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Representations in Dolbeault Cohomology

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Introduction

A real form G_0 of a complex semisimple Lie group G acts on a complex flag variety for G with a finite number of orbits. Irreducible representations of G_0 are constructed in terms of the geometry of these orbits. Of particular interest are the open orbits. The purpose of these lecture notes is to describe the structure of the open orbits and some tools for studying the associated representations. The lectures were aimed at graduate students who are comfortable with basic Lie theory and complex manifolds.

Suppose D is an open G_0–orbit in a complex flag variety. Then the representations associated to D occur in the Dolbeault cohomology spaces $H^s(D, \mathcal{L}_\chi)$ where $\mathcal{L}_\chi \to D$ are homogeneous line bundles and s is the dimension of a maximal compact complex subvariety of D. Under certain conditions (e.g., a negativity condition on $\mathcal{L}_\chi$) these representations are irreducible and infinitesimally equivalent to a unitary representation. Except for a small number of situations it is not known how to describe this unitary structure. We consider it a fundamental problem to construct a Hilbert space inside cohomology which is defined by a G_0–invariant inner product. As elliptic coadjoint orbits can be identified with open orbits in complex flag varieties, such a construction may be considered a quantization procedure for elliptic coadjoint orbits.

The first several lectures are on the structure of the G_0–orbits, especially the open orbits. As the examples in Lecture 3 show these orbits are interesting (generally noncompact) complex manifolds with indefinite invariant metrics. In the very special case where G_0 is compact the cohomology spaces are computed by the Bott–Borel–Weil Theorem. Here the unitary structure is given by the Hodge Theorem; each cohomology class is represented by a harmonic form and integration gives a G_0–invariant form on the space of harmonic forms. Lecture 5 discusses what goes wrong in general and how one can attempt to get around the difficulties. The remainder of the lectures focus on techniques for constructing intertwining maps between representations in cohomology and other better understood spaces. The

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intertwining maps serve several purposes. They provide realizations of the cohomology representations in function spaces which may be easier to understand than the cohomology spaces. The intertwining operator $\mathcal{S}$ of Lecture 6, Section 4, gives an integral formula for harmonic forms on D. This formula is explicit enough to study the growth, hence sqare integrability, of these harmonic forms. This is an important step in the unitarization/quantization procedure suggested in Lecture 5. A by–product is a Hodge type theorem on the existence of harmonic forms representing cohomology for the noncompact indefinite hermitian manifolds D.

LECTURE 1
Complex Flag Varieties and Orbits Under a Real Form

1.1. Complex flag varieties.

We let G be a connected complex semisimple Lie group. The Lie algebra of G will be denoted by $\mathfrak{g} = Lie(G)$. Recall that a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra consisting of semisimple elements. For a Cartan subalgebra we will generally denote the roots of $\mathfrak{h}$ in $\mathfrak{g}$ by $\Delta(\mathfrak{g},\mathfrak{h}), \Delta(\mathfrak{g})$ or Δ. For a root α the root space is $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H,X] = \alpha(H)X\}$, a one–dimensional subspace of $\mathfrak{g}$. For an $\mathrm{ad}(\mathfrak{h})$–stable subspace $\mathfrak{q}$ of $\mathfrak{g}$ we write $\Delta(\mathfrak{q},\mathfrak{h})$ or $\Delta(\mathfrak{q})$ for the roots of $\mathfrak{h}$ in $\mathfrak{q}$.

Definition 1.1. (i) A *Borel subalgebra* is a maximal solvable subalgebra of $\mathfrak{g}$.
(ii) A *parabolic subalgebra* is a subalgebra of $\mathfrak{g}$ which contains a Borel subalgebra.

Suppose $\mathfrak{b}$ is a Borel subalgebra. Then $\mathfrak{b}$ contains some Cartan subalgebra $\mathfrak{h}$ and $\mathfrak{b} = \mathfrak{h} + \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ for some positive system of roots Δ^+. Since all Cartan subalgebras are G–conjugate and all positive systems are conjugate under the Weyl group, all Borel subalgebras are G–conjugate.

