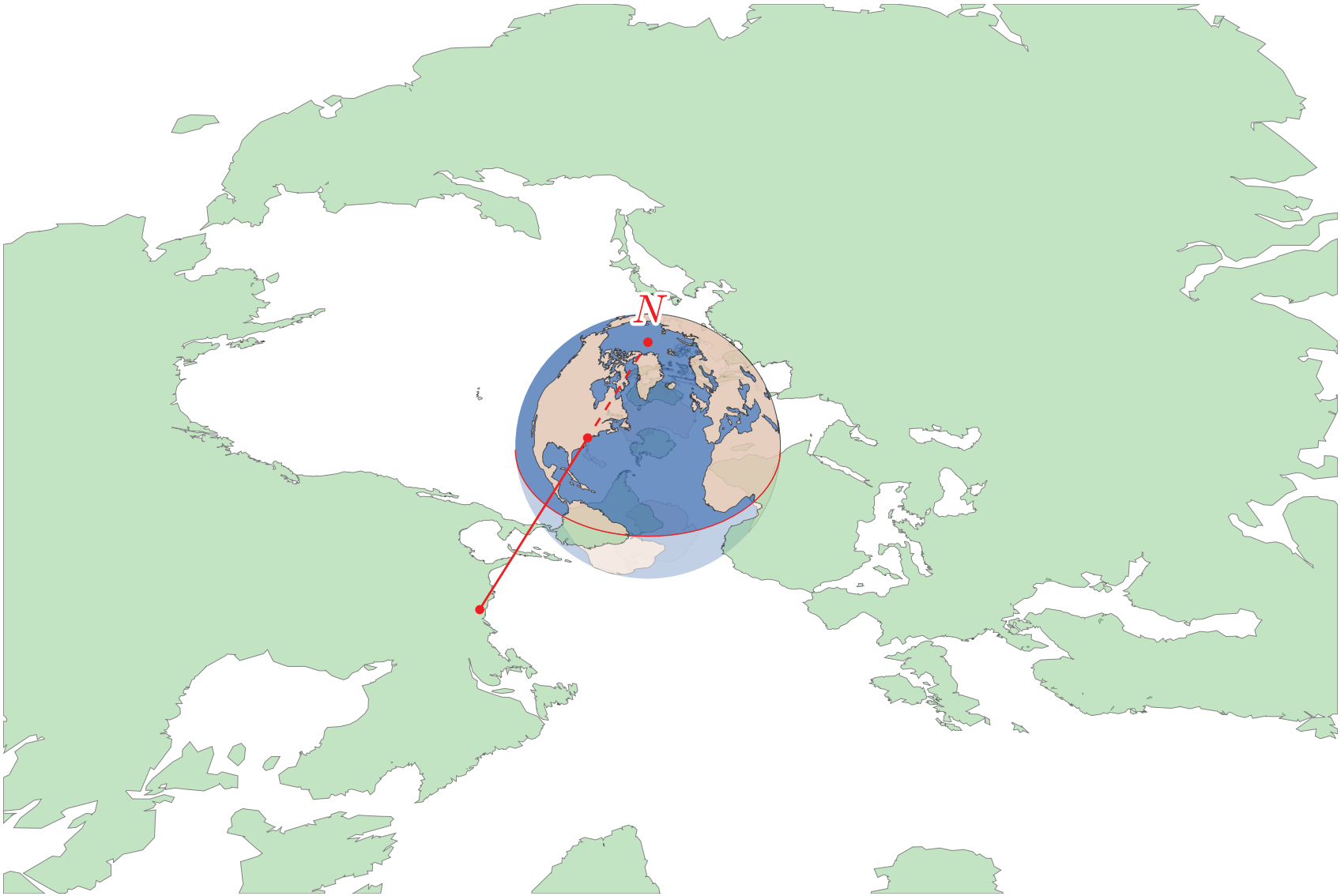
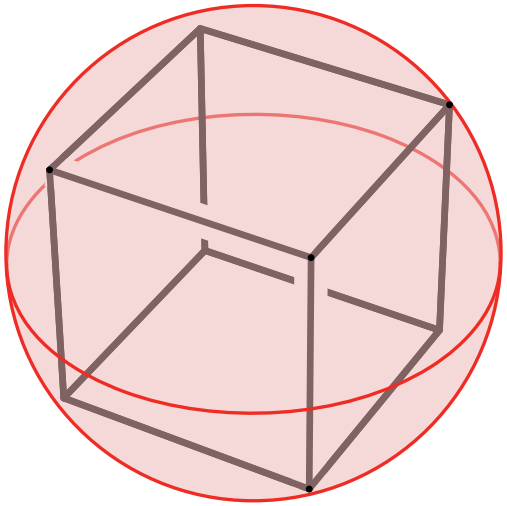


Example: cube



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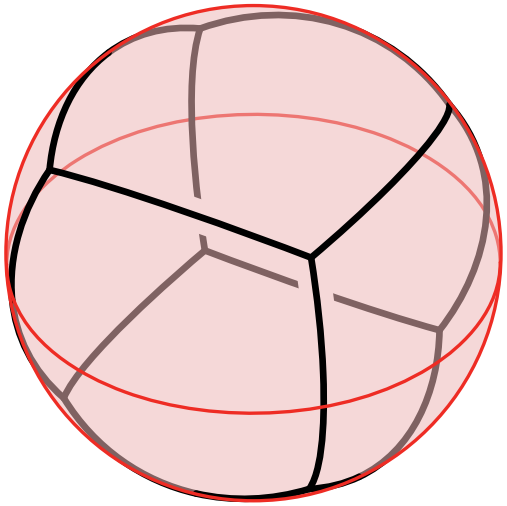

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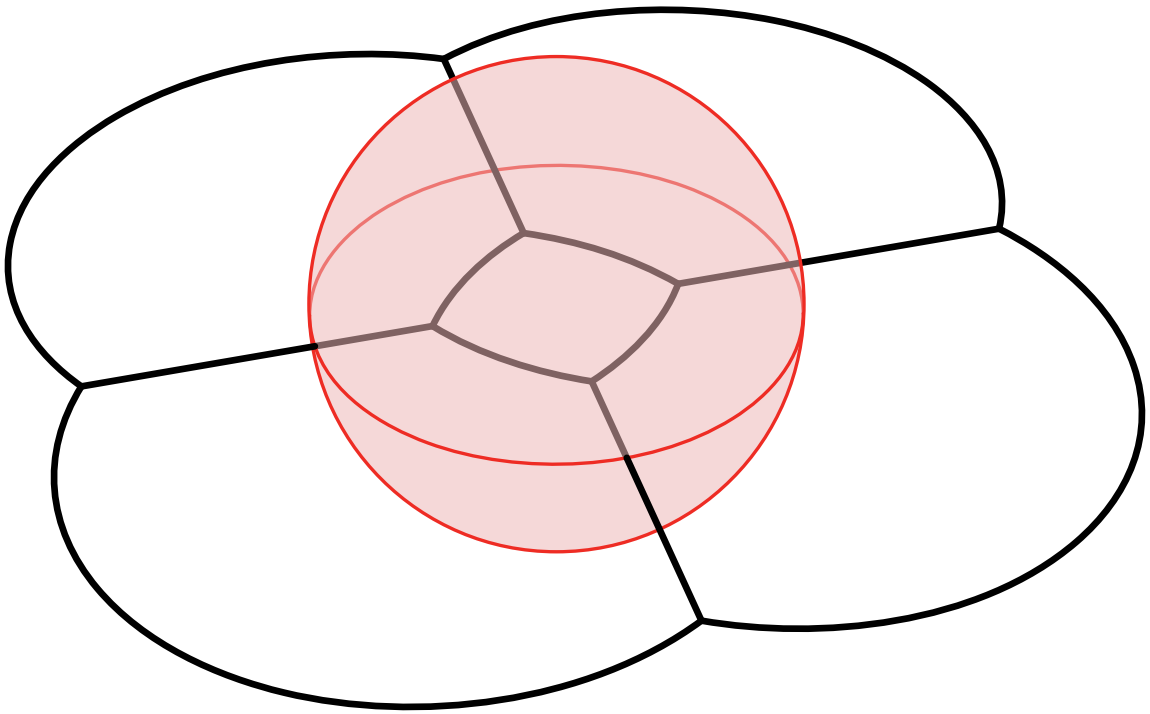
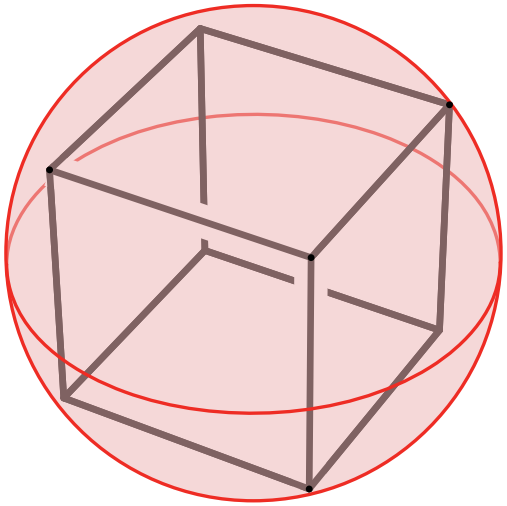


Example: cube, thickened



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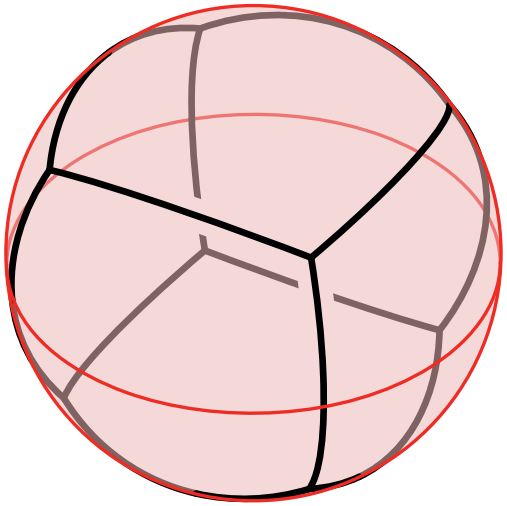

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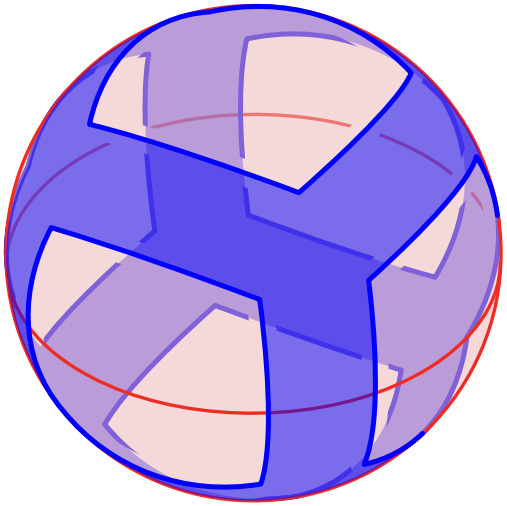

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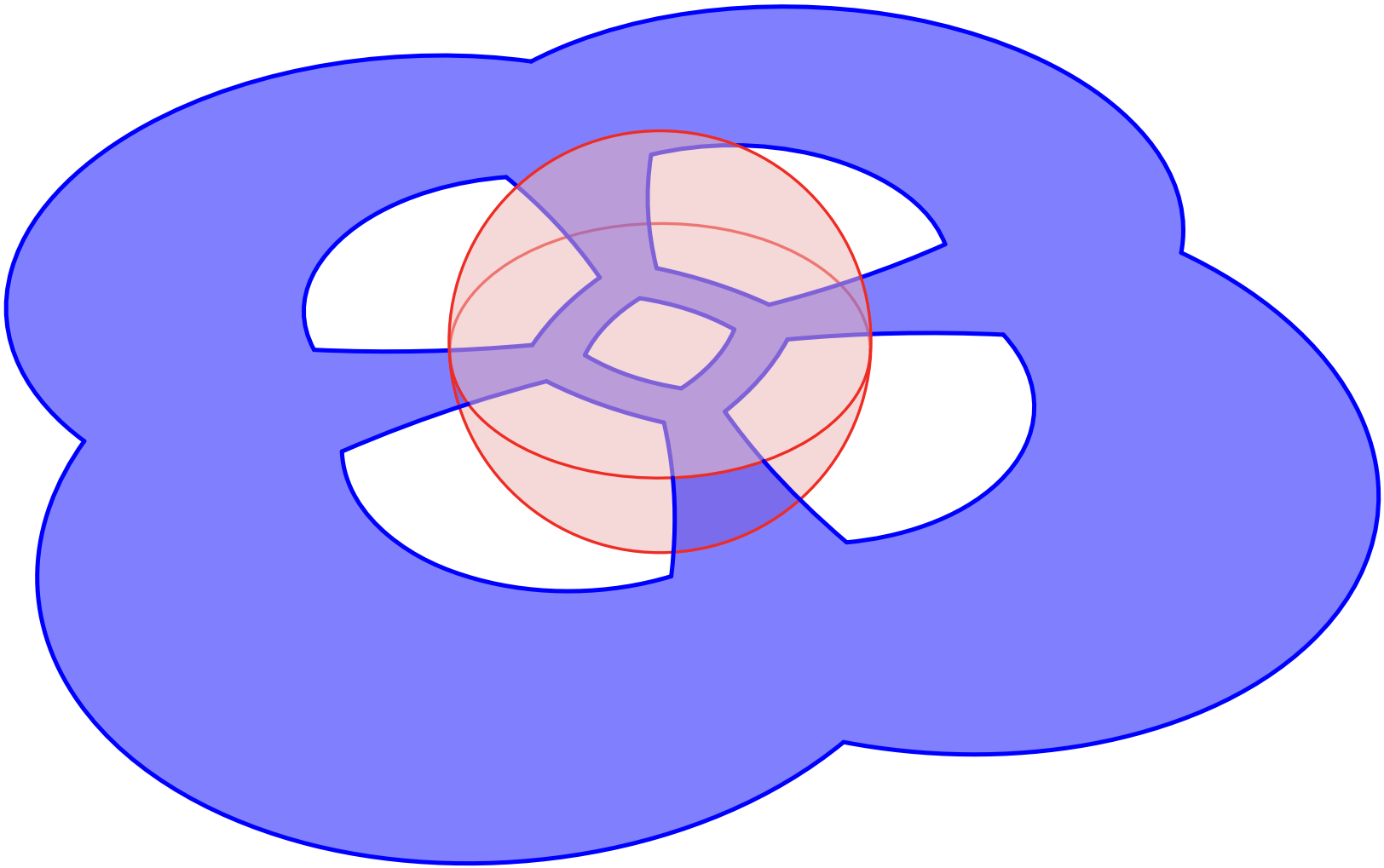
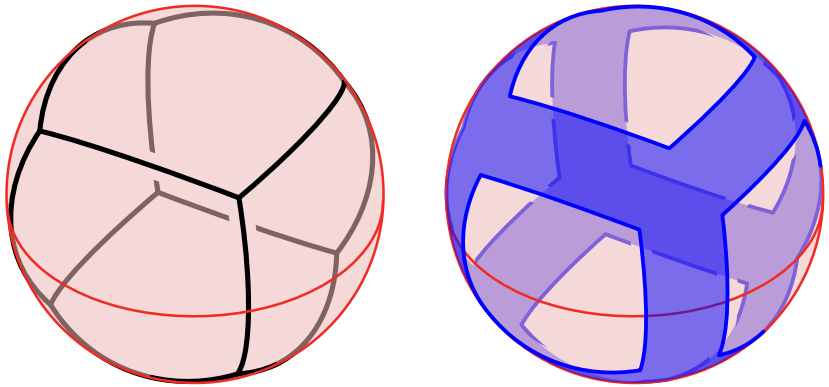


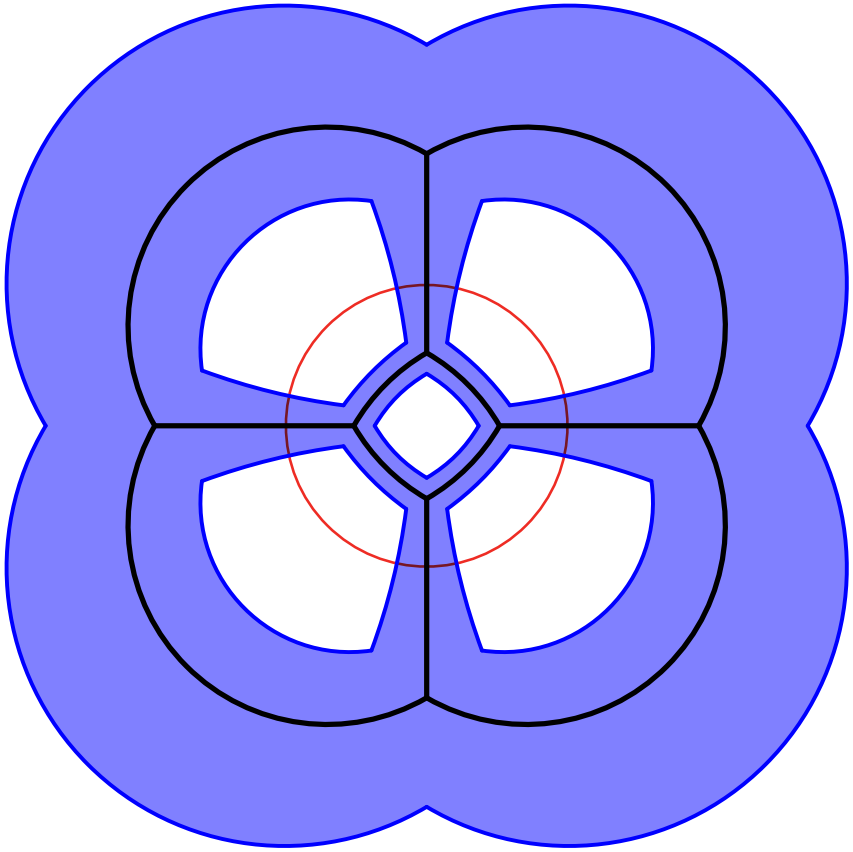
(Good) properties of stereographic projection:

- Circles in S^n map to circles or lines in $\mathbb{R}^n$.
- Stereographic projection is conformal, meaning that it preserves angles.

(Bad) properties of stereographic projection:

- Objects near the projection point become very big in the image.





3-dimensional stereographic projection

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$$\rho(x, y, z, w) = \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w} \right).$$

So this gives us a way to draw objects native to S^3.

We explored sculptures of two kinds of object:

- 4-dimensional polytopes
- Surfaces

In both cases, we need to thicken 1 or 2-dimensional objects to produce 3-dimensional objects that can be printed.

