A Sharp Version of Mahler's Inequality for Products of Polynomials *

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Abstract

In this note we give some sharp estimates for norms of polynomials via the products of norms of their linear terms. Different convex norms on the unit disc are considered.

1 Introduction

Let $p_1(z), \ldots, p_m(z)$ be complex polynomials such that their product $p := \prod_{j=1}^{m} p_j$ is of degree n. Then, by a well-known inequality of Mahler [6],

$$\|p_1\|_\infty \ldots \|p_m\|_\infty \le 2^n \|p\|_\infty, \quad n \in \mathbb{N}, \tag{1.1}$$

where $\|f\|_\infty := \max_{|z|=1} |f(z)|$ denotes the uniform norm on the unit circle. (A weaker version of (1.1) appeared earlier in Gel'fond [4].) Choosing $p(z) = z^n + 1$, $m = n$, and $p_j(z)$ $(1 \le j \le n)$ to be the linear factors of $z^n + 1$, one can easily see that the constant in (1.1) cannot be, in general, smaller than 2^{n-1}, i.e., 2^n in (1.1) is sharp up to the factor 2. Based on this observation, it was conjectured by Sarantopoulos [7] that the constant 2^n in (1.1) can be replaced by 2^{n-1}. We shall verify this conjecture in the present note. In fact, this will be accomplished in the context of generalized polynomials, and other norms on the unit circle will be discussed as well. It should be noted that, for $m = o(\sqrt{n})$, the constant in (1.1) was substantially improved by Boyd [3], see also Borwein [1] and Borwein-Erdélyi [2] for some recent developments in this area.

*1991 *Mathematics Subject Classification:* 30C10, 11C08.

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2 Results

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_+$, denote

$$\rho_n(\alpha) := \min_{|a_j|=1} \left\| \prod_{j=1}^n |z - a_j|^{\alpha_j} \right\|_\infty \quad \text{and} \quad d_n(\alpha) := \sum_{j=1}^n \alpha_j. \tag{2.2}$$

Now we can state the next

Theorem 2.1 *For any $\alpha \in \mathbb{R}^n_+$ and any set $\{a_j \in \mathbb{C}, 1 \le j \le n\}$, we have*

$$\prod_{j=1}^n \| |z - a_j|^{\alpha_j} \|_\infty \le \frac{2^{d_n(\alpha)}}{\rho_n(\alpha)} \left\| \prod_{j=1}^n |z - a_j|^{\alpha_j} \right\|_\infty . \tag{2.3}$$

Moreover, the equality in (2.3) *holds if and only if $\{a_j,\ 1 \le j \le n\}$ is a solution of the minimization problem* (2.2).

Functions $p(z) = \prod_{j=1}^n |z - a_j|^{\alpha_j}$ are usually called generalized polynomials of degree $\sum_{j=1}^n \alpha_j = d_n(\alpha)$ ($\alpha \in \mathbb{R}^n_+$) (see [2]). Using Theorem 2.1, we can easily derive the following

Corollary 2.2 *Let $p_1(z), \dots, p_m(z)$ be generalized complex polynomials such that $p(z) := \prod_{j=1}^m p_j(z) = \prod_{j=1}^n |z - a_j|^{\alpha_j}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_+$ and $\{a_j \in \mathbb{C},\ 1 \le j \le n\}$ are arbitrary. Then*

$$\|p_1\|_\infty \dots \|p_m\|_\infty \le \frac{2^{d_n(\alpha)}}{\rho_n(\alpha)} \|p\|_\infty, \tag{2.4}$$

with equality being attained only if p is a solution of the minimization problem (2.2).

It is well-known that when $\alpha^* = (1, \dots, 1)$, the solution of the minimization problem (2.2) is given by $\rho_n(\alpha^*) = 2$, with $z^n + 1$ being the unique (up to a rotation) extremal polynomial for (2.2). Hence we obtain an improvement of Mahler's inequality (1.1) from (2.4).