Conversely, if $\mathfrak{h}$ is a Cartan subalgebra and Δ^+ is a positive system of roots then $\mathfrak{b} = \mathfrak{h} + \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ is a Borel subalgebra. Let Π be the system of simple roots Δ^+. For any subset S of Π, set $\Delta_S = \mathrm{span}_{\mathbf{Z}}(S) \cap \Delta$. Then S determines a parabolic subalgebra

$$\mathfrak{q}_S = \mathfrak{l} + \mathfrak{u}, \text{ where } \mathfrak{l} = \mathfrak{h} + \bigoplus_{\alpha\in\Delta_S} \mathfrak{g}_\alpha \text{ and } \mathfrak{u} = \bigoplus_{\alpha\in\Delta^+\setminus\Delta_S} \mathfrak{g}_\alpha. \tag{1.2}$$

Note that $\mathfrak{b} \subset \mathfrak{q}_S$. Every parabolic subalgebra is G–conjugate to exactly one parabolic subalgebra $\mathfrak{q}_S$. Thus there is a 1-1 correspondence between the subsets of Π and G-conjugacy classes of parabolic subalgebras.

Definition 1.3. (i) A subgroup B of G is a *Borel subgroup* if it is a maximal connected solvable subgroup, that is, if and only if B is connected and $Lie(B)$ is a Borel subalgebra.
(ii) A subgroup Q of G is a *parabolic subgroup* if Q contains a Borel subgroup.

By the decomposition (1.2) of the Lie algebra of a parabolic subalgebra there is a decomposition $Q = LU$ of a parabolic subgroup with L reductive, U the maximal nilpotent subgroup and $L \cap U = \{e\}$.

Definition 1.4. A *complex flag variety* is a homogeneous space of the form $Z = G/Q$ where Q is a parabolic subgroup.

Since G is connected the normalizer of Q (or of $\mathfrak{q}$) in G is connected and is equal to Q. It follows from this connectedness of Q that flag varieties are simply connected. We will see that flag varieties are also compact. It is sometimes useful to identify $Z = G/Q$ with the G–conjugates of the parabolic subalgebra $\mathfrak{q}$. The coset gQ corresponds to the parabolic subalgebra $\mathrm{Ad}(g)\mathfrak{q}$. This is a bijection since the normalizer $N_G(Q)$ is Q.

The complex flag varieties for the classical groups may be realized as follows.
Type A_n. Let $G = SL(n+1, \mathbf{C})$ be the group of invertible linear transformations of determinant one of an $n+1$–dimensional vector space V. Let $\tilde{m} = (m_1, \dots, m_k) \in \mathbf{Z}^k$ so that $1 \le m_1 < \dots < m_k \le n$. Then

$$Z_{\tilde{m}} = \{E_1 \subset \dots \subset E_k \subset V : \dim(E_j) = m_j \text{ for all } j = 1, \dots, k\} \tag{1.5}$$

is a complex flag variety and all complex flag varieties are of this form for some $\tilde{m}$. Such a sequence of subspaces $(E_j) \subset Z_{\tilde{m}}$ is called a *flag*. The action is transitive. By computing the stabilizers of convenient base points, and comparing with (1.2), one checks that (1.5) gives all complex flag varieties. If $\tilde{m} = (m)$ (i.e., $k = 1$) then $Z_{(m)}$ is the Grassmannian of m–dimensional subspaces of V. At the other extreme, if $\tilde{m} = (1, 2, \dots, n-1)$ then $Z_{\tilde{m}}$ is the *full flag* variety $G/(\mathrm{Borel})$.
Types B_n and D_n. Let V be an N–dimensional vector space with $N = 2n+1$ or $N = 2n$. Fix a nondegenerate symmetric bilinear form $(\ ,\ )$ on V. The isometry group is the *orthogonal group* $G' = O(N, \mathbf{C}) = \{g \in GL(V) \,|\, (gv, gw) = (v, w) \text{ for all } v, w \in V\}$. G' has two connected components; the component containing the identity is the *special orthogonal* group $G = SO(N, \mathbf{C}) = O(N, \mathbf{C}) \cap SL(N, \mathbf{C})$. Letting $\tilde{m}$ be as above, consider the complex varieties

$$\begin{aligned} Z_{\tilde{m}} = \{E_1 \subset \dots \subset E_k \subset V : \dim(E_j) = m_j \text{ and} \\ E_j \text{ is isotropic for all } j = 1, \dots, k\}. \end{aligned}$$