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- 4-dimensional polytopes
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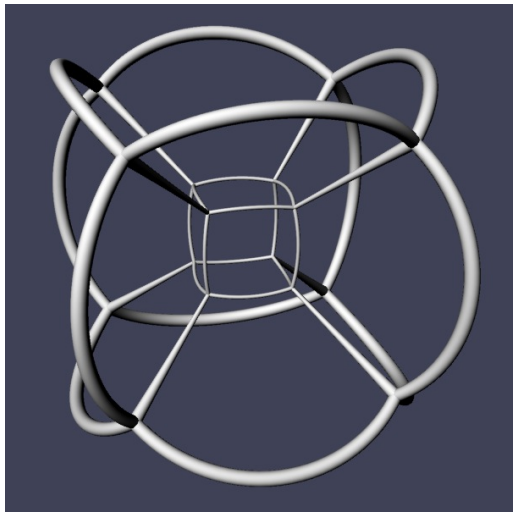
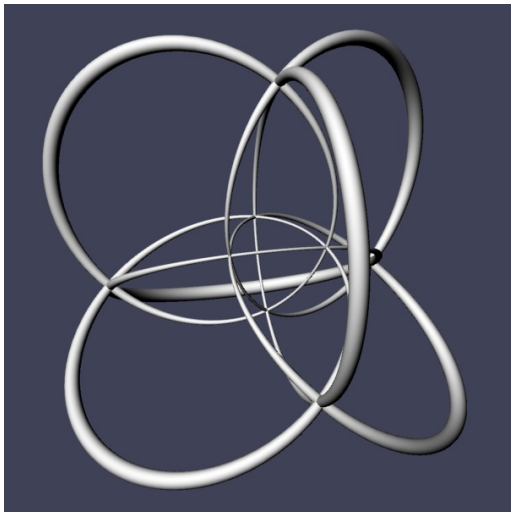
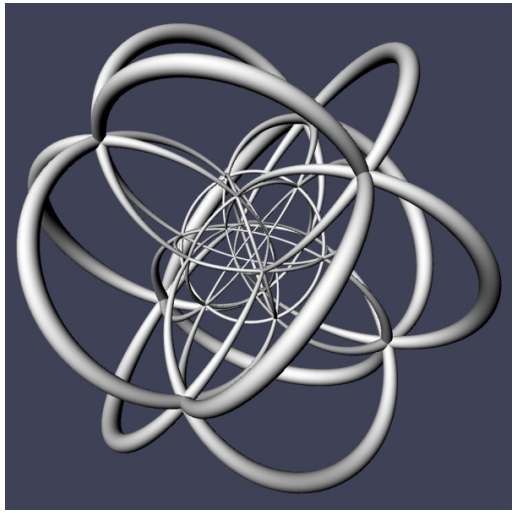
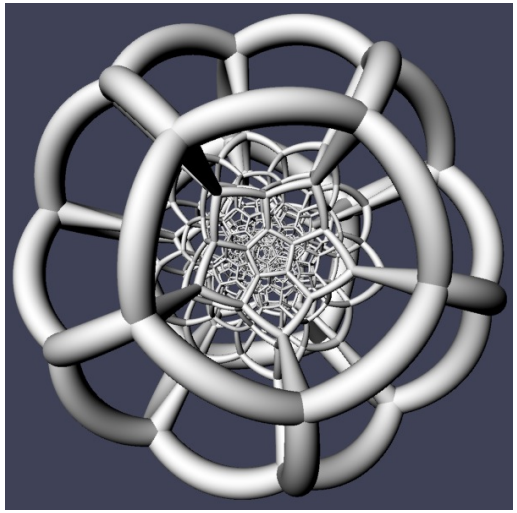
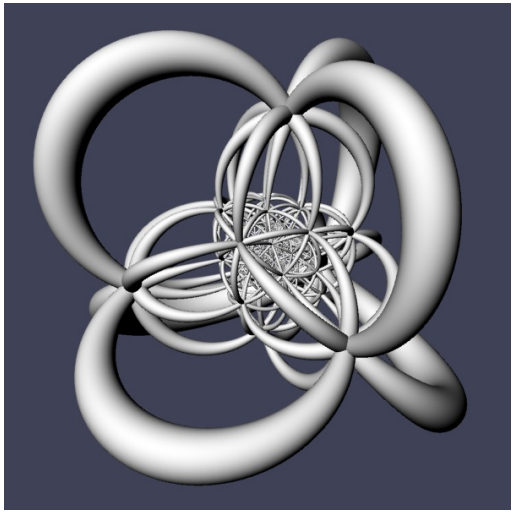
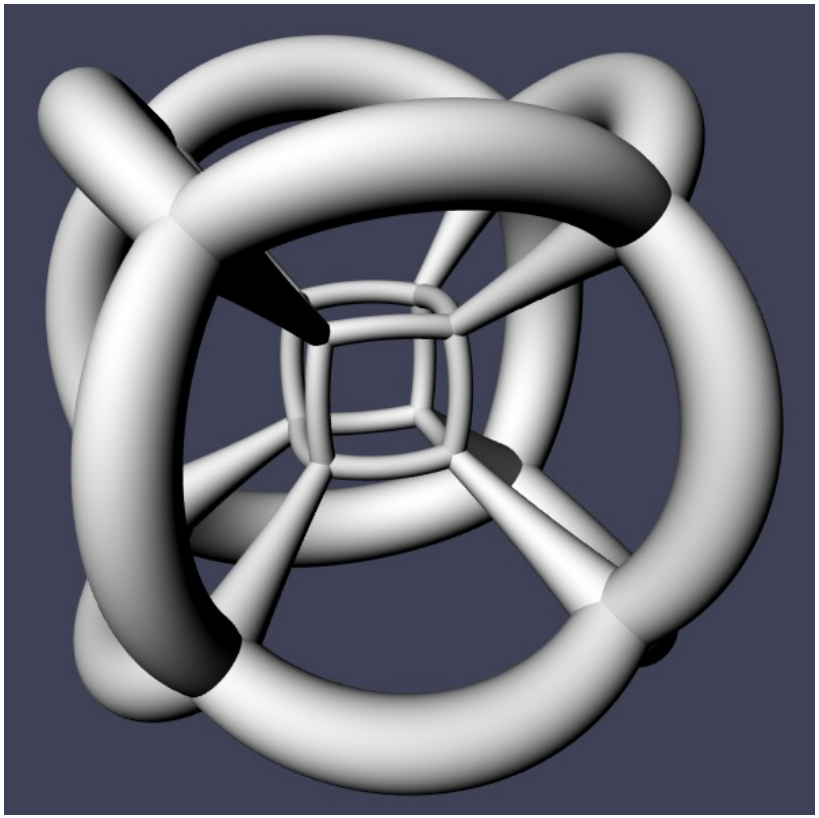
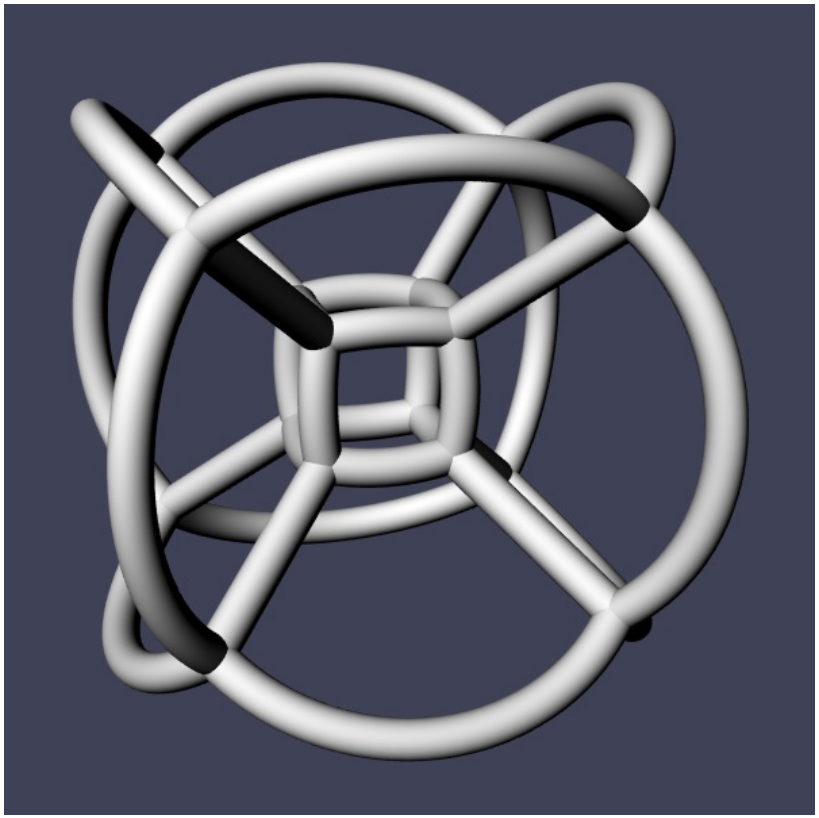
4-dimensional polytopes (drawn with thickened edges)

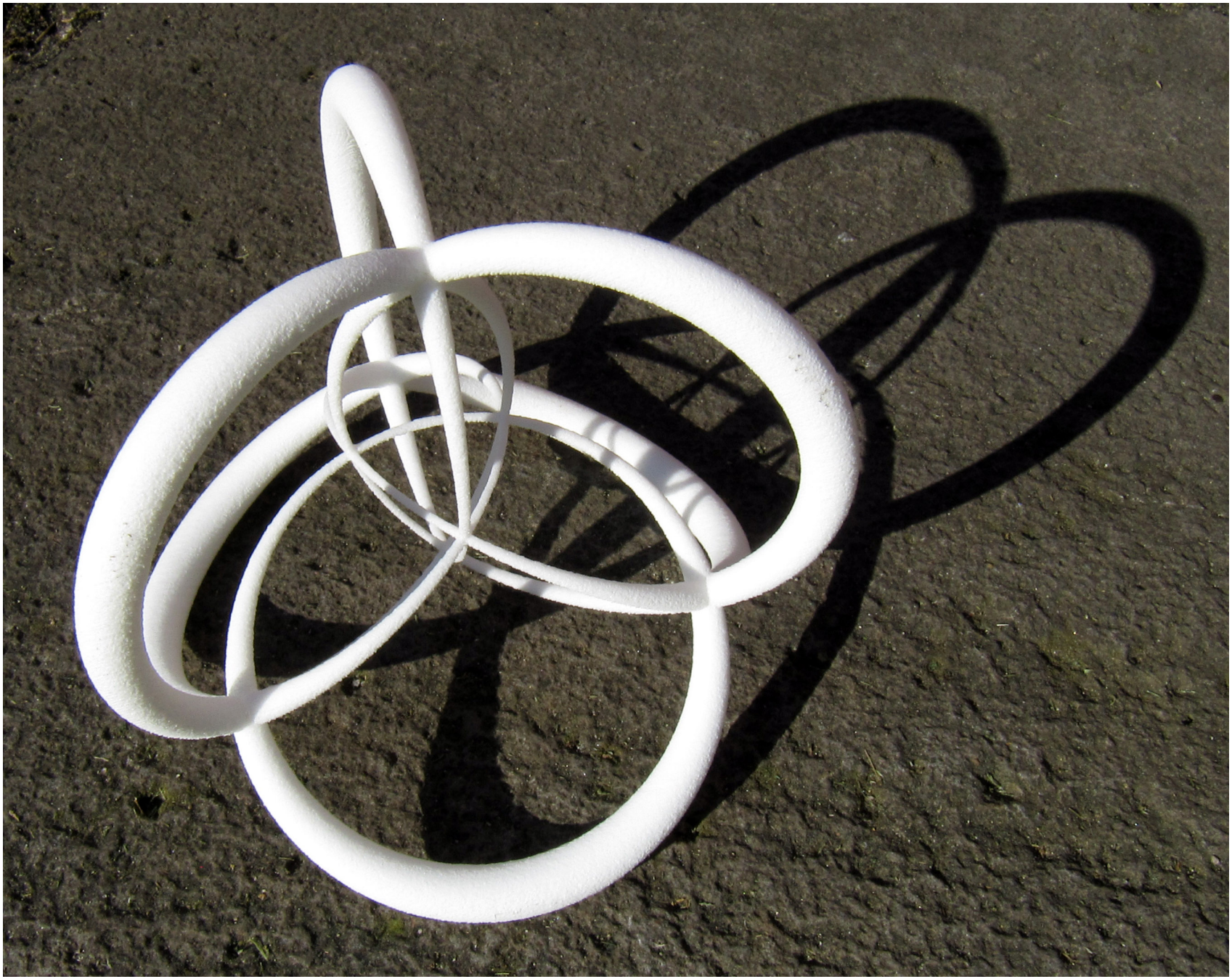
5-cell	8-cell	16-cell
		
24-cell	**120-cell**	**600-cell**
		

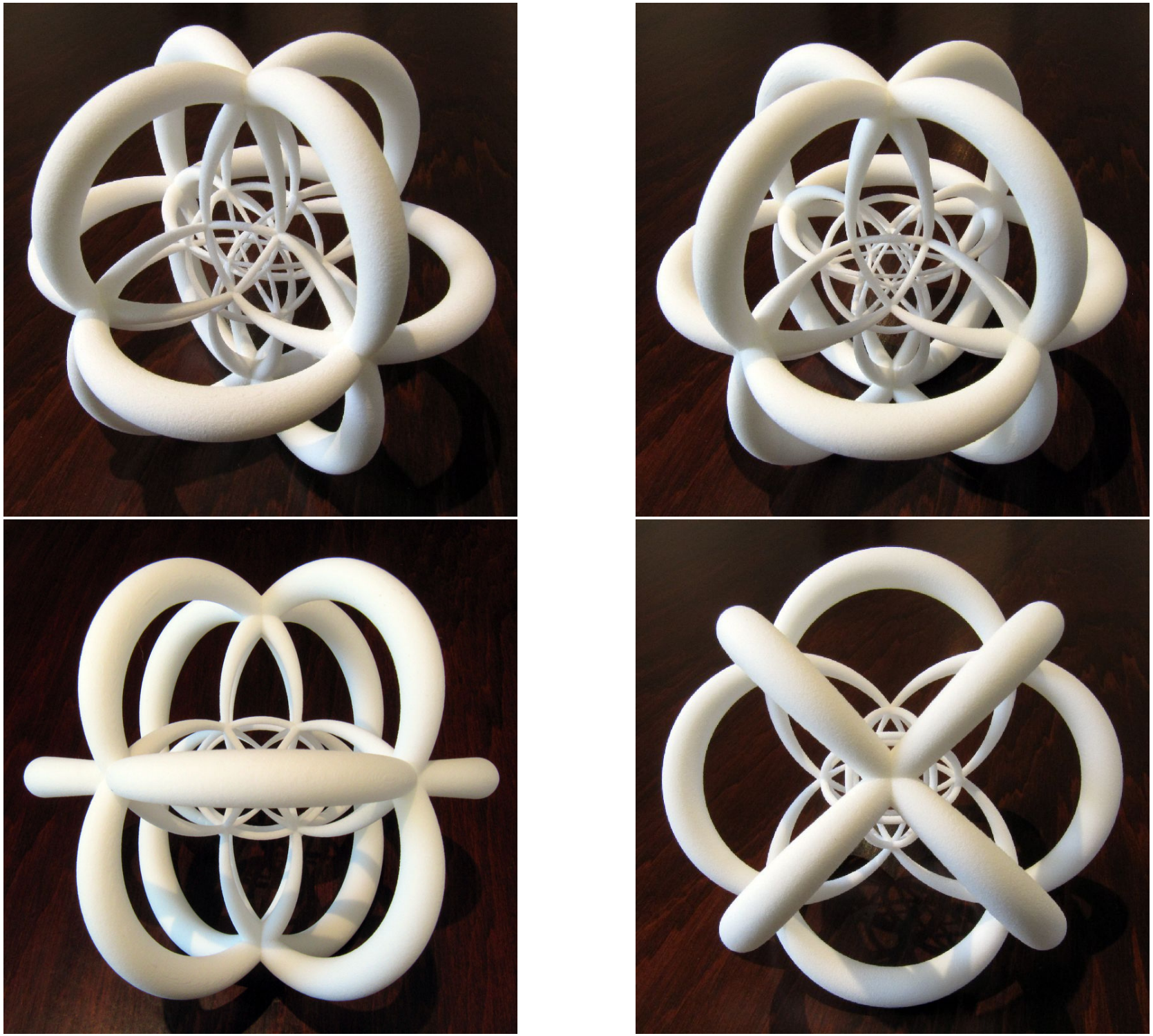
Thickening in S^3 is better than thickening in $\mathbb{R}^3$

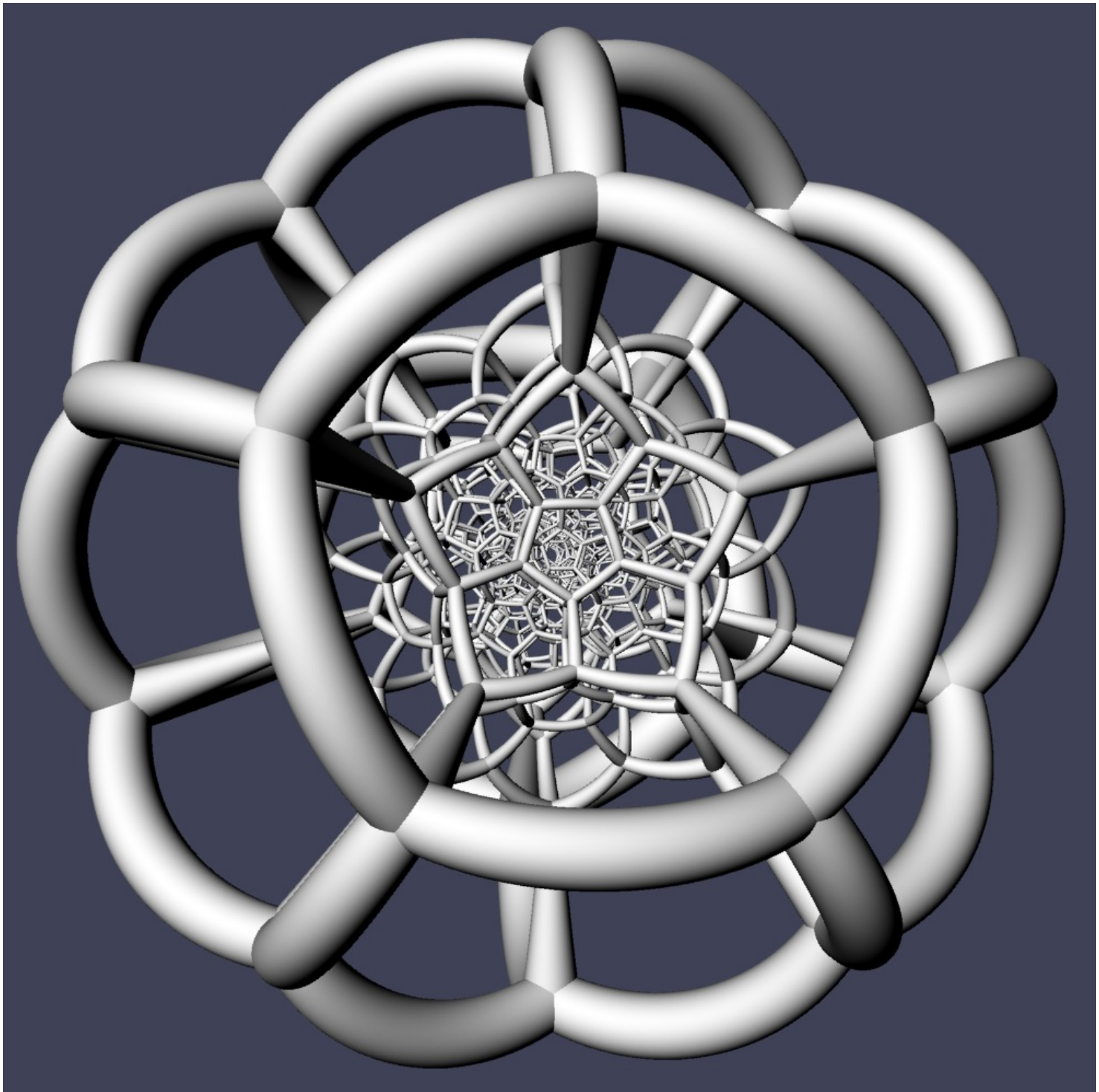


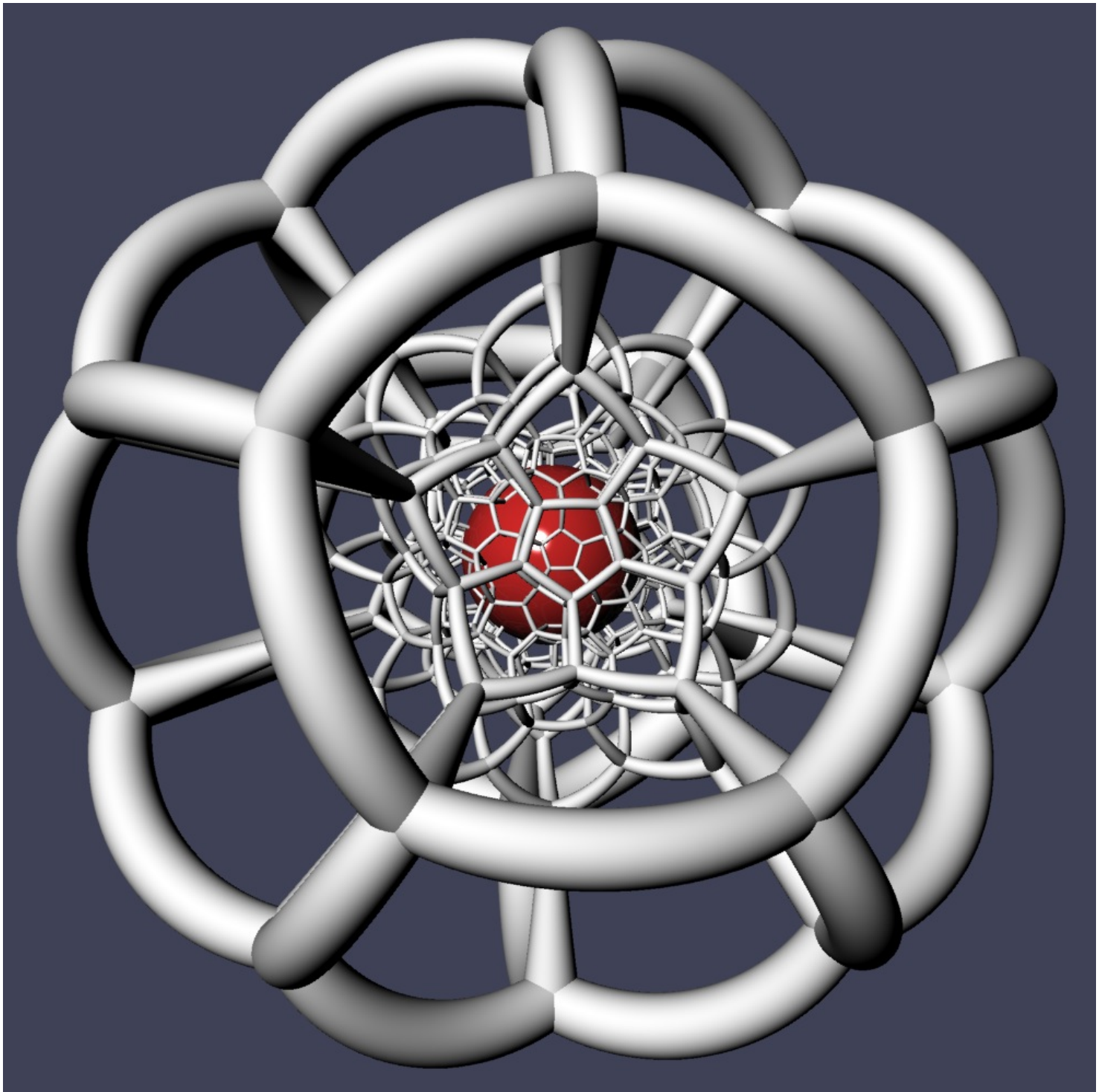


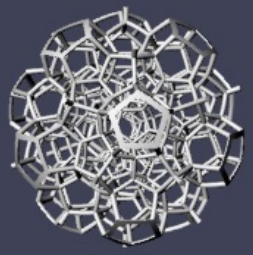
- The ratio of distances between objects to thicknesses of objects is constant if we thicken in S^3, but varies if we thicken in $\mathbb{R}^3$. So the former retains more symmetry.
- There are both pros and cons in terms of cost and strength of the printed object.