Corollary 2.3 *Let $p_1, \dots, p_m$ be complex polynomials such that their product $p = \prod_{j=1}^m p_j$ is of degree n. Then*

$$\|p_1\|_\infty \dots \|p_m\|_\infty \le 2^{n-1} \|p\|_\infty, \tag{2.5}$$

and equality in (2.5) *is attained if and only if $p(z) = z^n + \rho$, with $|\rho| = 1$, and $m = n$.*

Inequalities (2.3)-(2.5) provide sharp estimates for the products of generalized polynomials in the uniform norm. Next, we present the L_2-version of Theorem 2.1 for ordinary complex polynomials. Consider the L_q-norm on the unit circle given by

$$\|f\|_q = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^q d\theta\right)^{1/q}, \quad 0 < q < \infty.$$

Theorem 2.4 *For any set* $\{a_j \in \mathbb{C},\ 1 \le j \le n\}$,

$$2^{(1-n)/2}\prod_{j=1}^n \|z-a_j\|_2 \le \left\|\prod_{j=1}^n (z-a_j)\right\|_2 \le 2^{-n/2}\binom{2n}{n}^{1/2}\prod_{j=1}^n \|z-a_j\|_2. \tag{2.6}$$

Moreover, the estimates (2.6) *are sharp; the lower bound is attained for* z^n+1*, the upper bound is attained for* $(z+1)^n$*, and these extremal polynomials are unique up to a rotation.*

Note that inequality (2.3) can be written in an equivalent form

$$\left\|\prod_{j=1}^n |z-a_j|^{\alpha_j}\right\|_\infty \ge \rho_n(\alpha)2^{-d_n(\alpha)}\prod_{j=1}^n (1+|a_j|)^{\alpha_j}. \tag{2.7}$$

Let us also mention an interesting explicit form of (2.6): for $p_n(z) = \prod_{j=1}^n (z-a_j) = \sum_{k=0}^n c_k z^k$, where $c_n = 1$, we have

$$2^{-n+1}\prod_{k=1}^n (1+|a_k|^2) \le \sum_{k=0}^n |c_k|^2 \le 2^{-n}\binom{2n}{n}\prod_{k=1}^n (1+|a_k|^2), \tag{2.8}$$

and both of these bounds are sharp. Although the coefficients c_k of a polynomial p_n can be expressed explicitly via its zeros, the estimates (2.8) do not seem to follow directly from these expressions.

Now we shall address the question of extending (2.7) for the L_q-norms.

Let $\rho_{n,q}(\alpha)$ be defined as follows:

$$\rho_{n,q}(\alpha) := \min_{|a_j|=1}\left\|\prod_{j=1}^n |z-a_j|^{\alpha_j}\right\|_q, \quad \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}_+^n. \tag{2.9}$$

Theorem 2.5 *Let* $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}_+^n$ *and* $0 < q \le \infty$ *be such that* $q\alpha_j \ge 1,\ 1 \le j \le n$. *Then, for any set* $\{a_j \in \mathbb{C},\ 1 \le j \le n\}$*, we have*

$$\left\| \prod_{j=1}^{n} |z - a_j|^{\alpha_j} \right\|_q \geq \rho_{n,q}(\alpha)\ 2^{-d_n(\alpha)} \prod_{j=1}^{n} (1 + |a_j|)^{\alpha_j}. \tag{2.10}$$

The equality in (2.10) *is attained for solutions of the minimization problem* (2.9). *If, in addition,* $1 < q\alpha_j < \infty,\ 1 \leq j \leq n$, *then this equality holds only for solutions of* (2.9).

Note that the estimate (2.10) is more general than (2.3), but on the other hand the additional claim of Theorem 2.1, that equality in (2.3) can be attained *only* when the roots of generalized polynomials are on the unit circle, does not follow from Theorem 2.5. Thus, Theorems 2.1 and 2.5 complement each other. In addition, their proofs are based on different methods. Theorem 2.1 (as well as Theorem 2.4) will follow by variational arguments, while the proof of Theorem 2.5 will be based on applications of symmetry and convexity.