By Witt's Theorem (Theorem 3.4), given any two isotropic subspaces of the same dimension one can be mapped to the other by an isometry of V. (See [**3**].) Thus G' is transitive on any $Z_{\tilde{m}}$. Now consider the action of the connected component G of G' on $Z_{\tilde{m}}$. When $m_k \ne N/2$ G is transitive on $Z_{\tilde{m}}$, so $Z_{\tilde{m}}$ is a flag variety. In case $m_k = N/2$ (i.e., type D_n and $m_k = n$), $Z_{\tilde{m}}$ is the union of two G orbits (which are conjugate by an *outer automorphism*). These two orbits are complex flag varieties for G. By convenient choices of base points and $(\ ,\ )$ one can easily check that stabilizers of points are parabolic subgroups, and all parabolic subgroups arise this way. If $k = 1$ and $\tilde{m} = (1)$, then $Z_{(1)} = \{[z] \in \mathbf{CP}(N-1) : (z, z) = 0\}$, the quadric in projective space.
Type C_n. Let V be a vector space with a nondegenerate symplectic form ω. Then V is $2n$–dimensional. Let $G = Sp(n, \mathbf{C})$ be the isometry group of ω. Thus $g \in Sp(n, \mathbf{C})$ if and only if g is a linear transformation so that $\omega(gv, gw) = (v, w)$ for all $v, w \in V$. It is a fact that G is connected and each $g \in G$ has determinant one. With $\tilde{m}$ as above, the complex flag varieties for G are

$$\begin{aligned} Z_{\tilde{m}} = \{E_1 \subset E_2 \subset \dots \subset E_k \subset V : \dim(E_j) = m_j \text{ and} \\ E_j \text{ is } \omega\text{–isotropic for all } j = 1, 2, \dots, k\}. \end{aligned}$$

Again, by Witt's theorem G is transitive on each $Z_{\tilde{m}}$ and one can easily compute the stabilizers. A familiar example is the flag manifold $Z_{(n)}$ of maximal isotropic (i.e., *Lagrangian*) subspaces.

A general, but less explicit, method for realizing the complex flag varieties is as follows. Choose a Borel subgroup B containing a Cartan subgroup H. Let $\Delta^+ = \Delta(\mathfrak{b}, \mathfrak{h})$. For any dominant weight $\lambda \in \mathfrak{h}^*$ let V_λ be the irreducible G–representation with highest weight λ and v_+ a highest weight vector. Then $(g, v) \to [g \cdot v]$ defines a holomorphic action of G on the projective space $\mathbf{P}(V_\lambda)$. The stabilizer of $[v_+]$ contains B, so is a parabolic subgroup. Thus $[G \cdot v_+] \subset \mathbf{P}(V_\lambda)$ is a flag variety. The Lie algebra of the stabilizer is $\mathfrak{q}_S$ (in the notation of (1.2)) with $S = \{\alpha \in \Pi : \langle \lambda, \alpha \rangle = 0\}$. Therefore we obtain all the flag varieties in terms of the fundamental weights λ_j by taking $\lambda = \sum_j a_j \lambda_j$, with $a_j = 0$ or 1. The orbit $[G_0 \cdot v_+]$ is an orbit of minimal dimension, so is closed and hence a projective variety.

1.2. Algebraic groups and flag varieties.

We have given a fairly direct definition of parabolic subgroup and complex flag variety for a connected complex semisimple Lie group. We outline a more geometric treatment of the basic facts in the context of algebraic groups. This is done in terms of some very basic algebraic geometry. For now G will be a connected (affine) algebraic group (not necessarily reductive) defined over a field $\mathbf{F}$. We note that a connected complex semisimple Lie group is an algebraic group defined over $\mathbf{C}$. Everything in this section is contained in [**13**].

Let Z be an algebraic variety defined over $\mathbf{F}$. If the projection $Z \times X \to X$ is a closed map for all algebraic varieties X, then Z is called *complete*. This is the algebraic analogue of the notion of compact topological space. The corresponding statement is that a Hausdorff space Z is compact if and only if $Z \times X \to X$ is a closed map for all Hausdorff topological spaces X.