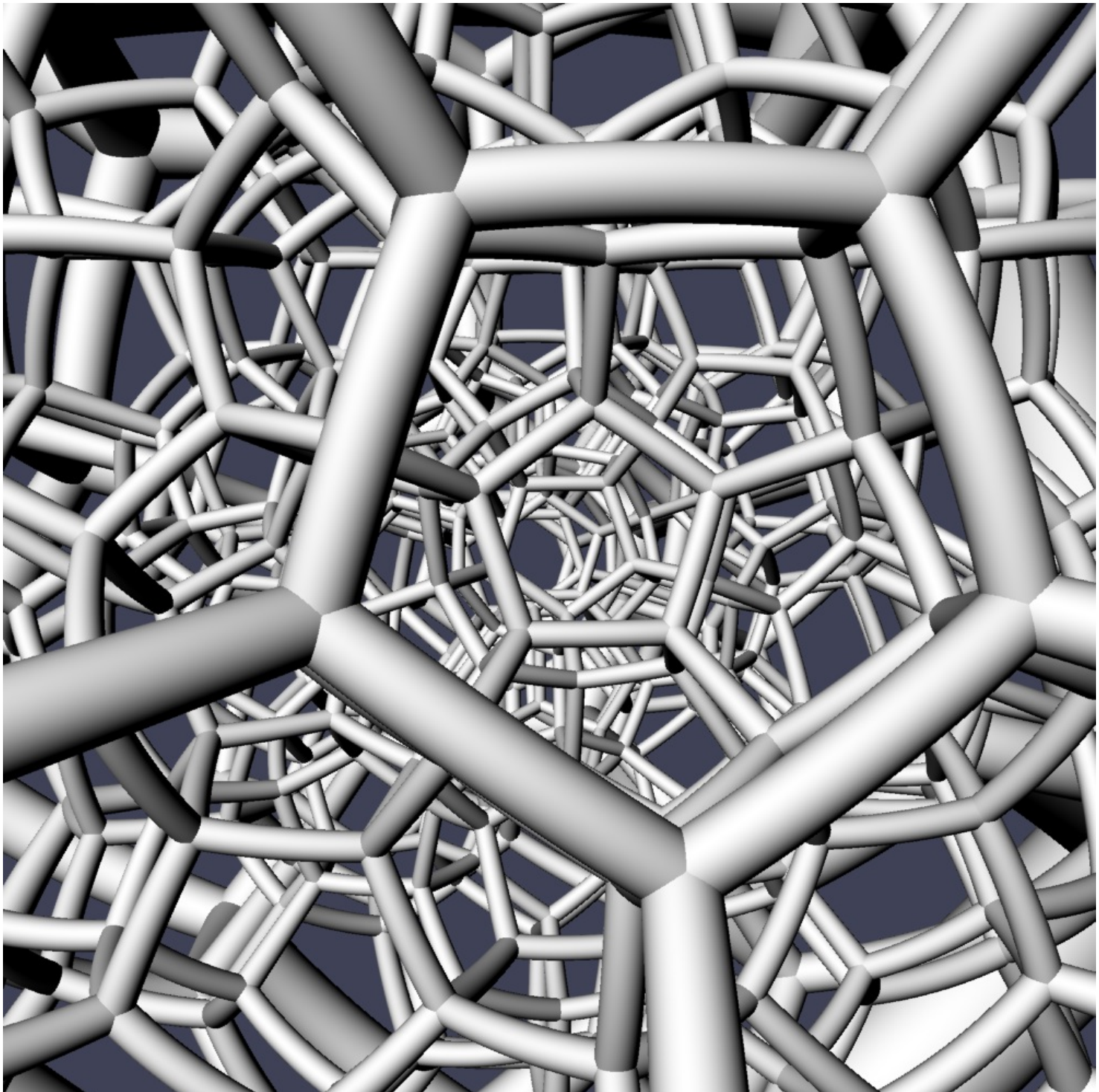


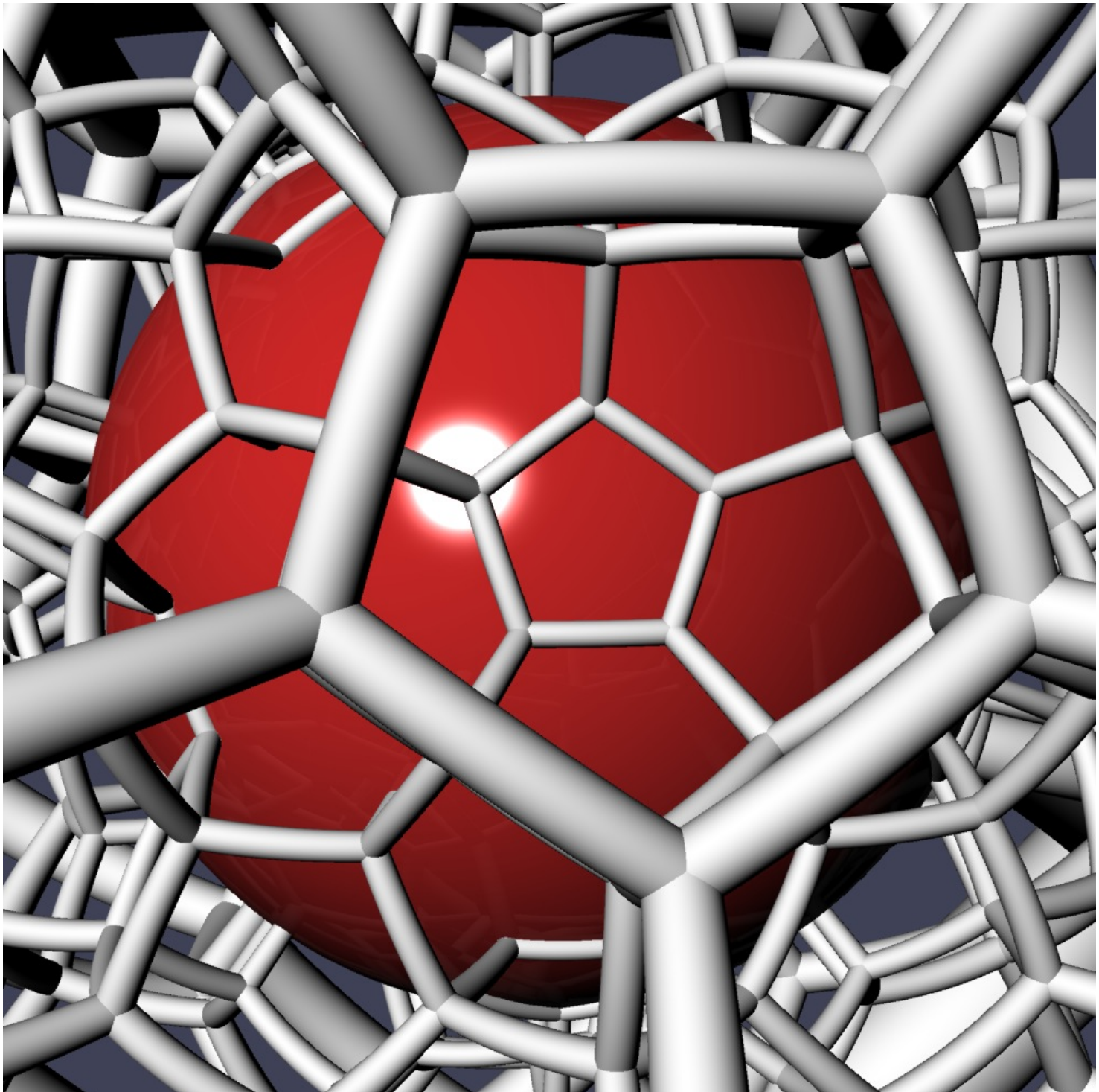


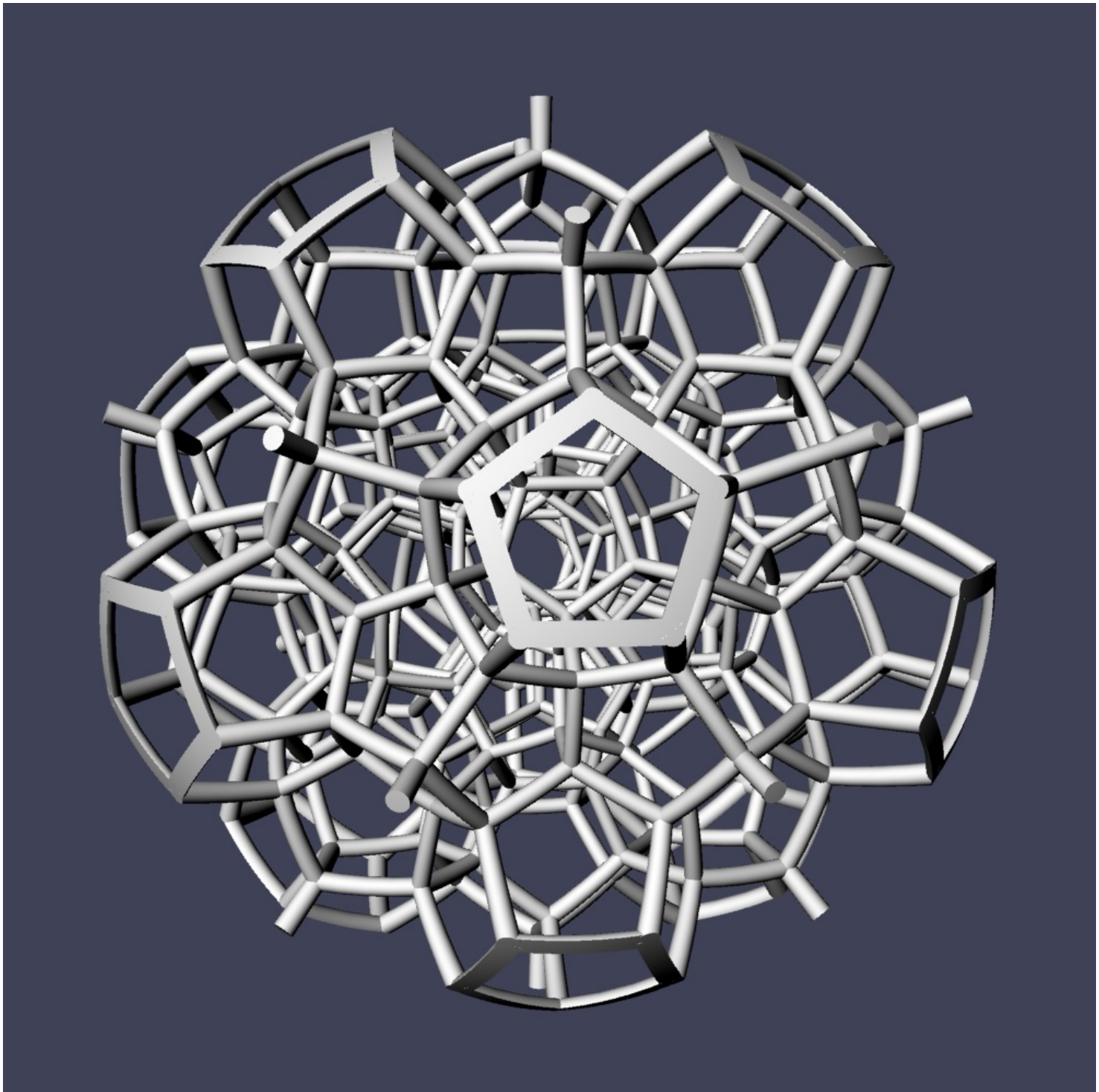


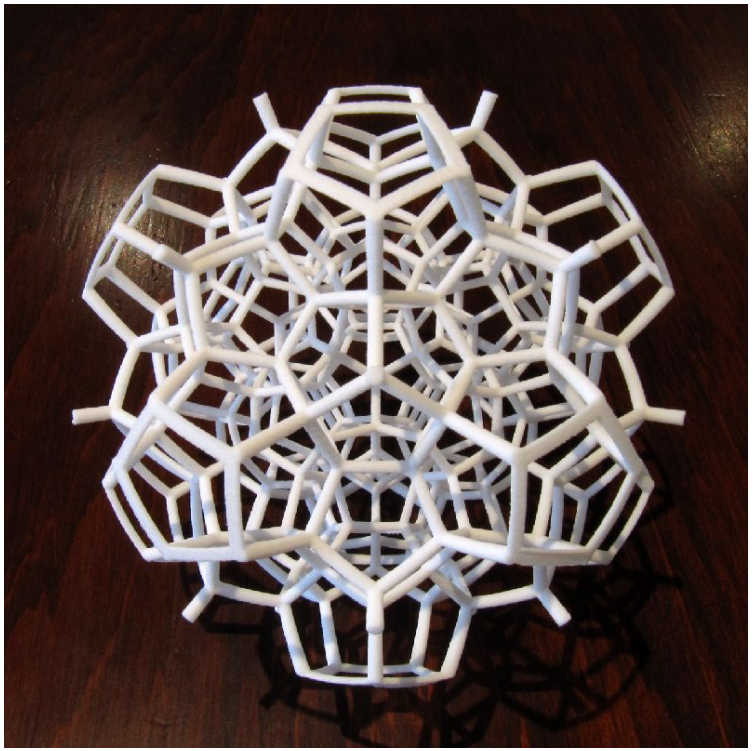


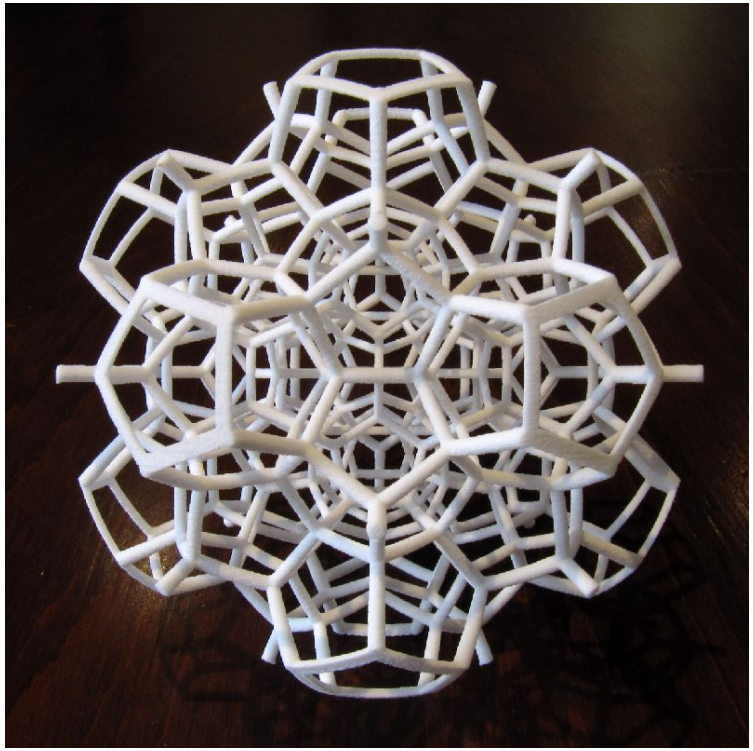


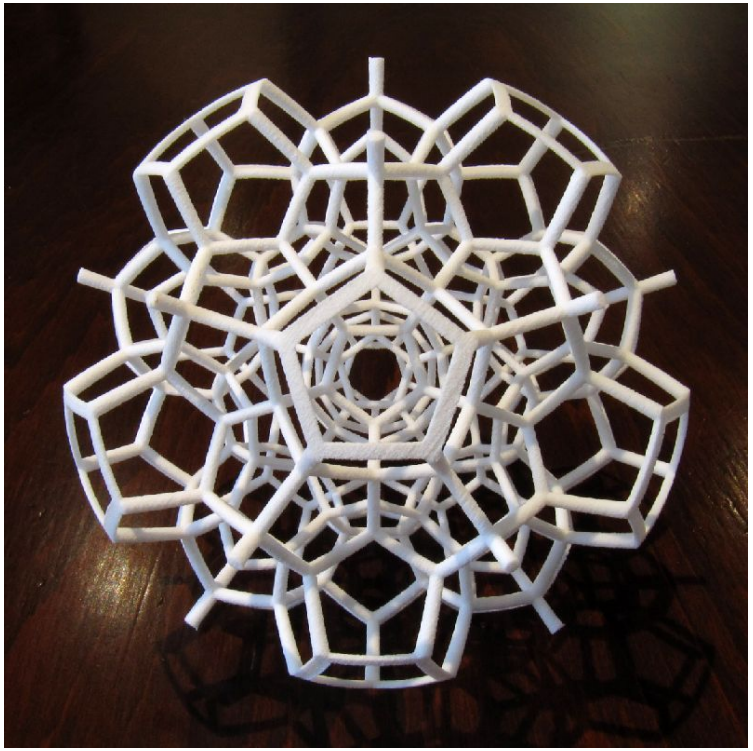


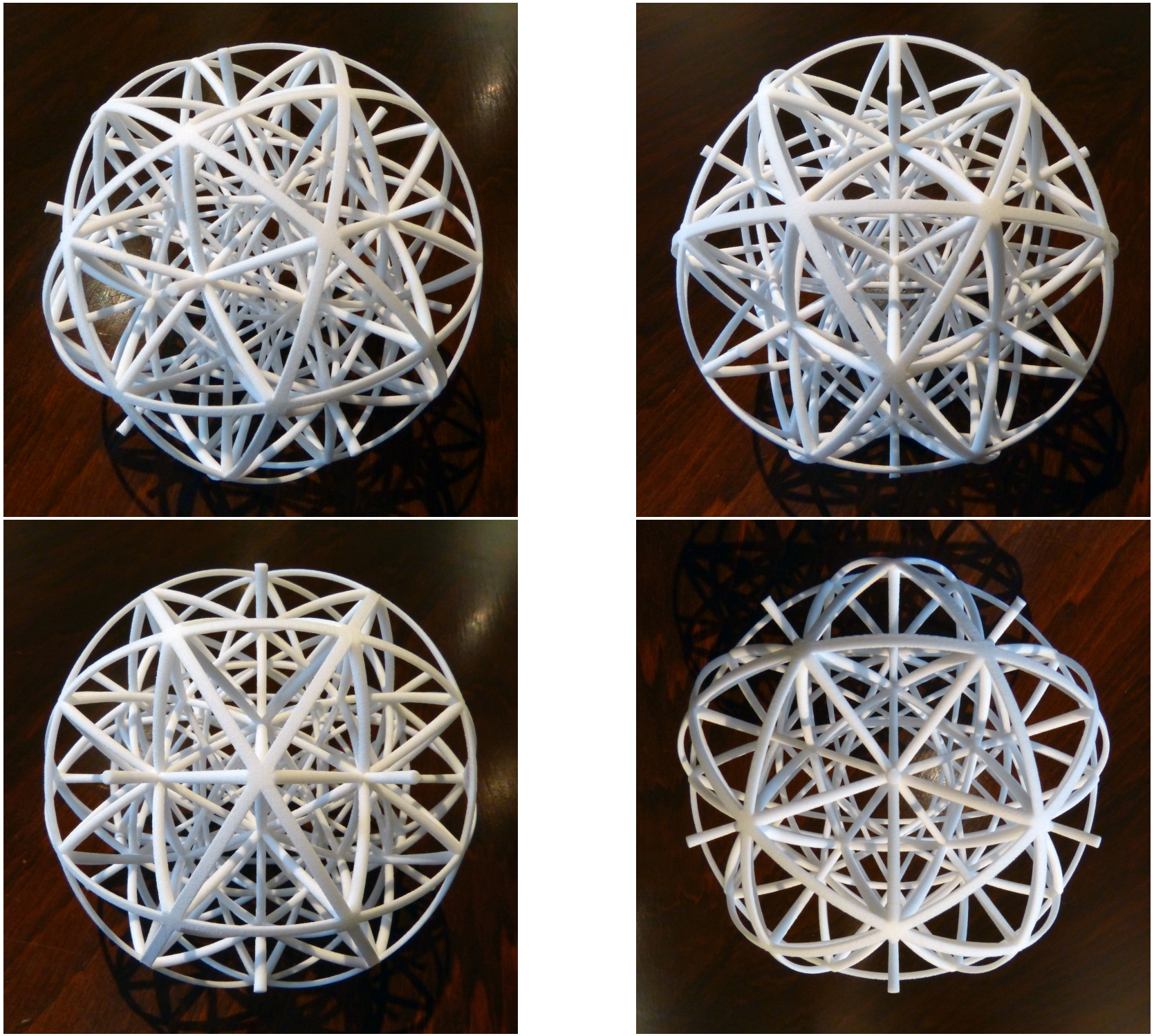


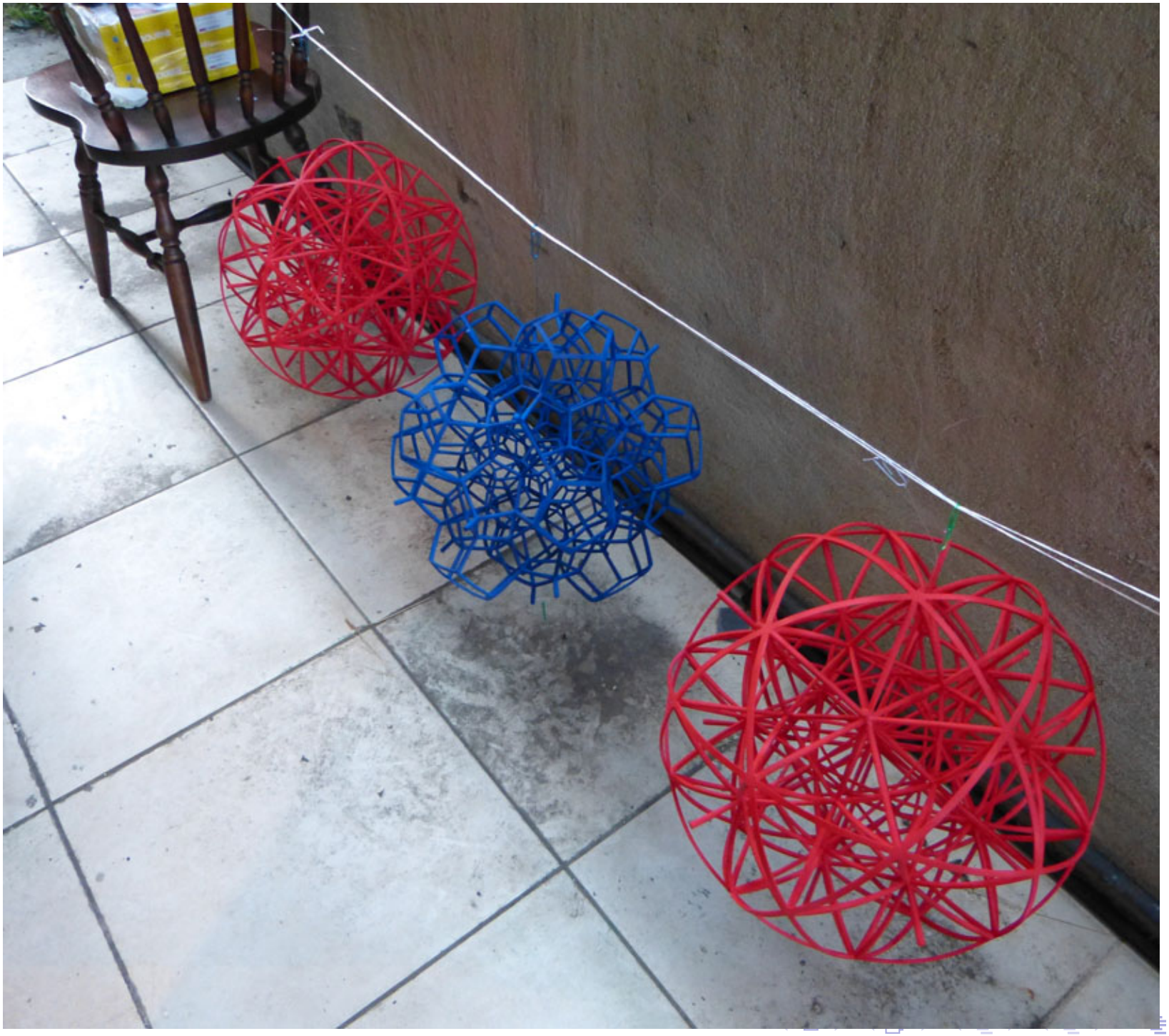






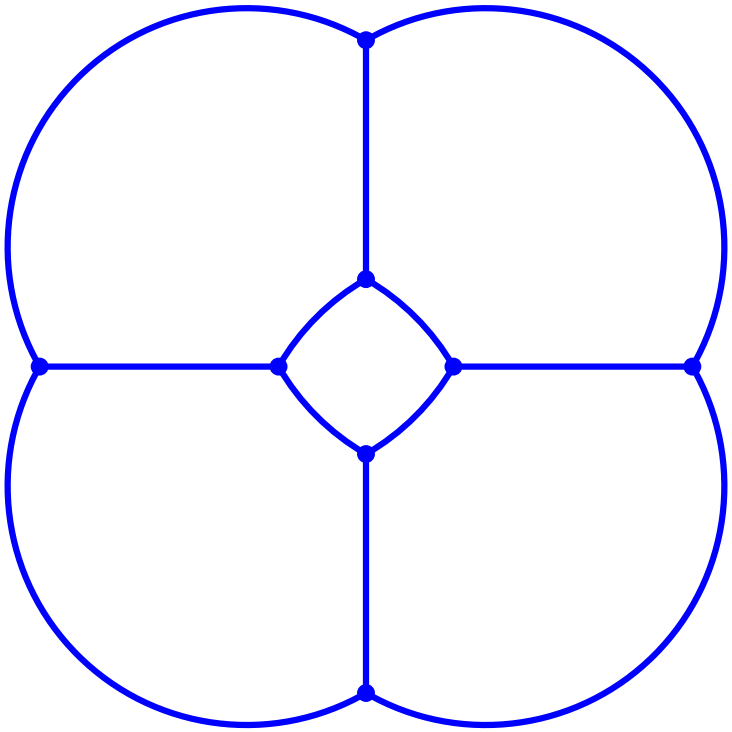






Duality for 3-dimensional polyhedra

vertices	$\leftrightarrow$	faces
edges	$\leftrightarrow$	edges
faces	$\leftrightarrow$	vertices