The above results indicate that the sharp constants, appearing in inequalities for the norms of products of polynomials, depend on the solution of the extremal problems (2.2) and (2.9). For ordinary polynomials, the explicit solution of such extremal problem can be given for a wide class of norms. Let

$$\rho_n := \min_{|a_j|=1} \left\| \prod_{j=1}^{n} (z - a_j) \right\|, \tag{2.11}$$

where $\|\cdot\|$ is an arbitrary convex norm in the space of polynomials on the unit circle. We shall say that $\|\cdot\|$ is *rotation invariant* if for any fixed $\varphi_0 \in \mathbb{R}$ and any polynomial p

$$\|p(e^{i(\varphi+\varphi_0)})\| = \|p(e^{i\varphi})\| \quad \text{and} \quad \|p\| = \||p|\|.$$

Our next result extends Theorem 2.5 for rotation invariant norms in the case of ordinary polynomials.

Theorem 2.6 *For any rotation invariant convex norm* $\|\cdot\|$ *on the unit circle and any set* $\{a_j \in \mathbb{C},\ 1 \leq j \leq n\}$, *we have* $\rho_n = \|z^n + 1\|$ *and*

$$\left\| \prod_{j=1}^{n} (z - a_j) \right\| \geq 2^{-n}\, \|z^n + 1\| \prod_{j=1}^{n} (1 + |a_j|). \tag{2.12}$$

The equality in (2.12) *is attained for solutions of the minimization problem* (2.11). *If, in addition,* $\|\cdot\|$ *is strictly convex, then this equality holds only for solutions of* (2.11).

3 Proofs

3.1 *Proof of Theorem 2.1.*

First, we note that (2.3) is trivial for $n = 1$ and assume that $n \geq 2$. Let us consider the following minimization problem:

$$\Gamma_n(\alpha) := \inf_{a_j \in \mathbb{C}} \frac{\left\| \prod_{j=1}^{n} |z - a_j|^{\alpha_j} \right\|_\infty}{\prod_{j=1}^{n} (1 + |a_j|)^{\alpha_j}}, \quad \alpha \in \mathbb{R}_+^n. \tag{3.13}$$

Since the functional minimized in (3.13) is invariant with respect to the transformation $a_j \to 1/\overline{a}_j$, it is clear that the inf in (3.13) is attained for $|a_j| \leq 1,\ 1 \leq j \leq n$. We are going to verify now that this inf can be attained *only* when $|a_j| = 1,\ 1 \leq j \leq n$. This will immediately imply the statement of Theorem 2.1. We shall show that the inf in (3.13) can be attained only for a_j's on the unit circle, by using variational arguments based on the following well-known formula for the directional derivative of the L_∞-norm (see [5]).

$$\lim_{t \to 0^+} \frac{\|f + tg\|_\infty - \|f\|_\infty}{t} = \max_{z \in E(f)} \operatorname{Re}\, g \operatorname{sgn} f, \tag{3.14}$$

where $E(f) = \{|z| = 1 : |f(z)| = \|f\|_\infty\}$, $\operatorname{sgn} f = \overline{f}/|f|$.

Since $\left\| \prod_{j=1}^{n} |z - a_j|^{\alpha_j} \right\|_\infty \leq (1+|a_1|)^{\alpha_1} \left\| \prod_{j=2}^{n} |z - a_j|^{\alpha_j} \right\|_\infty$, it easily follows that we do not lose generality by assuming that the inf in (3.13) is attained for $0 < |a_j| \leq 1,\ 1 \leq j \leq n$. Set

$$p(z) := \prod_{j=1}^{n} |z - a_j|^{\alpha_j},\ p_t(z) := \prod_{j=1}^{n} |z - a_j + tw_j|^{\alpha_j}$$

$$S := \prod_{j=1}^{n} (1 + |a_j|)^{\alpha_j},\ S_t := \prod_{j=1}^{n} (1 + |a_j - tw_j|)^{\alpha_j}$$

where $w_j \in \mathbb{C},\ 1 \leq j \leq n$, and $t > 0$ are arbitrary. Evidently,