If H is a (Zariski) closed algebraic subgroup of G, then the quotient space G/H has the structure of algebraic variety.

Definition 1.6. A *Borel subgroup* of G is a maximal connected solvable subgroup. A closed subgroup $Q \subset G$ is a *parabolic subgroup* if $Z = G/Q$ is a complete variety.

Remark 1.7. By a closed subgroup we mean closed in the Zariski topology, thus an algebraic subgroup. A smooth complex variety is complete if and only if it is compact as a complex analytic manifold. Thus an algebraic subgroup Q of G is a parabolic subgroup if and only if G/Q is compact. However there are non–algebraic subgroups with compact quotient (for instance, there is a discrete subgroup with this property).

The variety Z is *projective* if it is a closed subvariety of some projective space and is *quasi–projective* if it is an open subset of a projective variety. It follows from the definitions that closed subvarieties and products of complete varieties are complete. The same holds for projective varieties (see exercise 1.4(b)). If Z is complete and $Z \to X$ is an algebraic morphism, then the image is closed and complete. The regular functions on a complete variety are constants. It follows that the only morphisms from a complete variety to an affine variety are constants. We list a number of geometric results:

(1) Projective varieties are complete.
(2) A quasi–projective variety is complete if and only if it is projective.

(3) Homogeneous spaces are quasi–projective.

Corollary 1.8. *A closed subgroup $Q \subset G$ is a parabolic if and only if G/Q is a projective variety.*

An action of an algebraic group G on a variety Z is an *algebraic action* if $G \times Z \to Z$ is an algebraic morphism. Then

(4) Orbits are smooth varieties, each open in its closure. The boundary of an orbit is a union of orbits of smaller dimension. The orbits of smallest dimension are closed.
(5) If G is solvable and Z is complete, then there is a point of Z fixed by G.
(6) For any subgroup H of G there is a faithful representation $\pi : G \to GL(V)$ so that the corresponding action of G on $\mathbf{P}(V)$ has a point with stabilizer exactly H.

The following standard fact is a consequence of (5).

Corollary 1.9. *If (τ, V) is a representation of a connected solvable group G, then there is a (full) flag in V invariant under G. Equivalently, there is a basis of V for which each $\tau(g)$ is in upper triangular form.*

Theorem 1.10. *Suppose $B \subset G$ is a Borel subgroup. Then G/B is projective (so is complete and B is a parabolic subgroup). All Borel subgroups are conjugate to B.*

Proof. Let B be a Borel subgroup of greatest dimension. By (6) we may choose a faithful representation of G with a one dimensional subspace V_1 stable under exactly B. Applying Corollary 1.9 to V/V_1 construct a full flag $V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V$ of B–invariant subspaces. Let Z be the variety of all full flags in V. Because V is faithful, the stabilizer of any point in Z is solvable. As B has greatest dimension among the solvable subgroups of G we conclude that the orbit of (V_j) is of smallest dimension, so is closed by (4). The stabilizer if (V_j) is B, thus the orbit of (V_j) is G/B. Since the flag variety is projective so is G/B. By Corollary 1.9 G/B is complete.

Now let B' be another Borel subgroup of G. B' is solvable and acts on the complete variety G/B, so by (5) there is a fixed point. So, for some $g \in G$, $b'gB = gB$ for all $b' \in B'$. Therefore, $g^{-1}B'g \subset B$. Since B', hence $g^{-1}B'g$, is maximal solvable, we have $g^{-1}B'g = B$. □

Corollary 1.11. *The parabolic subgroups are precisely the closed subgroups which contain a Borel subgroup.*

Proof. If Q is a closed subgroup containing a Borel subgroup B then $G/B \to G/Q$ is an algebraic morphism. G/Q is the image of a complete variety, so is complete. Conversely, if Q is a parabolic subgroup, then any Borel subgroup B acts on G/Q and has a fixed point gQ (by fact (3) above). Thus $BgQ = gQ$, so $g^{-1}Bg \subset Q$. Therefore, Q contains the Borel subgroup $g^{-1}Bg \subset Q$. □

Thus the definition (1.6) of a parabolic subgroup of an arbitrary connected algebraic group coincides with the definition (1.1) of a parabolic subgroup of a connected complex reductive Lie group. Proofs of the following three theorems can be found in [**13**]. We use the notation $N_G(S)$ (respectively, $C_G(S)$) for the normalizer (respectively, centralizer) in G of a subset S of G.