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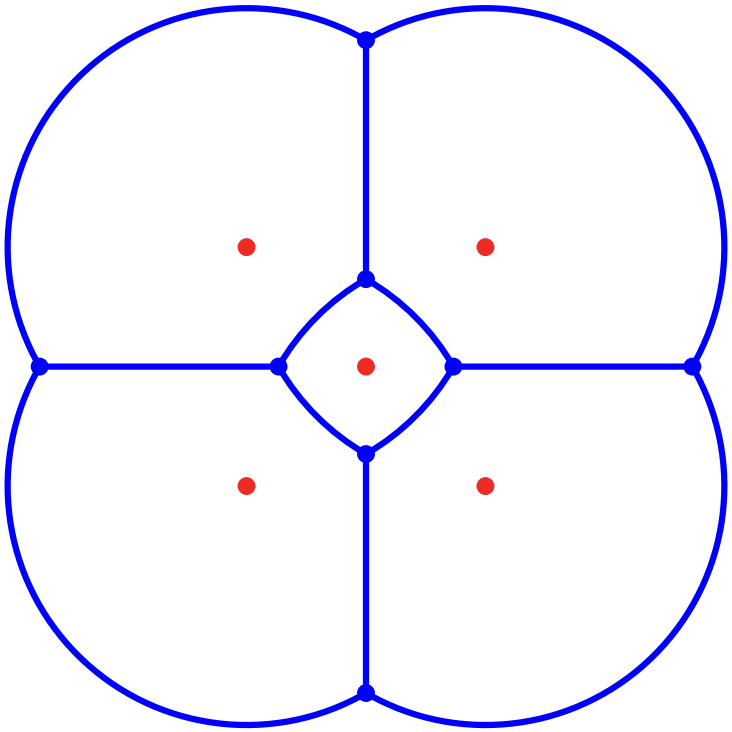

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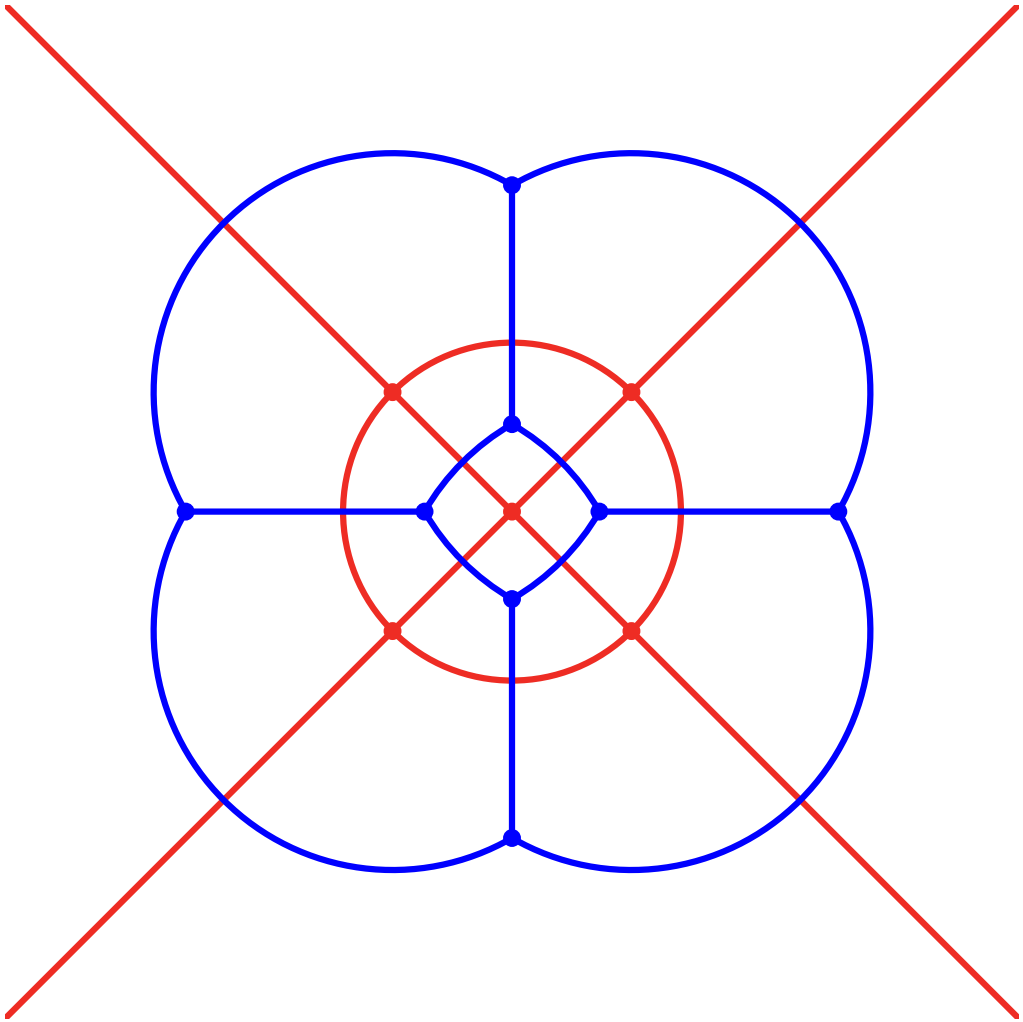

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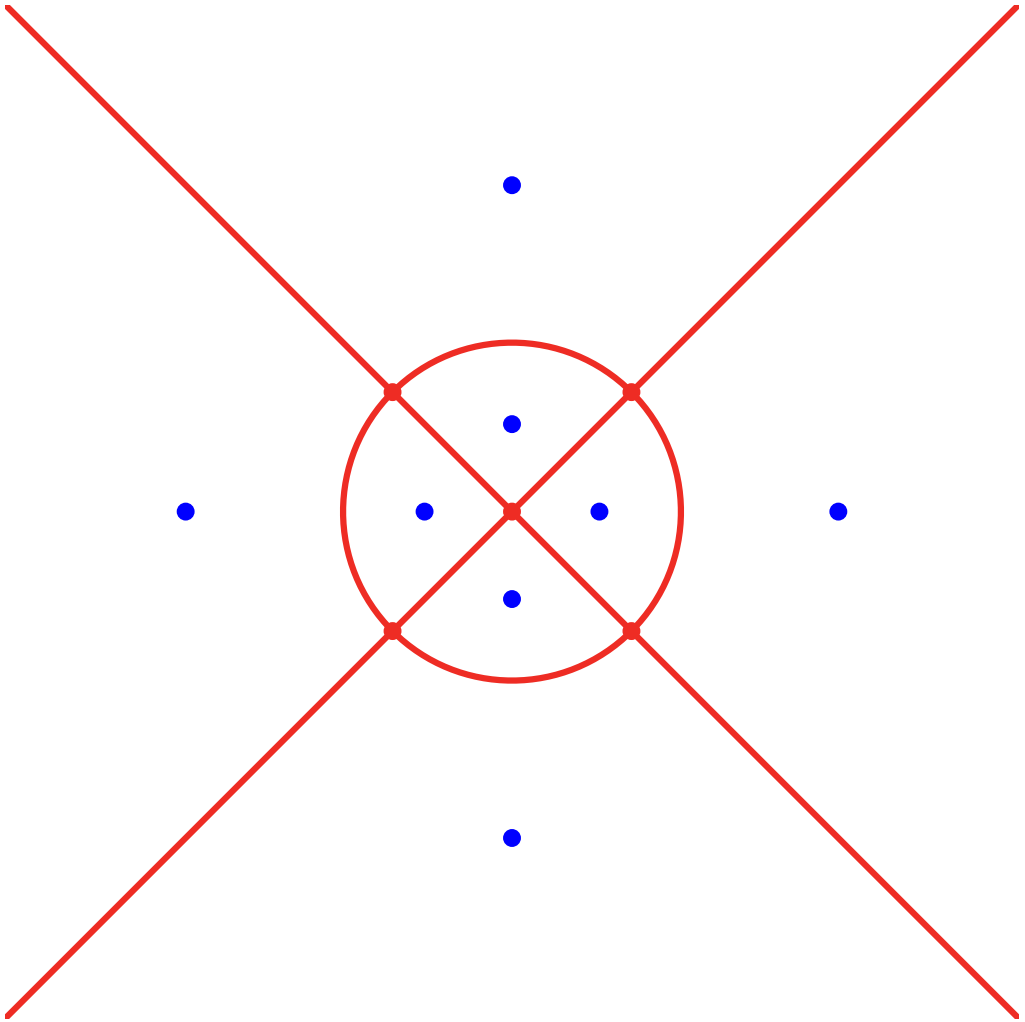

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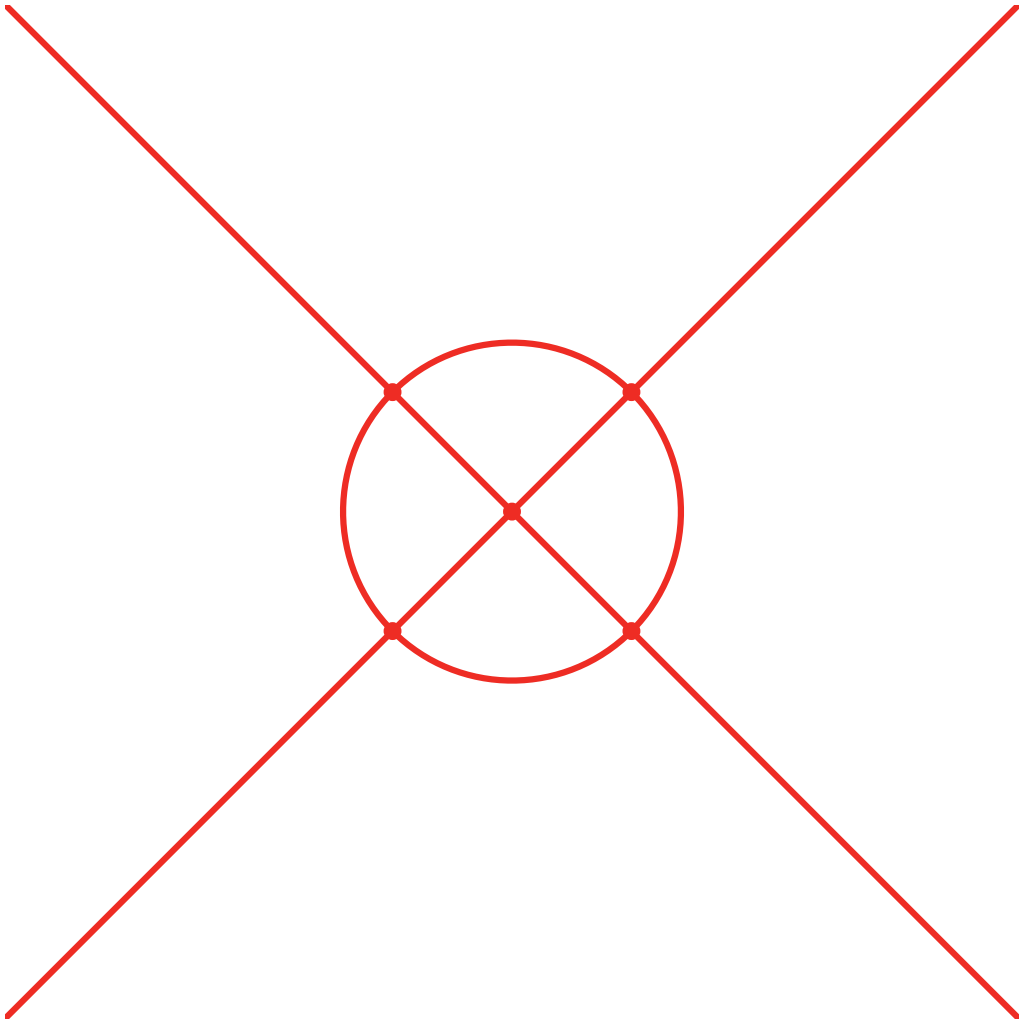

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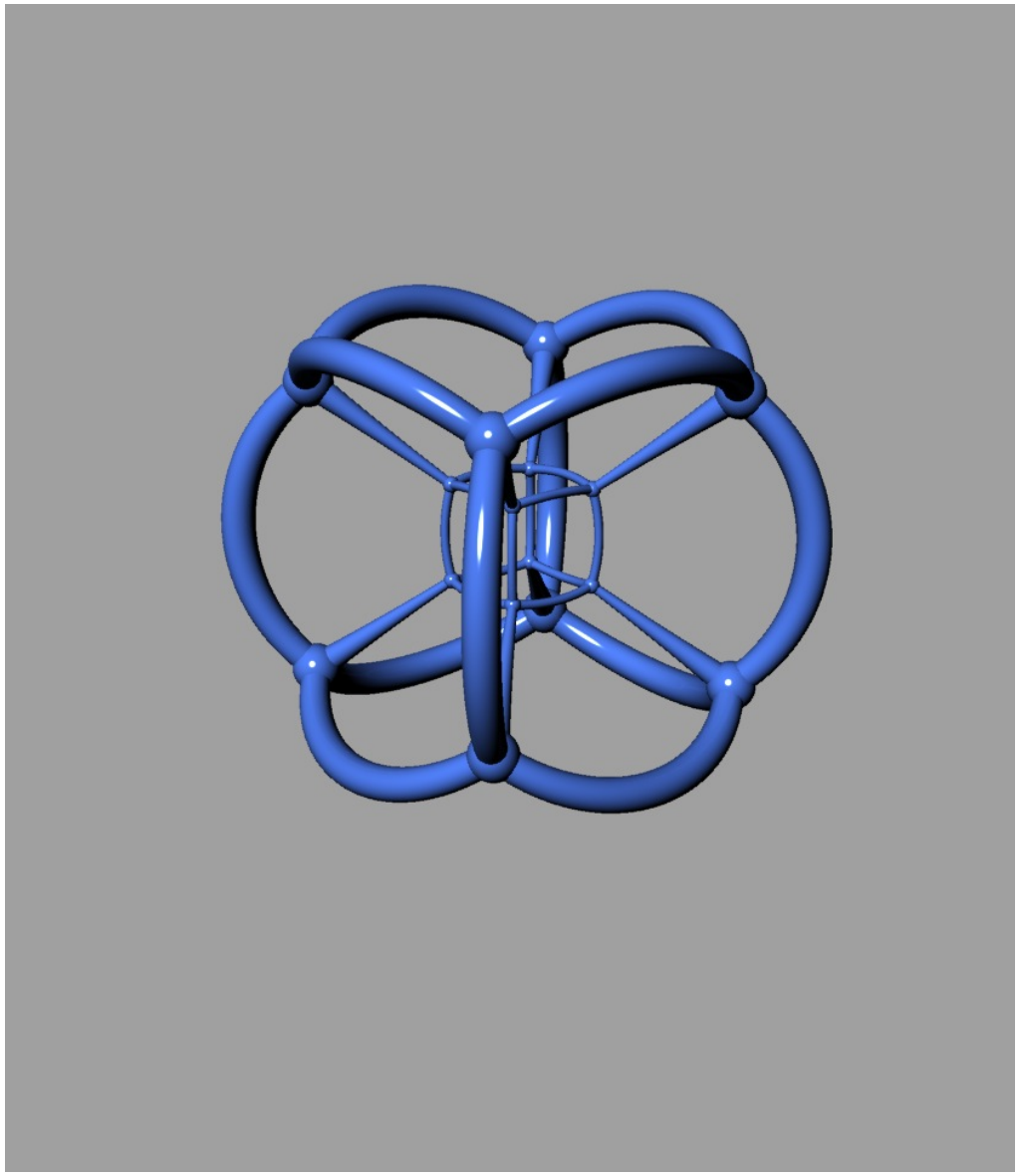
faces	$\leftrightarrow$	edges
cells	$\leftrightarrow$	vertices