$$\frac{1}{t}\left(S\|p_t\|_\infty - S_t\|p\|_\infty\right) \geq 0. \tag{3.15}$$

Furthermore, it can be easily shown that

$$|z-a_j+tw_j|^{\alpha_j}=|z-a_j|^{\alpha_j}+t\alpha_j|z-a_j|^{\alpha_j-2}\ \mathrm{Re}\ w_j(\overline{z-a_j})+O(t^2),\ 1\le j\le n,$$

where $O(t^2)$ above is uniform in z from compact subsets of $\mathbb{C}\backslash\{a_j\}_{j=1}^n$. Thus,

$$p_t(z)=p(z)+tp(z)\sum_{j=1}^n\alpha_j\ \mathrm{Re}\left(\frac{w_j}{z-a_j}\right)+O(t^2). \tag{3.16}$$

Similarly, for $a_j\neq 0$

$$(1+|a_j-tw_j|)^{\alpha_j}=(1+|a_j|)^{\alpha_j}-\alpha_j(1+|a_j|)^{\alpha_j-1}t\ \mathrm{Re}\ w_j\ \mathrm{sgn}\ a_j+O(t^2)$$

and therefore

$$S_t=S-tS\sum_{j=1}^n\frac{\alpha_j}{1+|a_j|}\ \mathrm{Re}\ w_j\ \mathrm{sgn}\ a_j+O(t^2). \tag{3.17}$$

Using (3.14)-(3.17) we obtain

$$0\le\lim_{t\to0^+}\frac{1}{t}\left(\|p_t\|_\infty-\|p\|_\infty\right)+\|p\|_\infty\sum_{j=1}^n\frac{\alpha_j}{1+|a_j|}\ \mathrm{Re}\ w_j\ \mathrm{sgn}\ a_j$$

$$=\|p\|_\infty\max_{z\in E(p)}\ \mathrm{Re}\sum_{j=1}^n\frac{\alpha_jw_j}{z-a_j}+\|p\|_\infty\sum_{j=1}^n\frac{\alpha_j}{1+|a_j|}\mathrm{Re}\ w_j\ \mathrm{sgn}\ a_j.$$

This means that for every $w_j\in\mathbb{C},\ 1\le j\le n$, there exists a $z\in E(p)$ so that

$$\sum_{j=1}^n\alpha_j\ \mathrm{Re}\left(w_j\left(\frac{1}{z-a_j}+\frac{\mathrm{sgn}\ a_j}{1+|a_j|}\right)\right)\ge 0. \tag{3.18}$$

It can be easily seen that

$$\frac{1}{z-a_j}+\frac{\mathrm{sgn}\ a_j}{1+|a_j|}=\frac{(1-|a_j|)(|a_j|+\mathrm{Re}\ a_j\overline{z})+i\ (1+|a_j|)\mathrm{Im}\ a_j\overline{z}}{a_j(1+|a_j|)|z-a_j|^2}$$

This and (3.18) yield that for every $w_j\in\mathbb{C},\ 1\le j\le n$, there exists a point $z\in E(p)$ so that

$$\sum_{j=1}^n\alpha_j\ \mathrm{Re}\left(\frac{w_j}{|z-a_j|^2}\left((1-|a_j|)(|a_j|+\ \mathrm{Re}\ a_j\overline{z})+i\ (1+|a_j|)\mathrm{Im}\ a_j\overline{z}\right)\right)\ge 0. \tag{3.19}$$

Assume now that $|a_j| \neq 1$ for $1 \le j \le k$ $(1 \le k \le n)$ and $|a_j| = 1$ if $k < j \le n$. Setting in (3.19) $w_j = 1/(|a_j| - 1)$ for $1 \le j \le k$, and $w_j = 0$ for $k < j \le n$, we obtain that for some $z_0 \in E(p)$

$$\sum_{j=1}^{k} \frac{\alpha_j}{|z_0 - a_j|^2} \left(|a_j| + \text{Re } a_j \overline{z}_0\right) \le 0. \tag{3.20}$$