Theorem 1.12. *If $Q \subset G$ is a parabolic subgroup, then $N_G(Q) = Q$ and Q is connected.*

Theorem 1.13. *If G is solvable, then any two maximal tori are conjugate by an element of the unipotent radical of G. If G is reductive, then any two Cartan subgroups are conjugate.*

For the next theorem suppose that G is reductive. Recall that the Weyl group is

$$W(G;H) = N_G(H)/C_G(H).$$

Each element of the Weyl group of the root system is represented by an element in $W(G;H)$.

Theorem 1.14. *(The Bruhat decomposition.) Let $B = HN$ be a Borel subgroup of the reductive group G. Then*

$$G = \bigcup_{w \in W(G;H)} NwB.$$

1.3. Orbits in a complex flag manifold under the action of a real form.

Let G be a connected complex algebraic group and $\mathfrak{g}$ its Lie algebra.

Definition 1.15. (1) A conjugation of a Lie algebra is a conjugate linear transformation of $\mathfrak{g}$ which preserves the bracket, i.e., $\tau([X,Y]) = [\tau(X), \tau(Y)]$ and satisfies $\tau^2 = 1$.
(2) The fixed point set of a conjugation of $\mathfrak{g}$ is called a *real form* of $\mathfrak{g}$. The corresponding analytic subgroup is a *real form* of G.

A real form of $\mathfrak{g}$ (respectively G) will usually be denoted by $\mathfrak{g}_0$ (respectively G_0). Then $\mathfrak{g} \cong \mathfrak{g}_0 \otimes \mathbf{C}$. If $\mathfrak{m}$ is a vector subspace of $\mathfrak{g}$ which is stable under τ (i.e., $\tau(\mathfrak{m}) = \mathfrak{m}$) then $\mathfrak{m}$ is defined over $\mathbf{R}$ in the sense that there is a real subspace $\mathfrak{m}_0$ of $\mathfrak{g}_0$ so that $\mathfrak{m} = \mathfrak{m}_0 \otimes \mathbf{C}$. Also, if such a subspace $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}$ then $\mathfrak{m}_0$ is a subalgebra of $\mathfrak{g}_0$.

We fix a connected complex semisimple algebraic group G and a real form G_0 with conjugation denoted by τ. Let θ be a Cartan involution of G_0. We also denote the differential of the Cartan involution by θ. There is an extension of θ to G; we again denote this extension and its differential by θ. Let $K_0 = (G_0)^\theta$ and $K = G^\theta$ be the fixed point groups of θ. Decompose $\mathfrak{g}_0$ and $\mathfrak{g}$ into ± 1 eigenspaces for θ:

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{k}_0 + \mathfrak{s}_0, \\ \mathfrak{g} &= \mathfrak{k} + \mathfrak{s}. \end{aligned} \tag{1.16}$$

Let $Z \simeq G/Q$ be a complex flag variety. We are interested in the orbits of G_0 on Z. We begin by considering the *full flag variety* $X = G/B$, with B a Borel subgroup.

Proposition 1.17. *Each Borel subalgebra contains a Cartan subalgebra preserved by τ (that is, a Cartan subalgebra defined over $\mathbf{R}$) and a Cartan subalgebra preserved by θ (called a θ–stable Cartan subalgebra).*

Proof. Write $B = HN$ where H is a Cartan subgroup and N the unipotent radical. Suppose that $\mathfrak{b}'$ is another Borel subgroup. We claim that $\mathfrak{b} \cap \mathfrak{b}'$ contains a Cartan subalgebra. By Theorem 1.10, there exists $g \in G$ so that $\mathfrak{b}' = \mathrm{Ad}(g)\mathfrak{b}$. By the Bruhat decomposition (1.14) we may write $g = nwb$, with $n \in N, b \in B$ and