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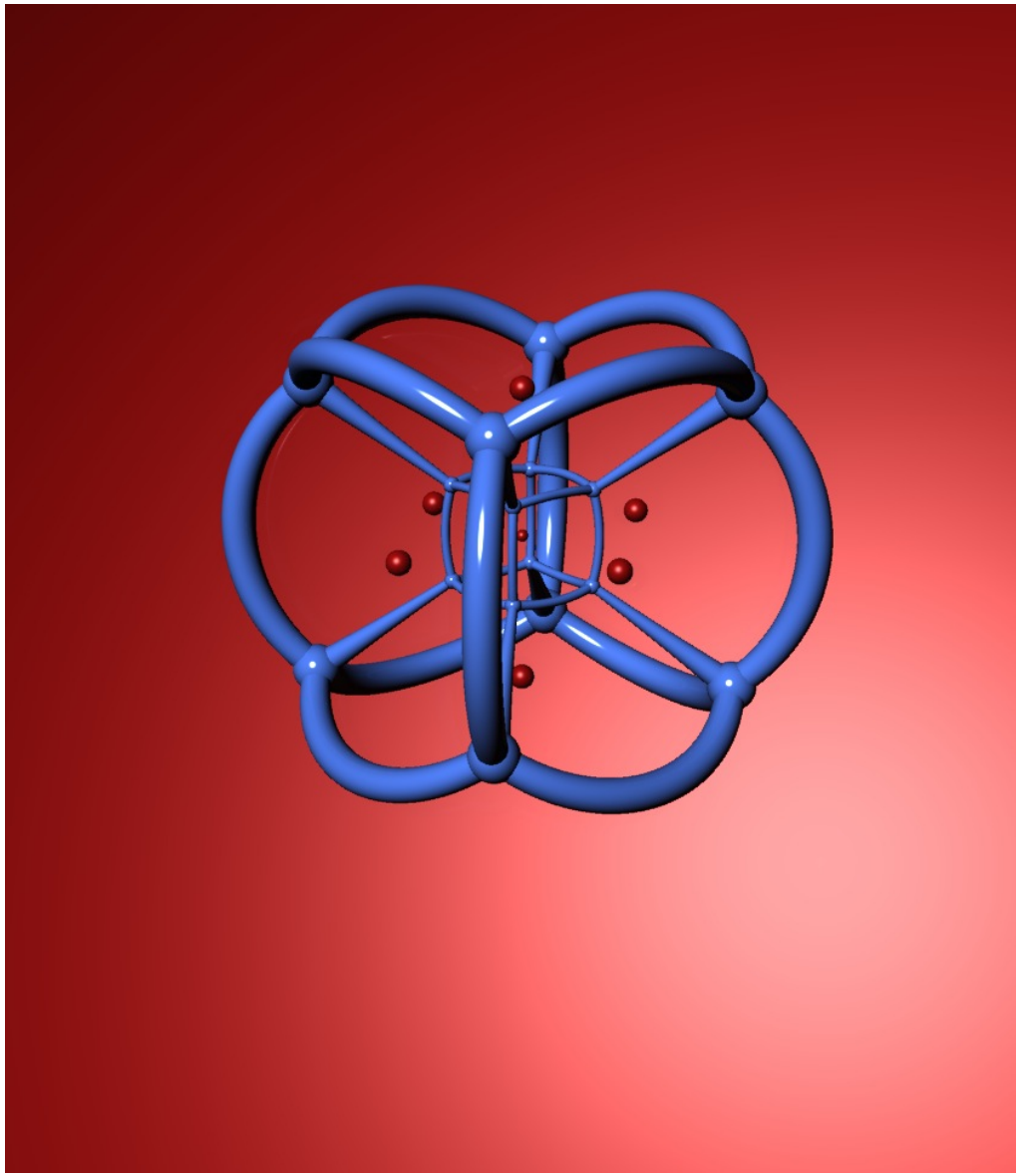

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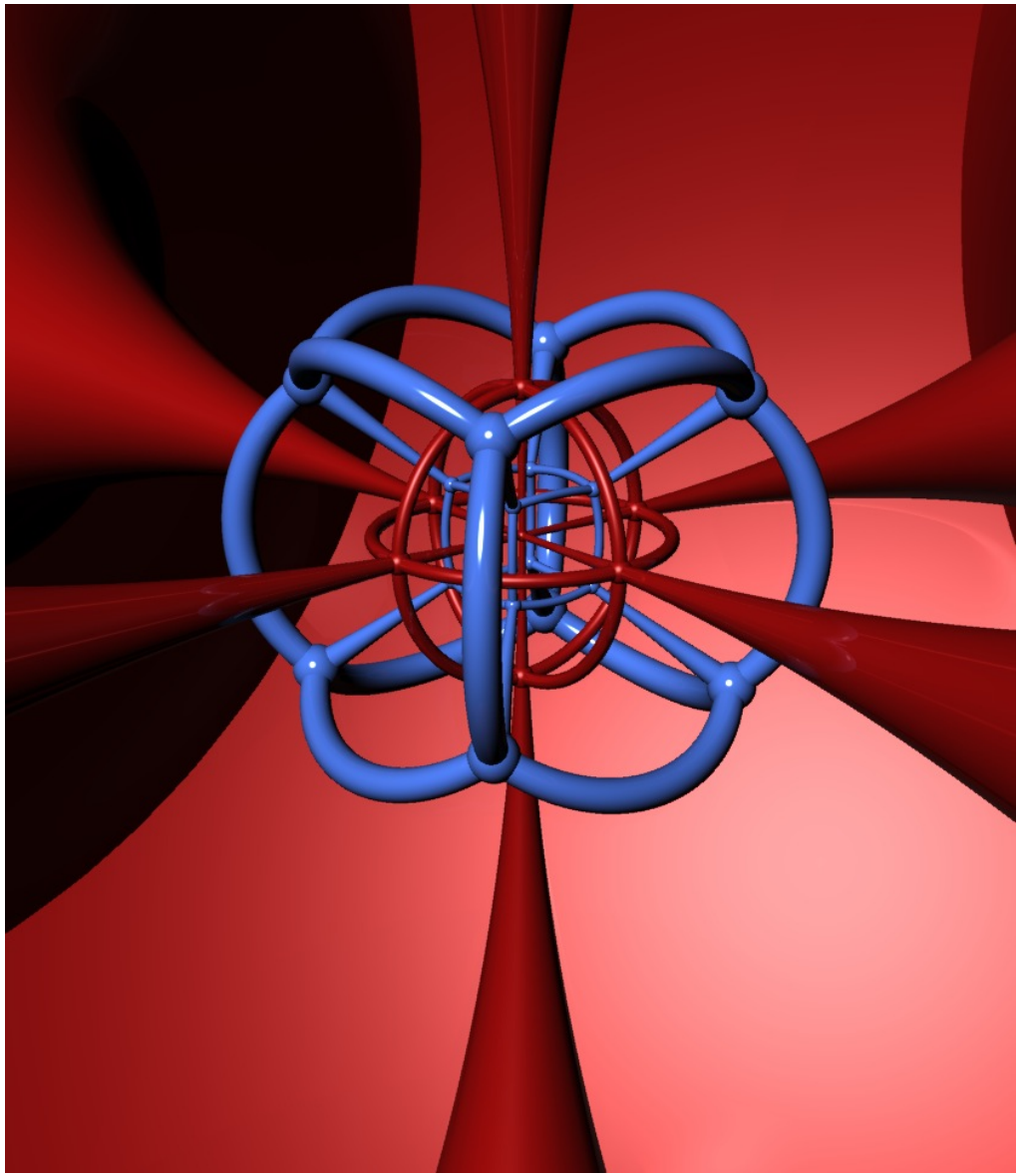

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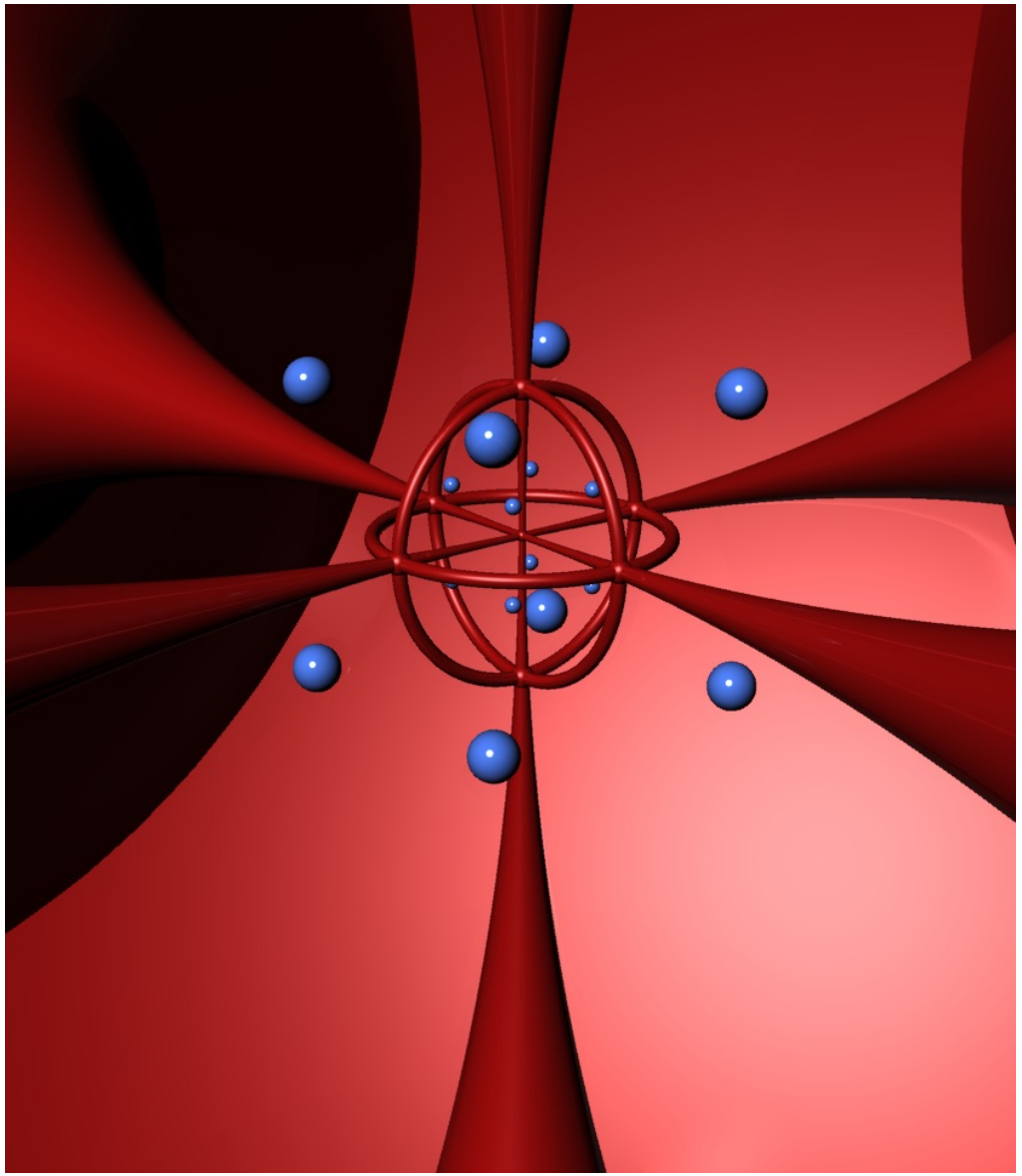
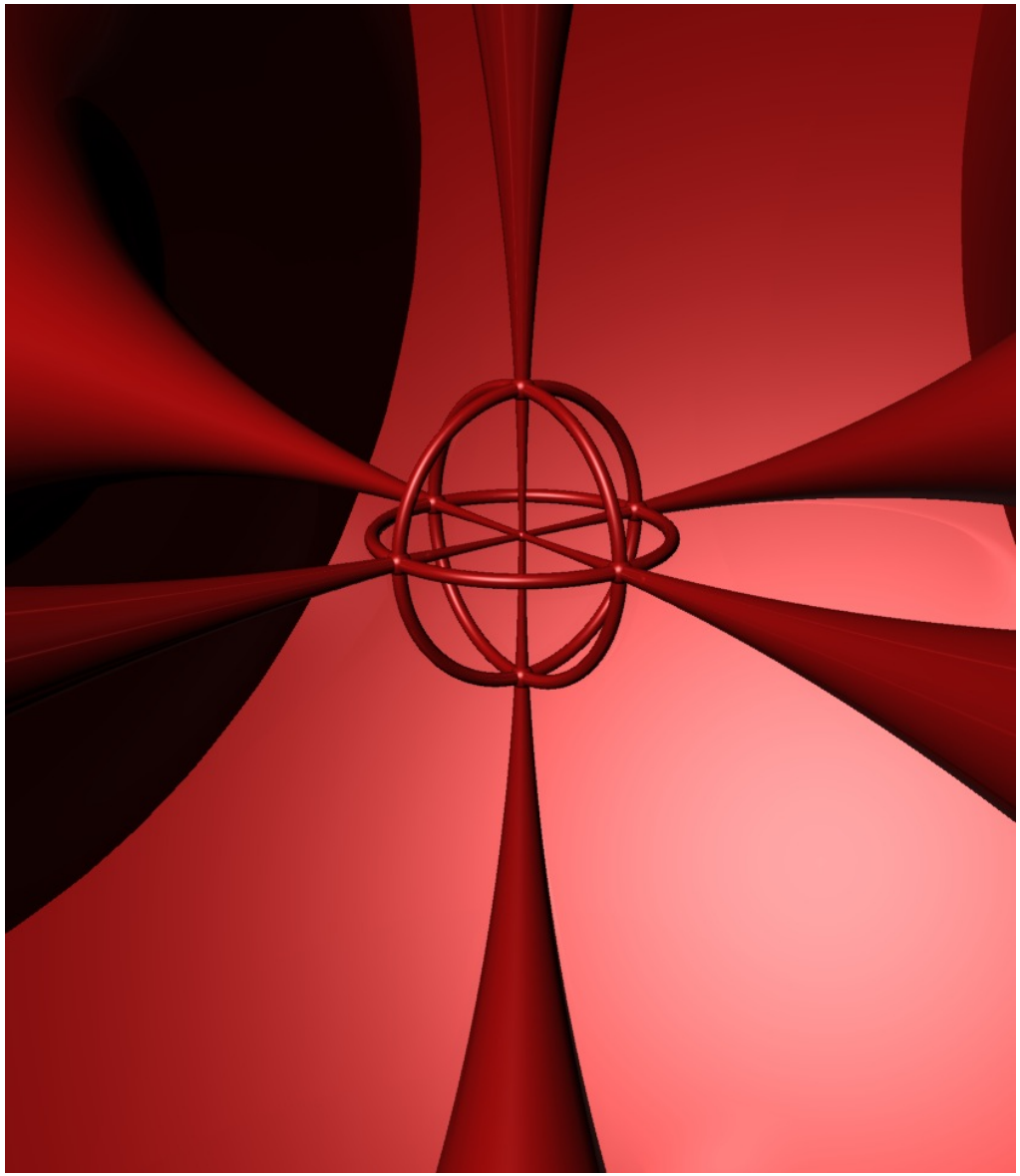
cells	$\leftrightarrow$	vertices

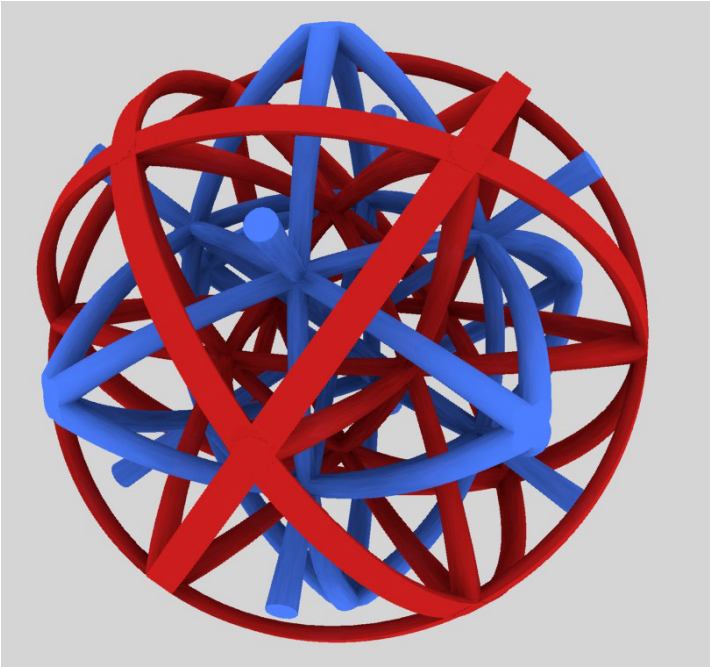


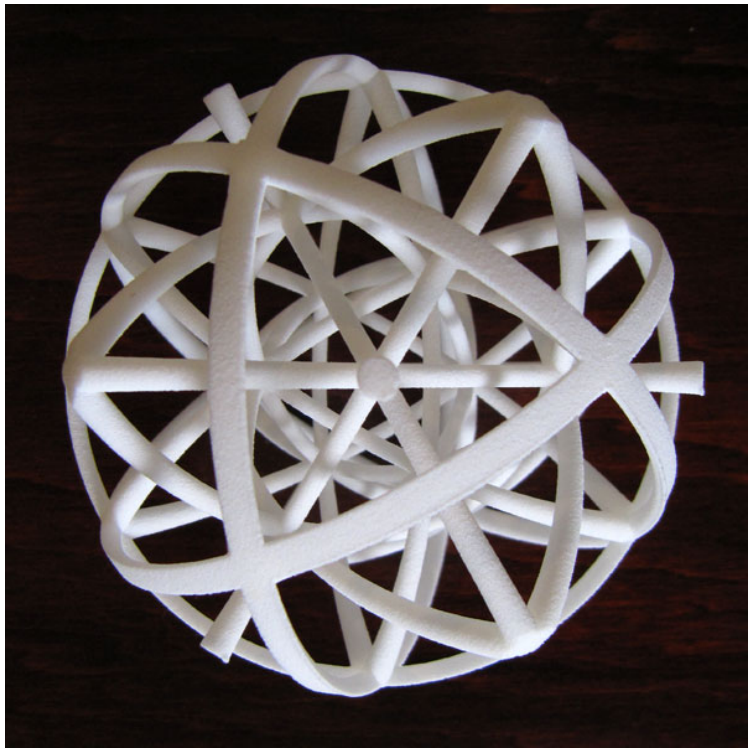
Duality for 4-dimensional polytopes

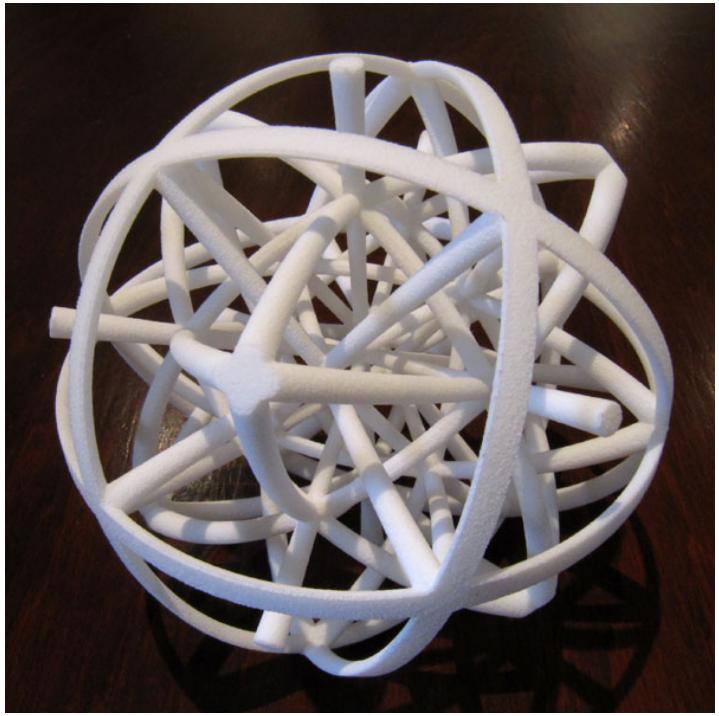
vertices	$\leftrightarrow$	cells
edges	$\leftrightarrow$	faces
faces	$\leftrightarrow$	edges
cells	$\leftrightarrow$	vertices