On the other hand $|z_0| = 1$ and, therefore, each term in the sum (3.20) is nonnegative, i.e., $z_0 = -a_j/|a_j|,\ 1 \le j \le k$. Thus, we may assume that $z_0 = -1$ and $a_j = x_j > 0$ for $1 \le j \le k$. Using, in addition, that

$$\text{Re } \frac{1+z}{z - x_j} = \frac{(1 - x_j)(1 + \text{Re } z)}{|z - x_j|^2},$$

we obtain from (3.18) that for every $b_j \in \mathbb{R},\ 1 \le j \le k$, and $w_j \in \mathbb{C},\ k < j \le n$, there exists a point $z \in E(p)$ so that

$$\sum_{j=1}^{k} \alpha_j b_j \frac{1 + \text{ Re } z}{|z - x_j|^2} + \sum_{j=k+1}^{n} \alpha_j \text{ Re} \left(w_j \frac{a_j + z}{a_j - z} \right) \ge 0.$$

Choose $b_j = -N,\ 1 \le j \le k$. Then, it follows that for some $z_N \in E(p)$

$$\sum_{j=k+1}^{n} \alpha_j \text{Re} \left(w_j \frac{a_j + z_N}{a_j - z_N} \right) \ge N \sum_{j=1}^{k} \alpha_j \frac{1 + \text{ Re } z_N}{|z_N - x_j|^2}. \tag{3.21}$$

But $1 + \text{Re } z_N \ge 0$ and $z_N \to z^* \in E(p)$ as $N \to \infty$ (for a proper subsequence). Thus we obtain from (3.21) that $z^* = -1$ and for every $w_j \in \mathbb{C},\ k+1 \le j \le n$

$$\sum_{j=k+1}^{n} \alpha_j \text{ Re} \left(w_j \frac{a_j - 1}{a_j + 1} \right) \ge 0.$$

Thus $a_j = 1,\ k+1 \le j \le n$, i.e., $p(z) = |z-1|^{\beta} \prod_{j=1}^{k} |z - x_j|^{\alpha_j}$ with some $\beta > 0$ and $x_j > 0,\ 1 \le j \le k$. Hence $\Gamma_n(\alpha)$, defined by (3.13), equals 1, which is an evident contradiction if $n \ge 2$. This completes the proof of Theorem 2.1. □

3.2 *Proof of Theorem 2.4.*

Set

$$M_n := \sup_{a_j \in \mathbb{C}} \frac{\left\| \prod_{j=1}^{n} (z-a_j) \right\|_2}{\prod_{j=1}^{n} \|z-a_j\|_2}; \quad m_n := \inf_{a_j \in \mathbb{C}} \frac{\left\| \prod_{j=1}^{n} (z-a_j) \right\|_2}{\prod_{j=1}^{n} \|z-a_j\|_2}.$$

Again, it is evident that sup and inf in the above expressions are attained.

Proposition 3.1 *If the set of points* $\{a_j,\ 1 \le j \le n\} \subset \mathbb{C}$ *is extremal for* M_n *or* m_n *and* $|a_1| \neq 1$*, then for* $p_n(z) = \prod_{j=1}^{n}(z-a_j)$

$$\|p_n\|_2 = \left\| \frac{p_n}{z-a_1} \right\|_2 \|z-a_1\|_2. \tag{3.22}$$

Proof. (3.22) is trivial for $a_1 = 0$, so we may assume that $a_1 \neq 0$. Consider the functional

$$\varphi(t) = \frac{\int_0^{2\pi} \left| p_n(z) - \frac{t a_1 p_n(z)}{z-a_1} \right|^2 d\theta}{1+(1+t)^2|a_1|^2}, \quad z = e^{i\theta}.$$

The extremality of the set $\{a_j,\ 1 \le j \le n\}$ yields that $\varphi'(0) = 0$. Therefore, by differentiating $\varphi(t)$, we obtain

$$\begin{aligned}(1+|a_1|^2) \int_0^{2\pi} |p_n(z)|^2 \ \mathrm{Re}\ \frac{a_1}{z-a_1} d\theta + |a_1|^2 \int_0^{2\pi} |p_n(z)|^2 d\theta \\ = \int_0^{2\pi} |p_n(z)|^2 \left(\mathrm{Re}\ \frac{a_1(1+|a_1|^2)}{z-a_1} + |a_1|^2 \right) d\theta = 0.\end{aligned} \tag{3.23}$$

Moreover, for $|z| = 1$

$$\mathrm{Re}\ \frac{a_1(1+|a_1|^2)}{z-a_1} + |a_1|^2 = \mathrm{Re}\ \frac{a_1 + z|a_1|^2}{z-a_1} = \frac{(1-|a_1|^2)\ \mathrm{Re}\ \overline{z}a_1}{|z-a_1|^2}.$$

Since $|a_1| \neq 1$, we obtain by substituting the above expression into (3.23) that

$$\int_0^{2\pi} \left| \frac{p_n(z)}{z-a_1} \right|^2 \ \mathrm{Re}\ \overline{z}a_1 \ d\theta = 0, \quad z = e^{i\theta}.$$

Finally, using this relation implies

$$\|p_n\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |p_n(z)|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{p_n(z)}{z-a_1}\right|^2 (z-a_1)(\overline{z}-\overline{a}_1)\ d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{p_n(z)}{z-a_1}\right|^2 (1+|a_1|^2 - 2\mathrm{Re}\ \overline{z}a_1) d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{p_n(z)}{z-a_1}\right|^2 (1+|a_1|^2)\ d\theta = \|z-a_1\|_2^2 \left\|\frac{p_n}{z-a_1}\right\|_2^2. \qquad \square$$

Corollary 3.2 *If the set of points $\{a_j,\ 1 \le j \le n\} \subset \mathbb{C}$ is extremal for M_n (or m_n) and $|a_1| \ne 1$, then $M_n = M_{n-1}$ (respectively, $m_n = m_{n-1}$).*

Proof. Clearly, $M_n \ge M_{n-1}$ and $m_n \le m_{n-1}$ for every $n \ge 2$. On the other hand, if $|a_1| \ne 1$ then (3.22) yields that $M_n \le M_{n-1}$ and $m_n \ge m_{n-1}$. Thus $M_n = M_{n-1}$ and $m_n = m_{n-1}$. $\square$

The next two statements are straightforward.

Proposition 3.3 *For every $n \in \mathbb{N}$*

$$M_n \ge 2^{-n/2}\|(z+1)^n\|_2 = \left(2^{-n}\sum_{k=0}^{n}\binom{n}{k}^2\right)^{1/2} = \left(2^{-n}\binom{2n}{n}\right)^{1/2},$$
$$m_n \le 2^{-n/2}\|z^n+1\|_2 = 2^{(1-n)/2}.$$

Proposition 3.4 *If the set of points $\{a_j,\ 1 \le j \le n\} \subset \{|z| = 1\}$ is extremal for M_n or m_n, then*

$$M_n = 2^{-n/2}\left\|\prod_{j=1}^{n}(z-a_j)\right\|_2 \le \left(2^{-n}\sum_{k=0}^{n}\binom{n}{k}^2\right)^{1/2} = \left(2^{-n}\binom{2n}{n}\right)^{1/2}.$$

Respectively,

$$m_n = 2^{-n/2}\left\|\prod_{j=1}^{n}(z-a_j)\right\|_2 \ge 2^{-n/2}\|z^n+1\|_2 = 2^{(1-n)/2}.$$

Let us verify now that

$$M_n = \left(2^{-n}\binom{2n}{n}\right)^{1/2} \quad \text{and} \quad m_n = 2^{(1-n)/2}.$$