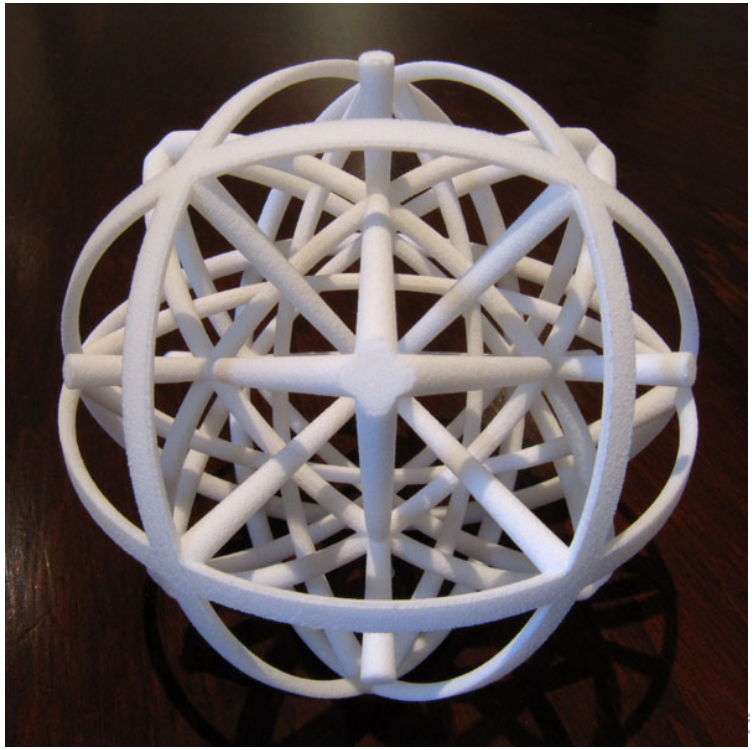


Dual half 24-cells

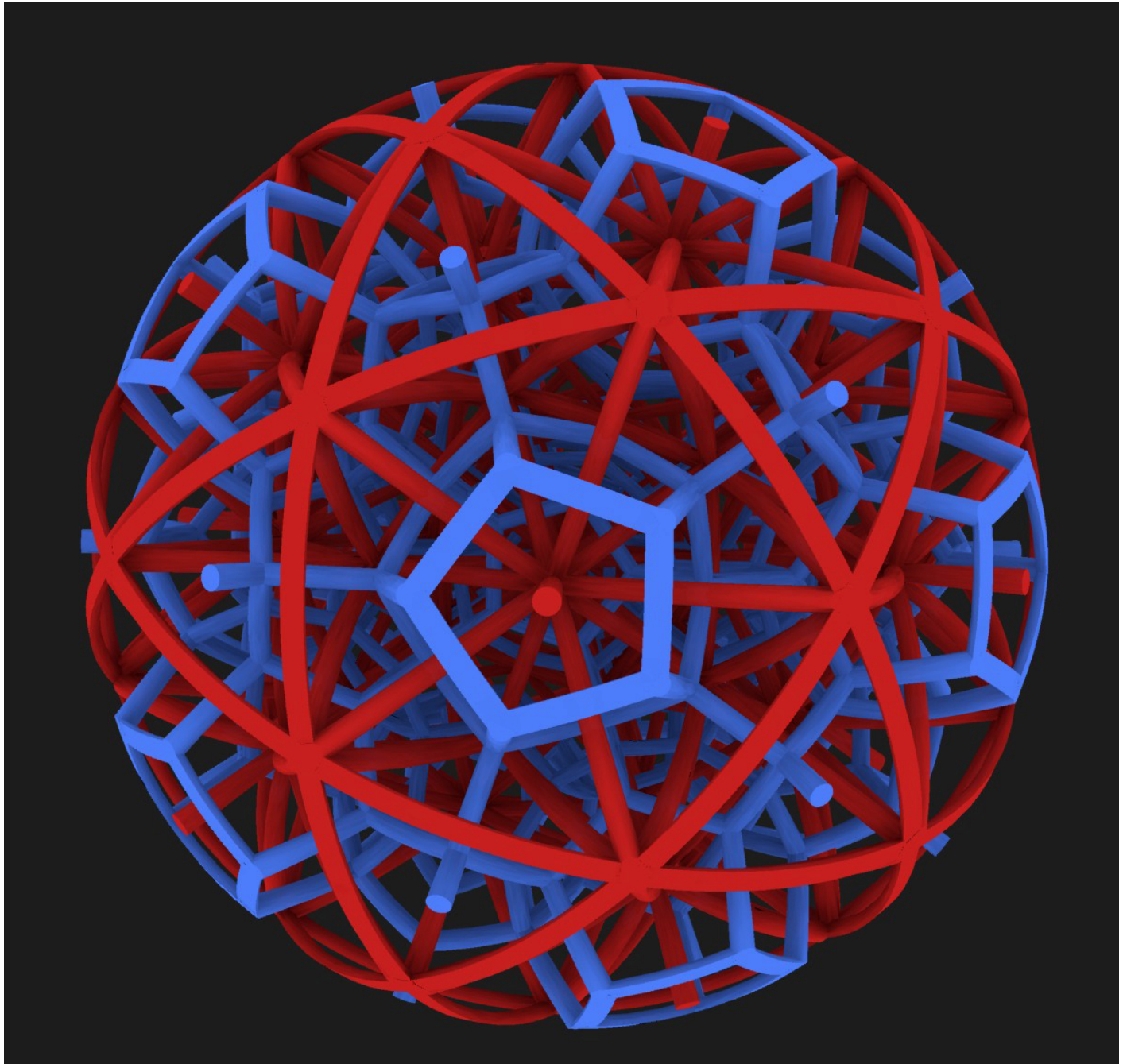


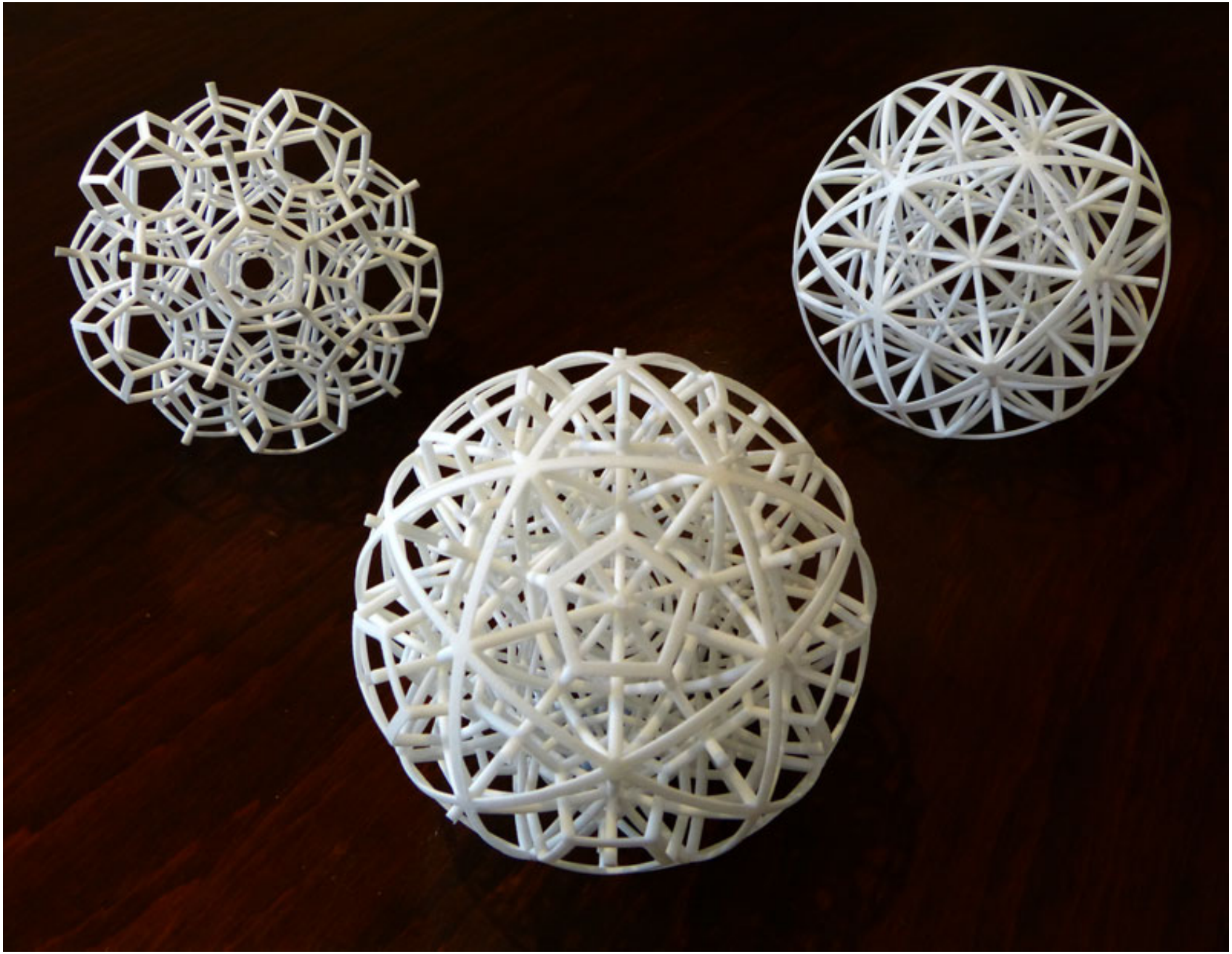


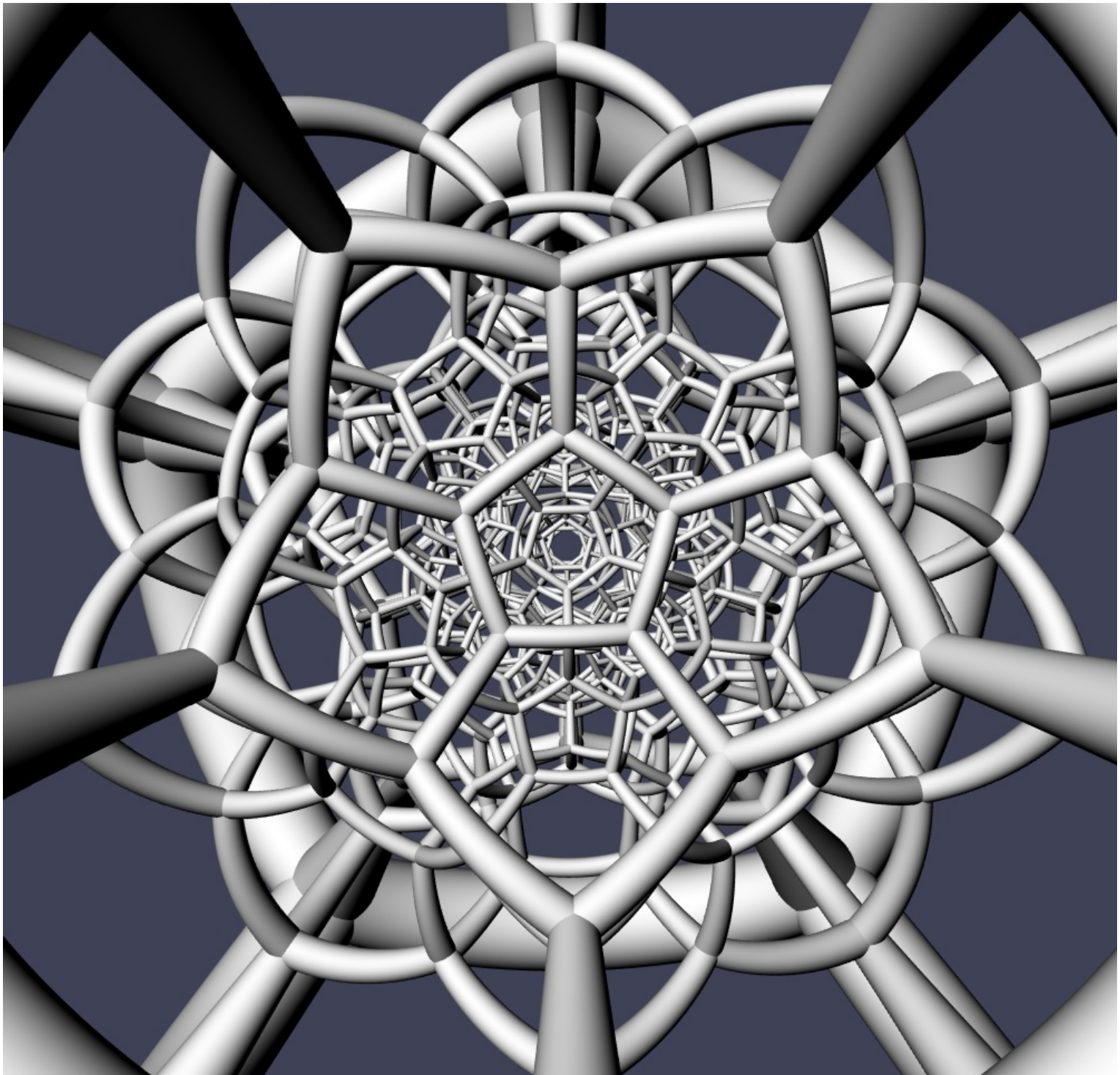


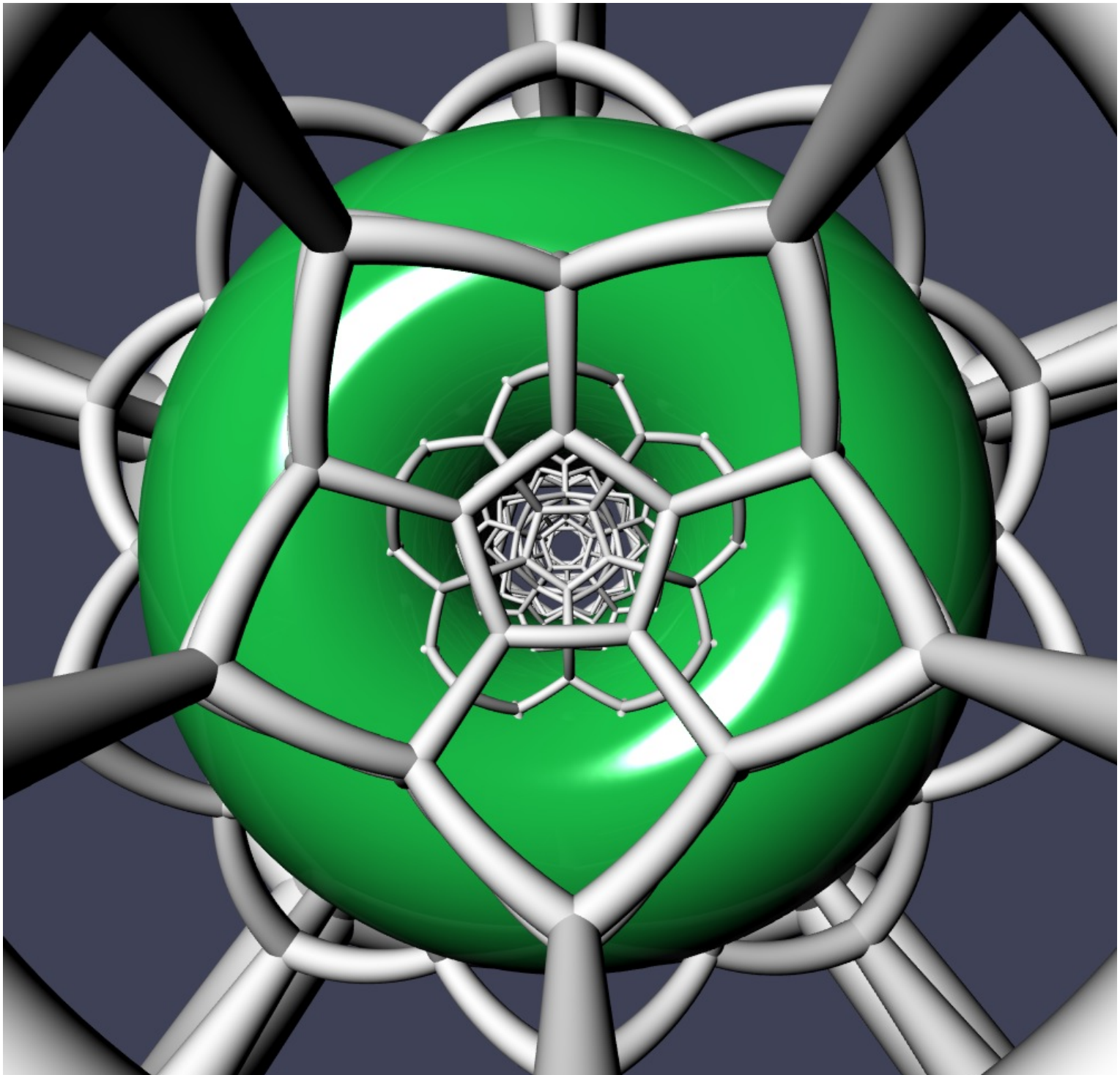


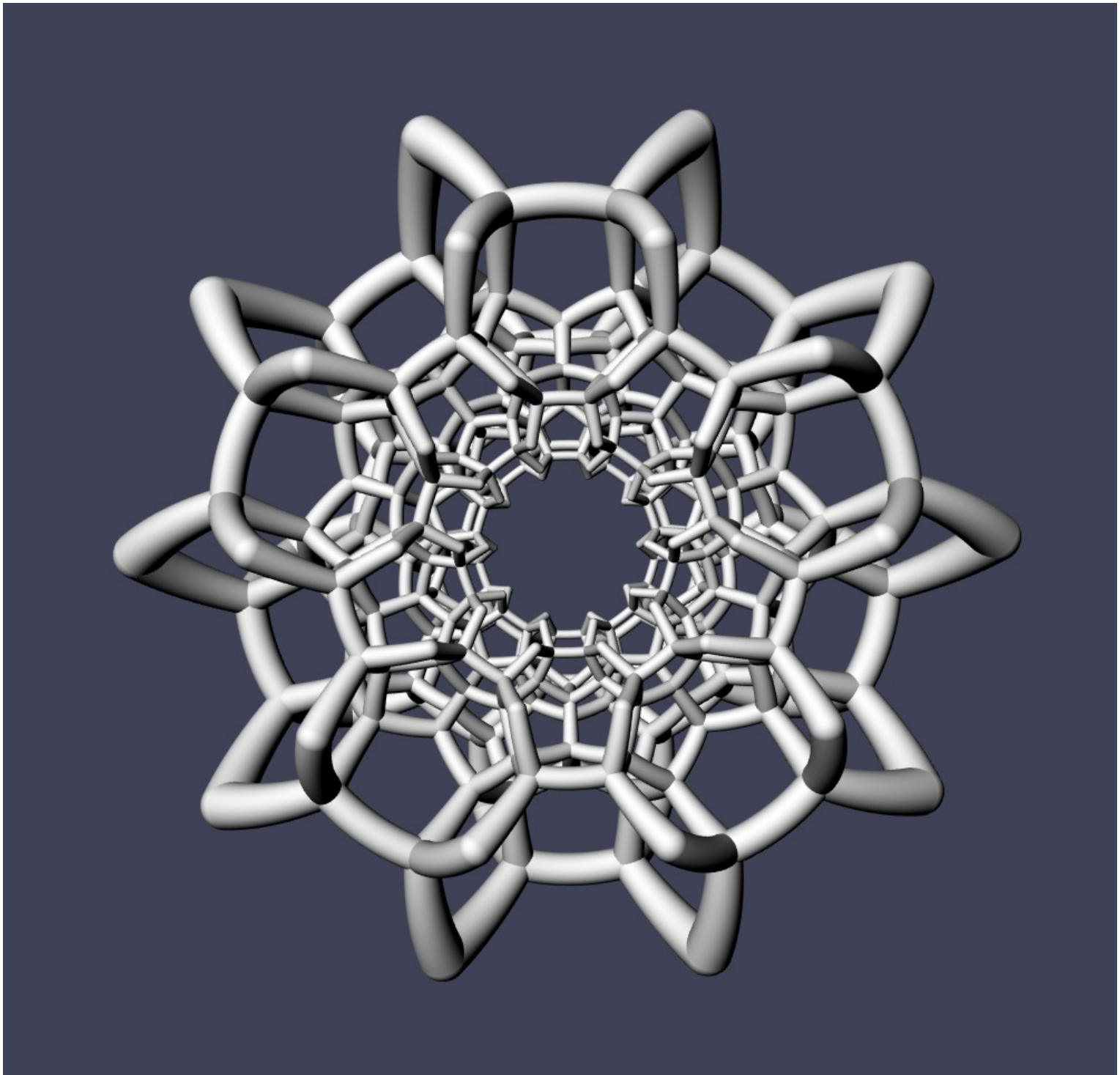
Dual half 120-cell and 600-cell

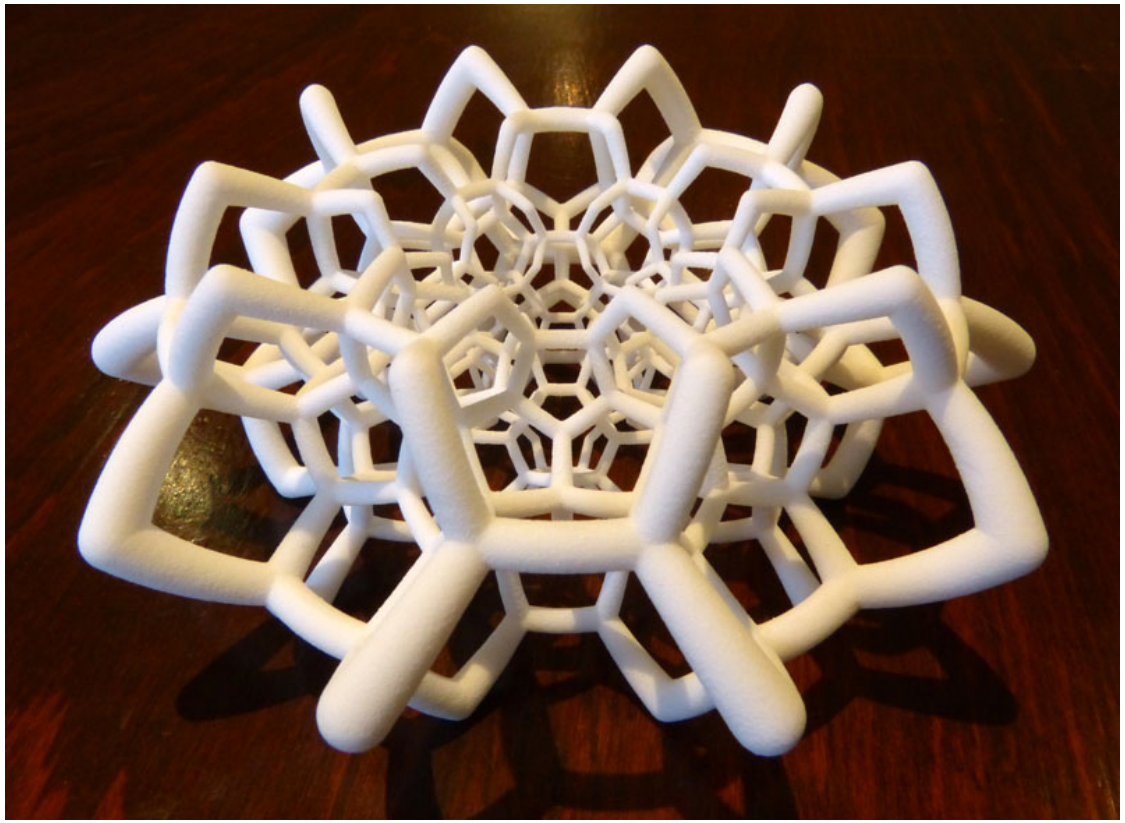


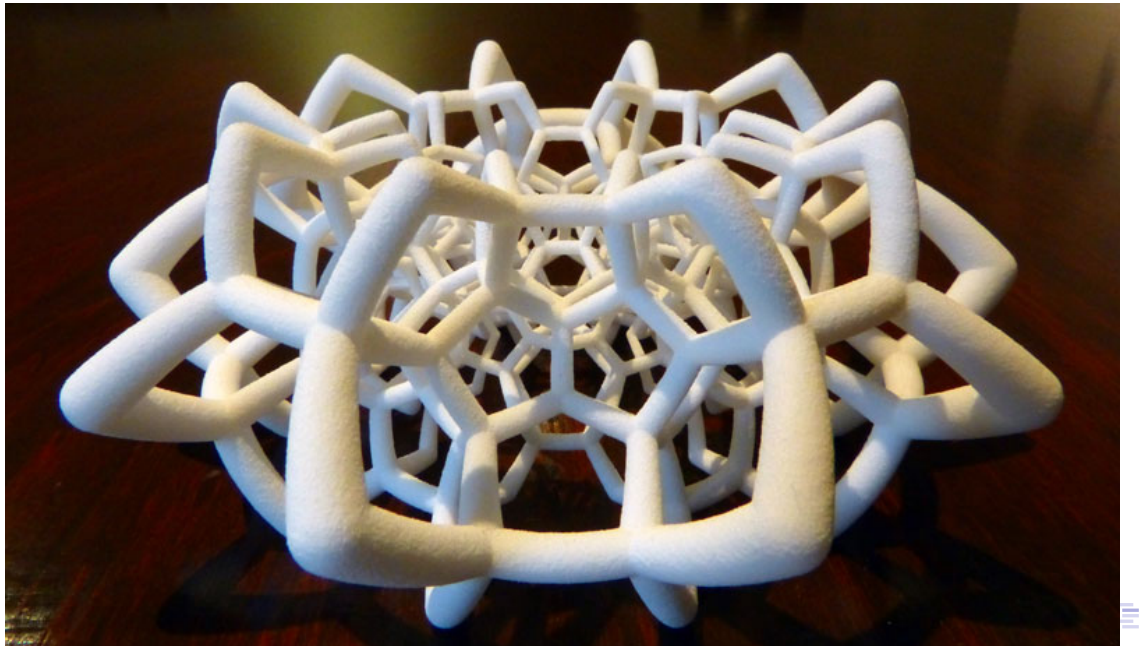




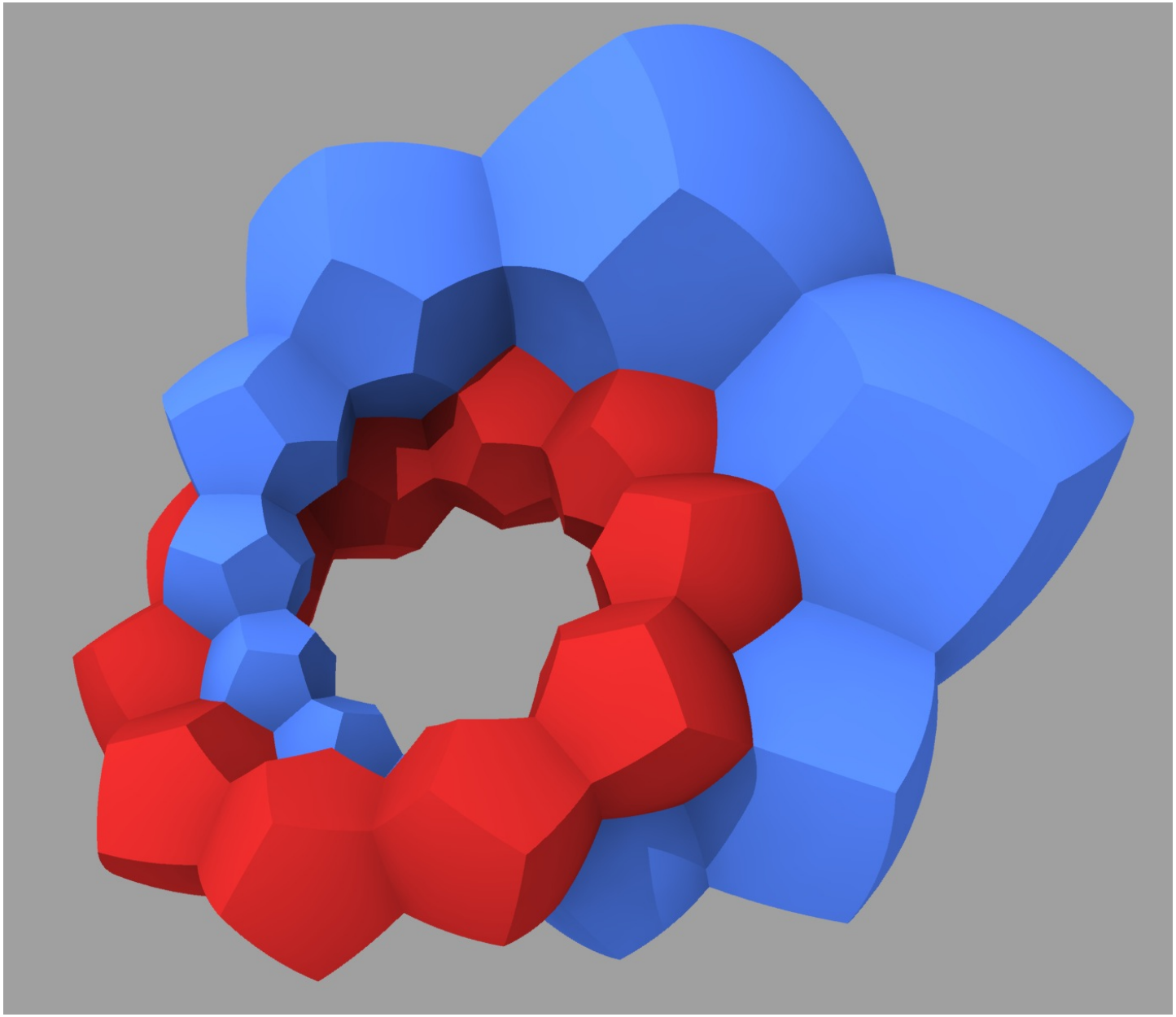


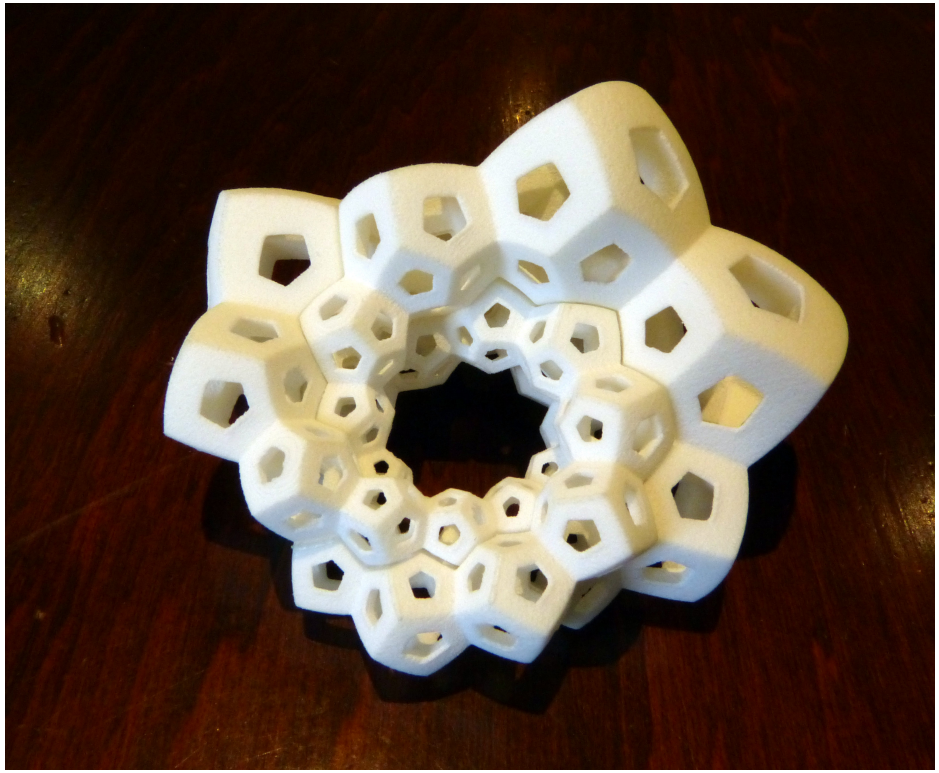


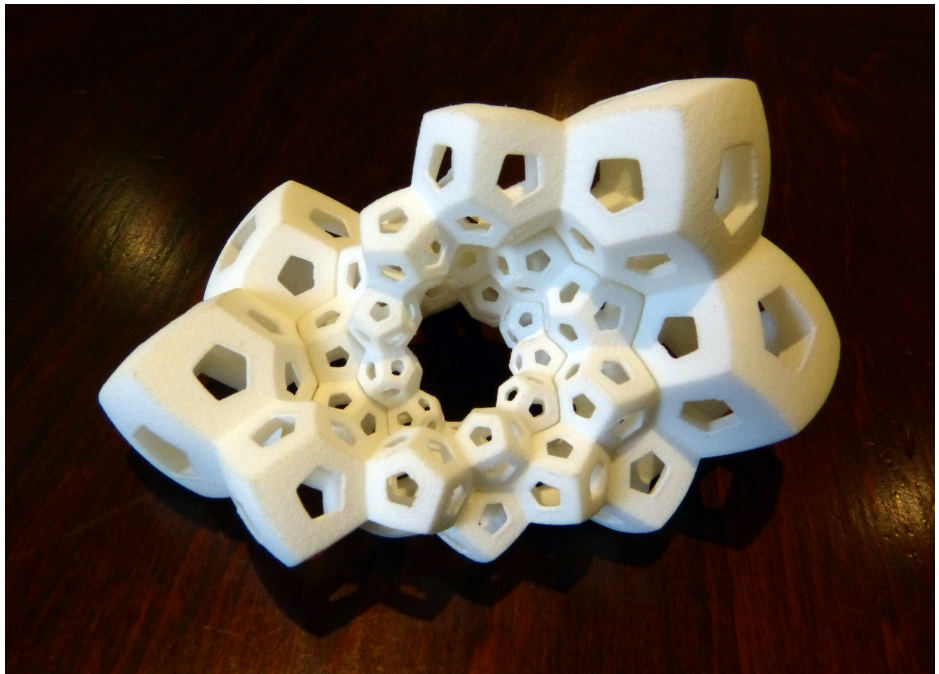


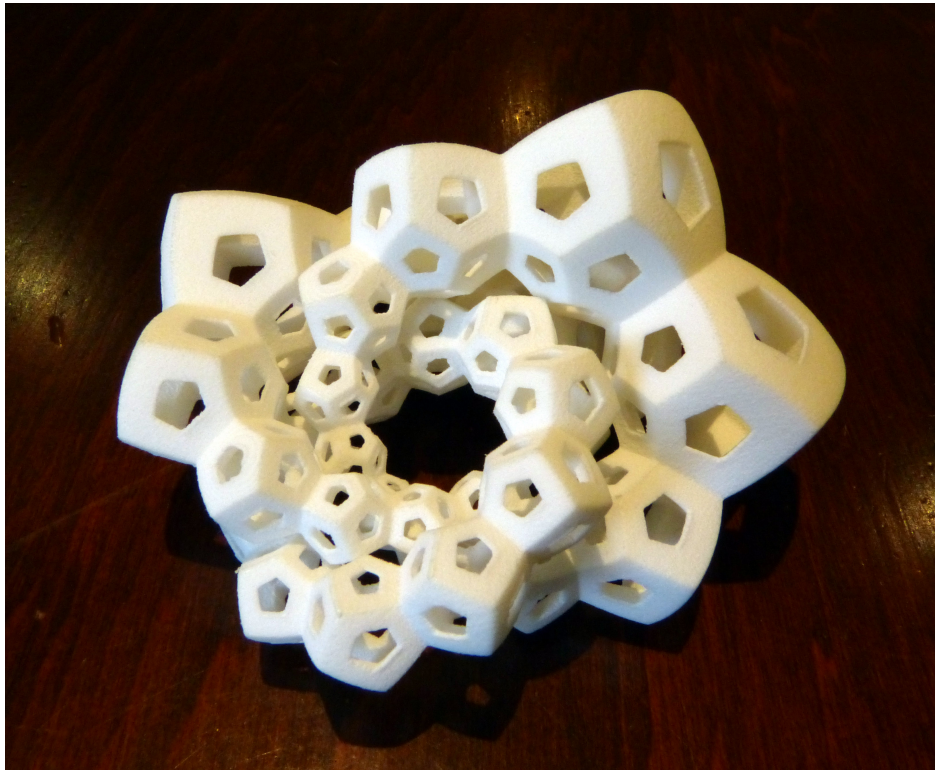


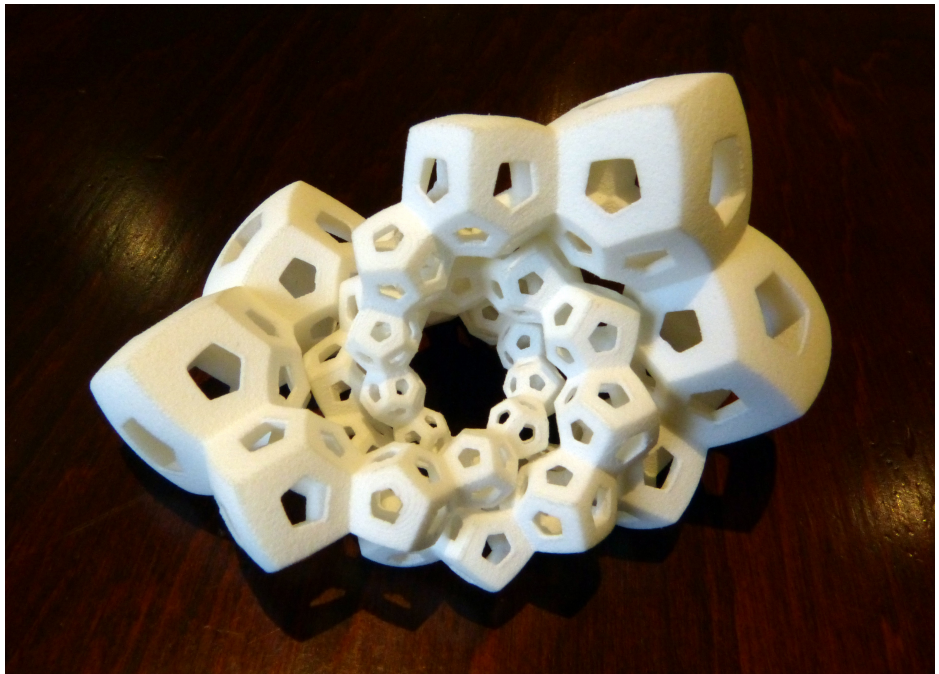
Chains of dodecahedra in the 120-cell











Surfaces in S^3

Some surfaces (and other objects) can be represented in a particularly natural way in S^3.

This is partly to do with circles being very natural in S^3. For example,

$$\left\{ \big(\cos(\alpha), \sin(\alpha), 0, 0\big) \;\middle|\; 0 \leq \alpha < 2\pi \right\} \subset S^3$$

is a circle (i.e. S^1) which is also a geodesic.

The torus, $\mathbb{T}$ can be formed as the product $S^1 \times S^1$, so we can parameterise it as the Clifford Torus:

$$\left\{ \frac{1}{\sqrt{2}}\big(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta)\big) \;\middle|\; 0 \leq \alpha < 2\pi,\ 0 \leq \beta < 2\pi \right\} \subset S^3$$

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Unfortunately, the curves we get by fixing α and varying β, or vice versa, are not geodesics (great circles).

A rotation and reparameterisation of this gives:

$$\left\{ \big(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi)\big) \;\middle|\; 0 \leq \theta < 2\pi, 0 \leq \phi < \pi \right\}$$

Now we do get geodesics by fixing θ and varying ϕ, or vice versa.

Also, the torus goes through the projection point, $(0, 0, 0, 1)$.

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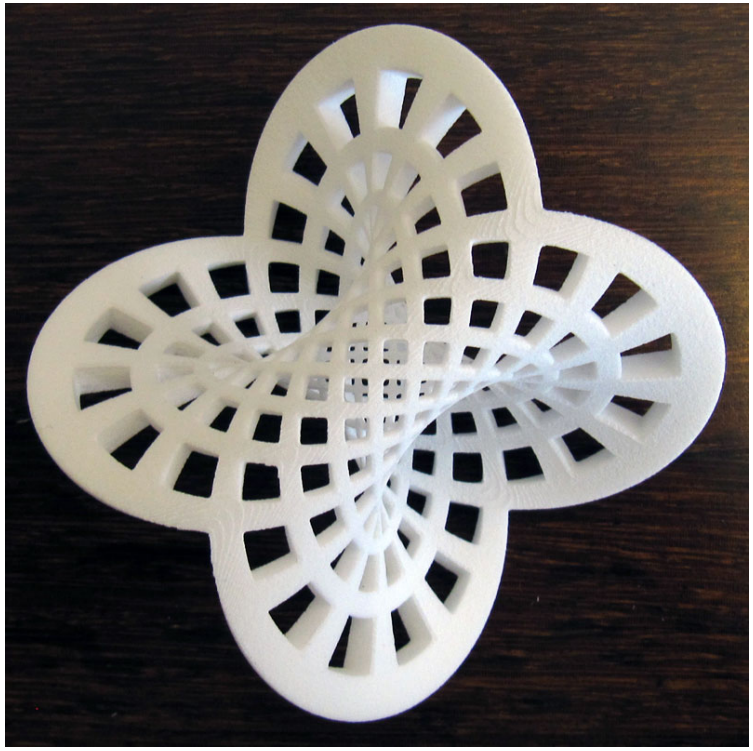

- All arcs shown are geodesics.
- We cut off the surface near the projection point, otherwise we would use an infinite amount of plastic.
- We get a pleasing symmetry: the Clifford torus cuts S^3 into two isometric solid tori, and this property is retained in $\mathbb{R}^3$.

Thickening

As before, thickening in S^3 is better than thickening in $\mathbb{R}^3$, for aesthetic reasons of retaining more symmetry.

We parameterise the normal to the surface in S^3, and then thicken in that direction.

How do we find a normal in S^3?

$$p(\theta, \phi) = \big(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi)\big)$$

$p(\theta, \phi)$ is perpendicular to both $\frac{\partial}{\partial\theta}p(\theta, \phi)$ and $\frac{\partial}{\partial\phi}p(\theta, \phi)$ in $\mathbb{R}^4$, so we define the normal vector $n(\theta, \phi)$ to be one of the unit vectors perpendicular to p, $\frac{\partial}{\partial\theta}p$ and $\frac{\partial}{\partial\phi}p$ in $\mathbb{R}^4$.

We introduce a new parameter, ψ for thickness, and move a distance ψ along the geodesic from $p(\theta, \phi)$ to $n(\theta, \phi)$ to reach

$$r(\theta, \phi, \psi) = \cos(\psi)p(\theta, \phi) + \sin(\psi)n(\theta, \phi).$$

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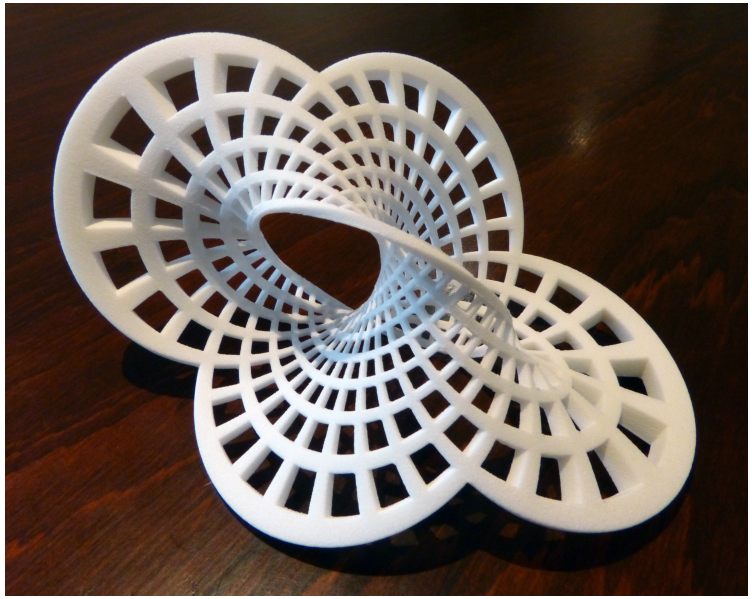
$$r(\theta, \phi, \psi) = \cos(\psi)p(\theta, \phi) + \sin(\psi)n(\theta, \phi).$$

A slight modification of the torus gives a "Round" Möbius strip:

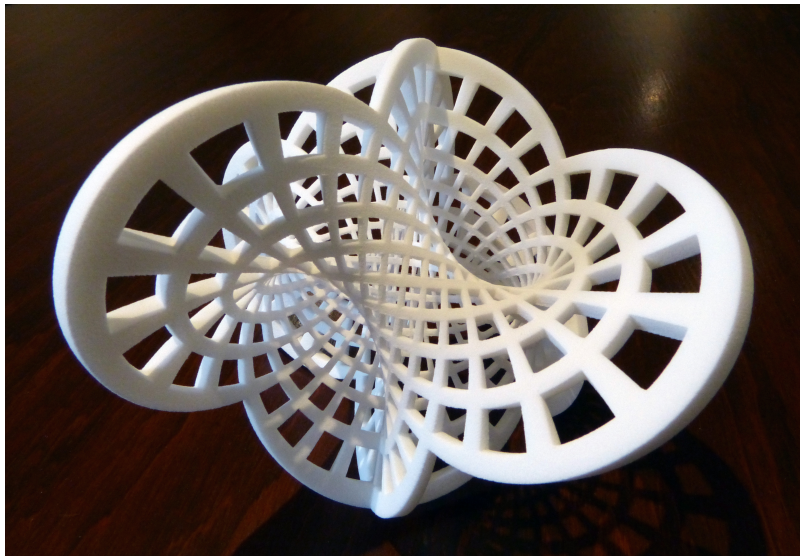
$$\left\{ \left(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta)\cos(2\phi), \sin(\theta)\sin(2\phi)\right) \;\middle|\; 0 \leq \theta < \pi, 0 \leq \phi < \pi \right\}$$

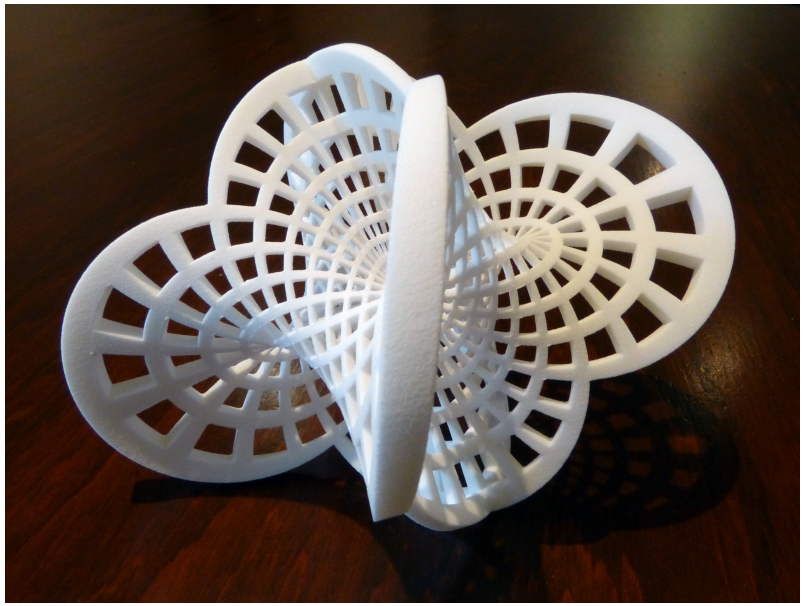


This is a version of the "Sudanese Möbius strip", but projected to $\mathbb{R}^3$ in such a way that the surface goes through infinity.