Assume that $M_n = \ldots = M_m > M_{m-1}$, where $2 \leq m \leq n$. Then, by Corollary 3.2, $|a_j| = 1,\ 1 \leq j \leq m$, where a_j's are extremal for M_m. Thus, by Propositions 3.3 and 3.4

$$2^{-n}\binom{2n}{n} \leq M_n^2 = M_m^2 \leq 2^{-m}\binom{2m}{m}.$$

It is easy to show that this inequality is possible only if $m = n$, i.e., $M_n = \left(2^{-n}\binom{2n}{n}\right)^{1/2}$. Similarly, if $m_n = \ldots = m_\ell < m_{\ell-1}$, we obtain by Propositions 3.3 and 3.4

$$2^{(1-n)/2} \geq m_n = m_\ell \geq 2^{(1-\ell)/2},$$

yielding that $n = \ell$ and $m_n = 2^{(1-n)/2}$. The above arguments show that $M_n > M_{n-1}$ and $m_n < m_{n-1}$ for every $n \geq 2$. Thus, by Corollary 3.2, the extremal sets for M_n and m_n must belong to the unit circle. Therefore, if $p_n(z) = \prod_{j=1}^{n} (z - a_j)$ is extremal for m_n, then we must have $|a_j| = 1,\ 1 \leq j \leq n$, and $\|p_n\|_2 = \sqrt{2}$. It is well-known that only $p_n(z) = z^n + \rho\ (\rho \in \mathbb{C},\ |\rho| = 1)$ can satisfy the above properties. Similarly, if $g_n(z) = \prod_{j=1}^{n} (z - b_j)$ is extremal for M_n, then $|b_j| = 1,\ 1 \leq j \leq n$, and $\|g_n\|_2 = \sqrt{\binom{2n}{n}}$. Again, it is easy to see that $g_n(z) = (z + \rho)^n\ (\rho \in \mathbb{C},\ |\rho| = 1)$.

The proof of Theorem 2.4 is now complete. □

3.3 *Proof of Theorem 2.5.*

Consider the functional

$$\psi(a_1, \ldots, a_n) := \frac{\left\| \prod_{j=1}^{n} |z - a_j|^{\alpha_j} \right\|_q}{\prod_{j=1}^{n} (1 + |a_j|)^{\alpha_j}}$$

and the corresponding extremal problem

$$\gamma_n(\alpha) := \inf_{a_j \in \mathbb{C}} \psi(a_1, \ldots, a_n). \tag{3.24}$$

Evidently, it suffices to show that (3.24) possesses a solution $\{a_j,\ 1 \leq j \leq n\} \subset \mathbb{C}$ such that $|a_j| = 1$, $1 \leq j \leq n$. First, observe that for every $a_j \neq 0,\ 1 \leq j \leq n$,

$$\frac{|z-a_j|^{\alpha_j}}{(1+|a_j|)^{\alpha_j}}=\frac{\left|z-\dfrac{1}{\overline{a}_j}\right|^{\alpha_j}}{\left(1+\dfrac{1}{|\overline{a}_j|}\right)^{\alpha_j}},$$

and therefore

$$\psi(a_1,\dots,a_j,\dots,a_n)=\psi\left(a_1,\dots,\frac{1}{\overline{a}_j},\dots,a_n\right). \tag{3.25}$$

(Note that this relation easily implies the existence of a solution of (3.24).)

Set

$$p(z):=\prod_{j=1}^{n}|z-a_j|^{\alpha_j},\ p_k(z):=p(z)/|z-a_k|^{\alpha_k},$$

$$S:=\prod_{j=1}^{n}(1+|a_j|)^{\alpha_j},\ S_k:=S/(1+|a_k|)^{\alpha_k}\quad(1\le k\le n).$$

Then, using that $\alpha_j q\ge 1$ $(1\le j\le n)$, we obtain for any $a_k\ne 0$ and $0\le t\le 1$:

$$\begin{aligned}
&\left\|\left|z-\left(ta_k+(1-t)\,\frac{1}{\overline{a}_k}\right)\right|^{\alpha_k}p_k\right\|_q^{1/\alpha_k}=\left\|\left|z-\left(ta_k+(1-t)\,\frac{1}{\overline{a}_k}\right)\right|p_k^{1/\alpha_k}\right\|_{\alpha_k q}\\
&\le t\left\||z-a_k|\,p_k^{1/\alpha_k}\right\|_{\alpha_k q}+(1-t)\left\|\left|z-\frac{1}{\overline{a}_k}\right|p_k^{1/\alpha_k}\right\|_{\alpha_k q}=t\|p\|_q^{1/\alpha_k}+(1-t)\left\|\left|z-\frac{1}{\overline{a}_k}\right|^{\alpha_k}p_k\right\|_q^{1/\alpha_k}\\
&=S_k^{1/\alpha_k}\left\{t\,(1+|a_k|)\,(\psi(a_1,\dots,a_n))^{1/\alpha_k}+(1-t)\left(1+\frac{1}{|\overline{a}_k|}\right)\left(\psi\left(a_1,\dots,\frac{1}{\overline{a}_k},\dots,a_n\right)\right)^{1/\alpha_k}\right\}.
\end{aligned} \tag{3.26}$$