If we extend the surface, taking $0 \leq \theta < 2\pi$, we get the union of two Möbius strips along their boundaries, which is...

A “Round” Klein bottle



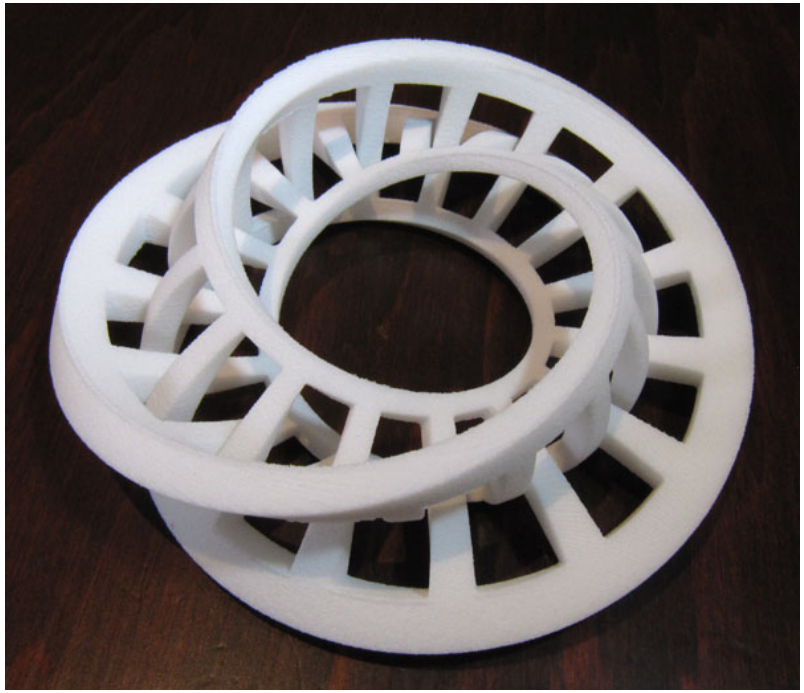


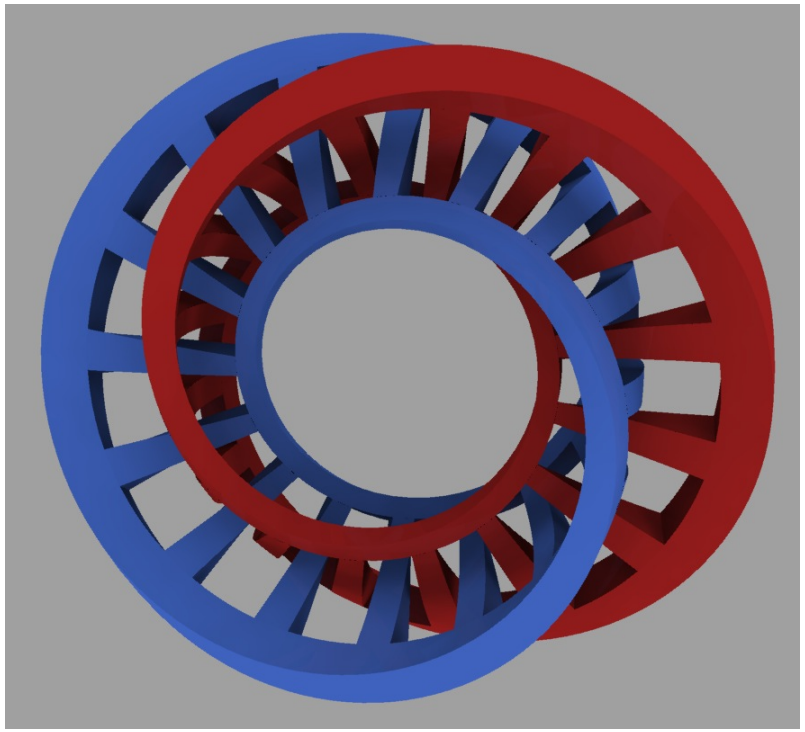
Another variant gives a parameterisation of a torus knot (a knot that can be drawn on a torus). In this case, the trefoil knot:

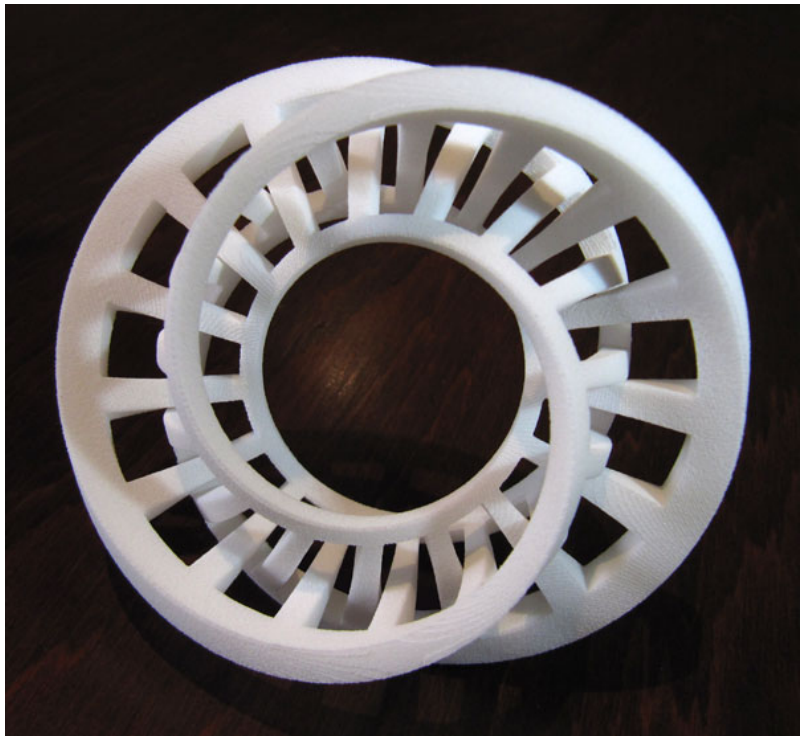
$$\left\{ \left(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta)\cos(3/2\phi), \sin(\theta)\sin(3/2\phi)\right) \;\middle|\; 0 \leq \phi < 4\pi \right\}$$

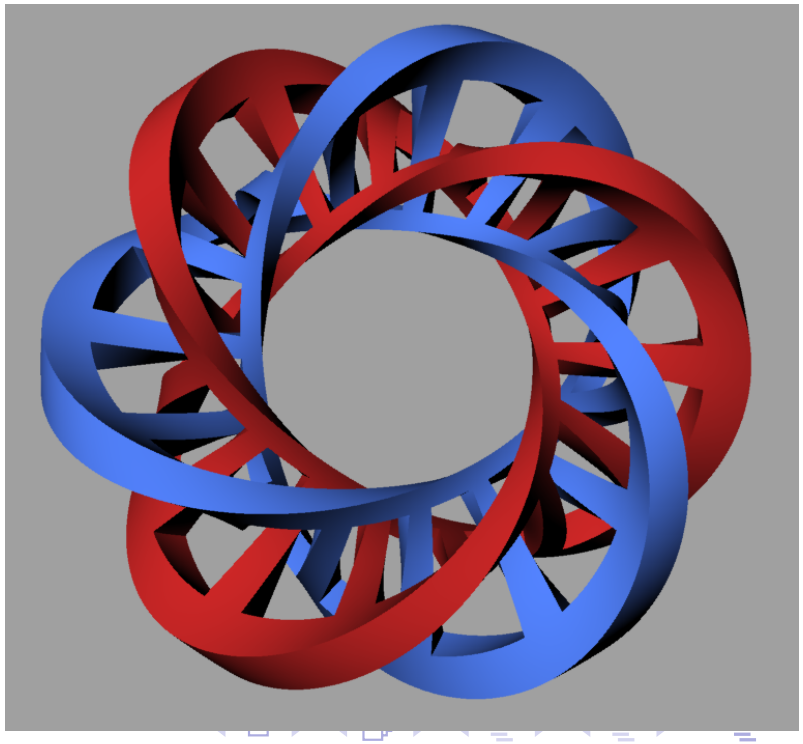
Here θ has a fixed value, greater than 0 and smaller than $\pi/2$. Altering the fraction $3/2$ will produce other torus knots.

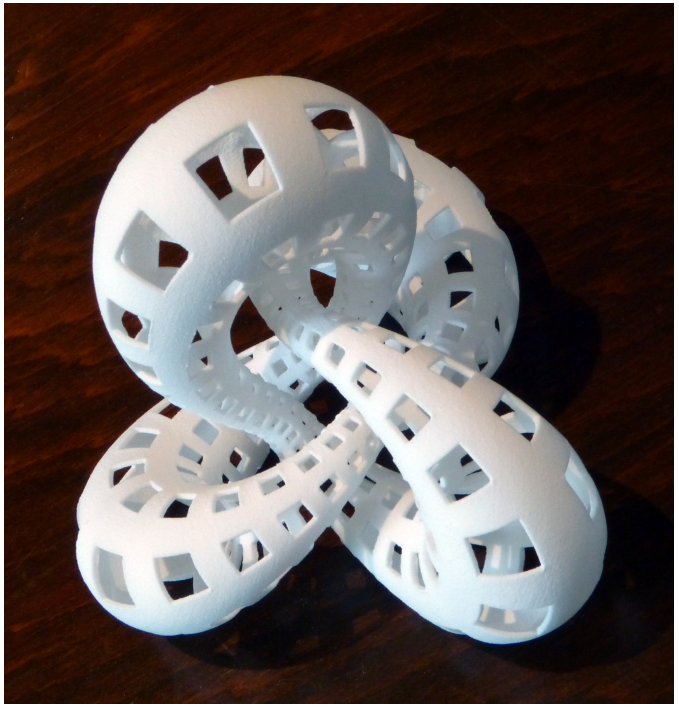
We can use the local coordinates $(\theta, \phi, \psi) : \mathbb{R}^3 \to S^3$ to add small features, using any shape we could define in ordinary 3-dimensional space, in this case cog teeth.

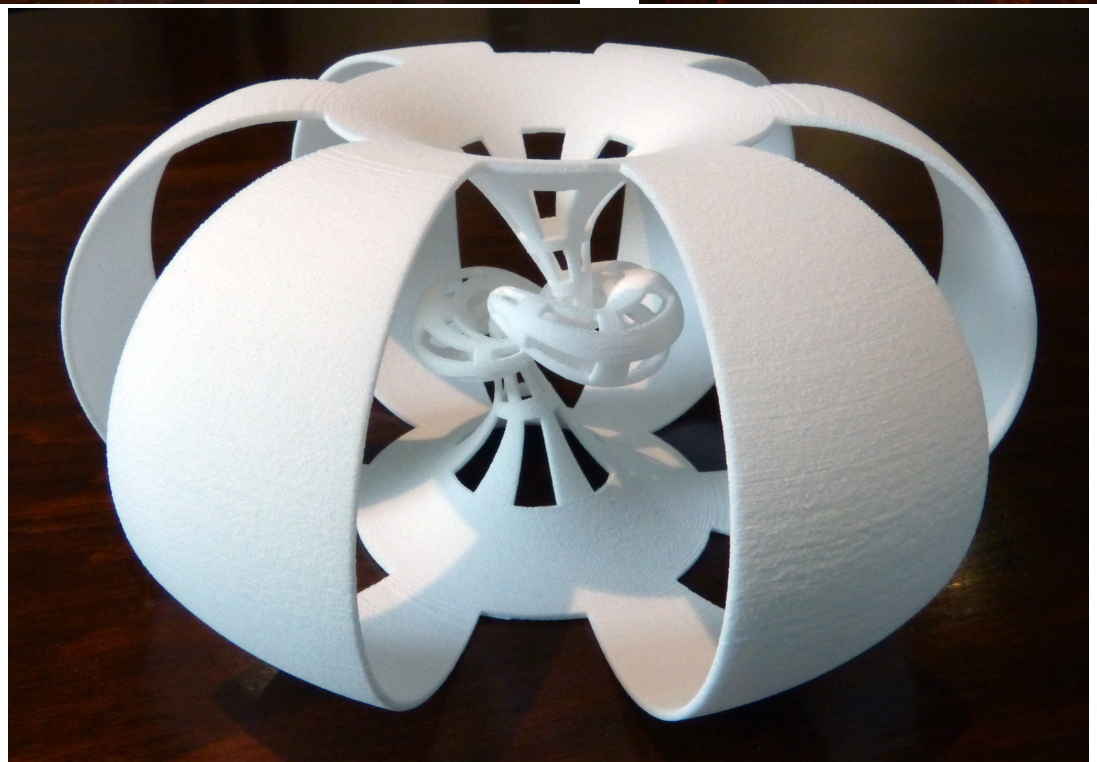


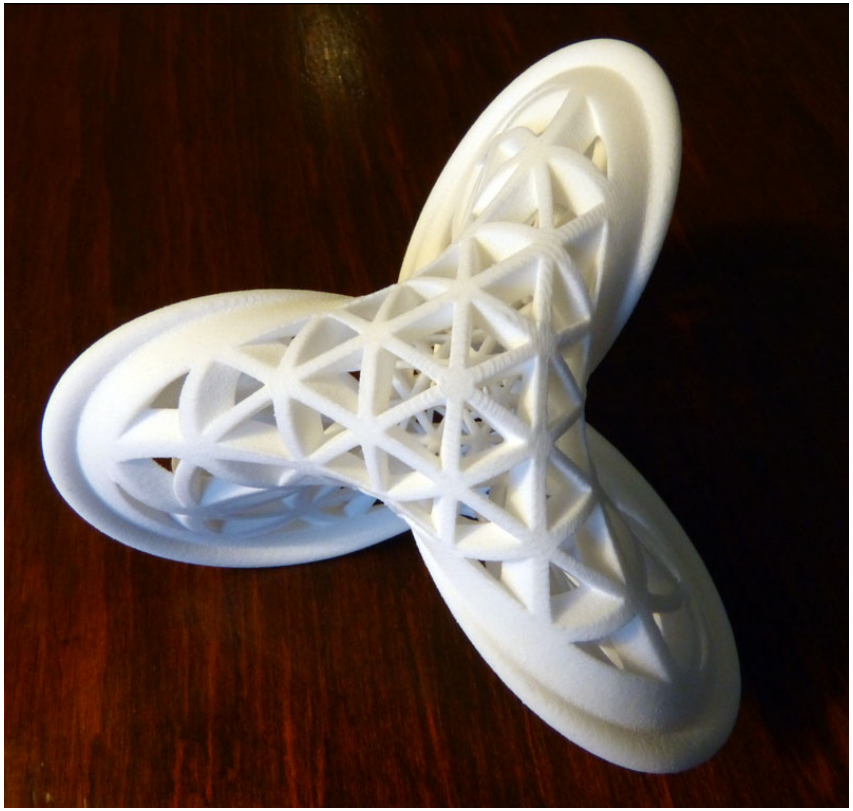


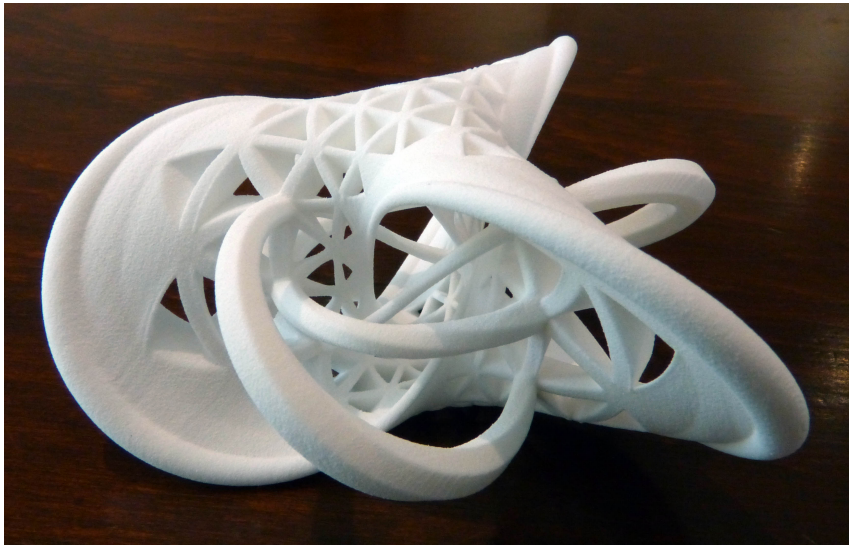


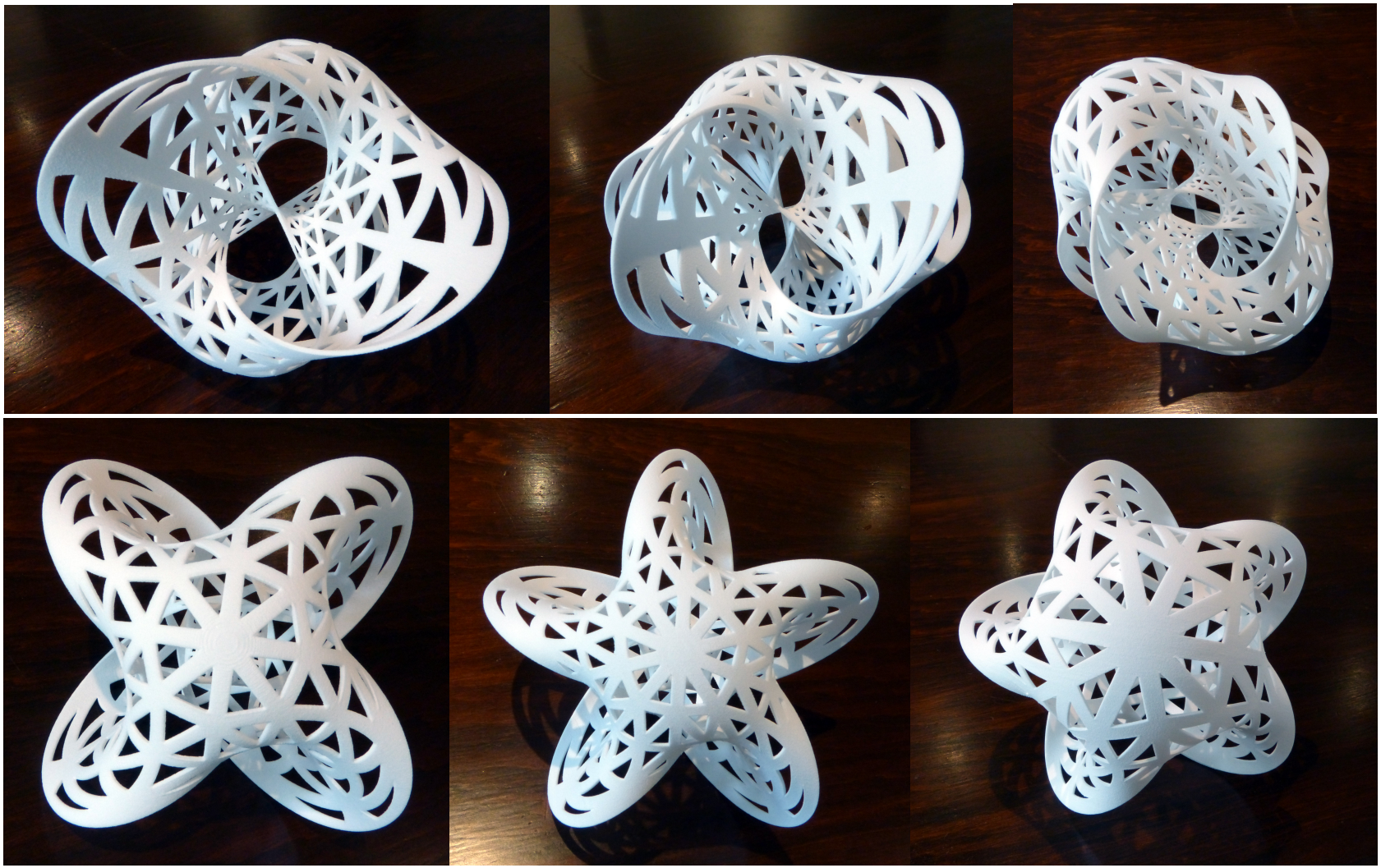












Thanks!

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ms.unimelb.edu.au/~segerman/

youtube.com/user/henryseg