Using (3.25) and the relation

$$t(1+|a_k|)+(1-t)\left(1+\frac{1}{|\overline{a}_k|}\right)=1+\left|ta_k+\frac{1-t}{\overline{a}_k}\right|,$$

we obtain from (3.26)

$$\psi\left(a_1,\dots,ta_k+\frac{1-t}{\overline{a}_k},\dots,a_n\right)\le\psi(a_1,\dots,a_n),\quad 1\le k\le n, \tag{3.27}$$

if $a_k \neq 0$ and $0 \leq t \leq 1$. Furthermore, the inequality $\|p\|_q \leq (1+|a_k|)^{\alpha_k}\|p_k\|_q$ yields that (3.24) possesses a solution $\{a_j,\ 1 \leq j \leq n\}$ such that $a_j \neq 0,\ 1 \leq j \leq n$. Moreover, (3.27) implies that this solution can be chosen so that $|a_j| = 1,\ 1 \leq j \leq n$.

Note that if $1 < \alpha_j q < \infty$, then $\|\cdot\|_{\alpha_j q}$ is *strictly* convex, $1 \leq j \leq n$. Therefore, it follows from (3.26) that the equality in (3.27) can hold only for $|a_k| = 1,\ 1 \leq k \leq n$, in this case. □

Remark 3.5 *It can be seen from the above proof that Theorem 2.5 could be generalized by replacing* $\prod_{j=1}^{n}(1+|a_j|)^{\alpha_j}$ *by a function* $g(|a_1|,\dots,|a_n|)$ *such that* $g_j(t) := g^{1/\alpha_j}(|a_1|,\dots,|a_{j-1}|,t,$ $|a_{j+1}|,\dots,|a_n|)$ *is concave and satisfies the symmetry property* $tg_j\left(\frac{1}{t}\right) = g_j(t)$ $(1 \leq j \leq n),\ t > 0$.

3.4 *Proof of Theorem 2.6.*

We need to verify first that $\rho_n = \|z^n+1\|$. The rest of the proof will then follow by the same arguments as in the proof of Theorem 2.5. Consider the following best approximation problem

$$\tilde{\rho}_n := \min_{c_j \in \mathbb{C}} \left\| z^n + 1 - \sum_{j=1}^{n-1} c_j z^j \right\|. \tag{3.28}$$

Clearly, $\tilde{\rho}_n \leq \rho_n$. Assume first that $\|\cdot\|$ is strictly convex. Then (3.28) possesses a unique solution $q(z) := \sum_{j=1}^{n-1} c_j^* z^j$. This and the rotation invariance of the norm yield that $q(e^{2\pi i/n}z) = q(z)$ for every $|z| = 1$, i.e., $q \equiv 0$. This verifies that $\tilde{\rho}_n = \rho_n = \|z^n+1\|$ when $\|\cdot\|$ is strictly convex. In the general case, set

$$\|p\|_\epsilon = \|p\| + \epsilon\|p\|_2 \quad (\epsilon > 0).$$

Evidently, $\|\cdot\|_\epsilon$ is rotation invariant and strictly convex, i.e., the above argument is applicable to $\|\cdot\|_\epsilon$. Letting $\epsilon \to 0$, we obtain that $\tilde{\rho}_n = \rho_n = \|z^n+1\|$. □

Acknowledgement. The authors are grateful to Prof. Y. Sarantopoulos for many helpful discussions concerning the content of this paper.

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