$w \in N_G(H)$. So $\mathfrak{b}' = \mathrm{Ad}(nw)\mathfrak{b}$. Thus $\mathrm{Ad}(nw)\mathfrak{h} \subset \mathfrak{b} \cap \mathfrak{b}'$. The claim is now proved. We may therefore assume that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b} \cap \tau(\mathfrak{b})$. By (1.13) there is an $X \in \mathfrak{n} \cap \tau(\mathfrak{n})$ so that $\mathrm{Ad}(\exp X)\mathfrak{h} = \tau(\mathfrak{h})$. Since $\mathrm{Ad}(\tau(\exp(X))\exp(X))\mathfrak{h} = \mathfrak{h}$ we have $\tau(\exp(X))\exp(X) \in N_G(H) \cap N = \{e\}$. Since the exponential map is one–to–one on N this means that $\tau(X) = -X$. Now let $\mathfrak{h}_1 = \mathrm{Ad}(\exp(\frac{1}{2}X))\mathfrak{h}$. We show that $\mathfrak{h}_1$ is τ–stable.

$$\begin{aligned}\tau(\mathfrak{h}_1) =& \tau(\mathrm{Ad}(\exp(\frac{1}{2}X))\mathfrak{h}) = \mathrm{Ad}(\exp(-\frac{1}{2}X))\tau(\mathfrak{h}) \\ &= \mathrm{Ad}(\exp(-\frac{1}{2}X))\mathrm{Ad}(\exp(X))\mathfrak{h} \\ &= \mathrm{Ad}(\exp(\frac{1}{2}X))\mathfrak{h} = \mathfrak{h}_1.\end{aligned}$$

The proof for θ–stable Cartans is the same, with θ replacing τ. □

The proposition does not guarantee the existence of a Cartan subalgebra stable under both τ and θ, i.e., θ–stable and defined over **R**. In fact, it is not true that every Borel subalgebra contains such a Cartan subalgebra.

Proposition 1.18. *Let $\mathfrak{b}$ be a Borel subalgebra. There is a $g \in G_0$ so that $Ad(g)\mathfrak{b}$ contains a θ–stable Cartan subalgebra defined over* **R**. *There also exists a $k \in K$ so that $Ad(k)\mathfrak{b}$ contains a θ–stable Cartan subalgebra defined over* **R**.

Remark 1.19. Another way to say the first part of the proposition is: any Borel subalgebra contains a Cartan subalgebra defined over **R** which is stable under *some* Cartan involution. This follows from the fact that all Cartan involutions are G_0–conjugate.

Proof. This follows from the fact that each Cartan subalgebra defined over **R** is G_0 conjugate to a θ–stable Cartan subalgebra. See [**20**], Prop. 6.59. □

Proposition 1.20. *There are a finite number of G_0–orbits on X.*

Proof. There are a finite number of G_0–conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$ ([**20**], Prop. 6.64). List them as $\mathfrak{h}^1, \dots, \mathfrak{h}^k$. If $\mathfrak{b} \in X$, then $\mathfrak{b}$ is G_0–conjugate to a Borel subalgebra containing some $\mathfrak{h}^i$ from the list. There are only a finite number of Borel subalgebras containing a given Cartan subalgebra. They are of the form $\mathfrak{b} = \mathfrak{h}^i + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ for some positive system Δ^+. Thus the number of G_0–orbits on X is bounded by $k \cdot |W(\mathfrak{g};\mathfrak{h})|$. □

Lemma 1.21. *Suppose $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ and $\mathfrak{b}' = \mathfrak{h}' + \mathfrak{n}'$ with $\mathfrak{h}$ and $\mathfrak{h}'$ θ–stable and defined over* **R**. *If $\mathfrak{b}$ and $\mathfrak{b}'$ are G_0–conjugate or K–conjugate, then (a) $\mathfrak{h}$ and $\mathfrak{h}'$ are K_0–conjugate and (b) $\mathfrak{b}$ and $\mathfrak{b}'$ are K_0–conjugate.*

Proof. Assume $g_0\mathfrak{b}' = \mathfrak{b}$. Then $g_0\mathfrak{h}'$ and $\mathfrak{h}$ are maximal tori in $\mathfrak{b} \cap \tau(\mathfrak{b})$). By Theorem 1.13 there is some $n \in N \cap \tau(N)$ so that $n\mathfrak{h} = g_0\mathfrak{h}'$. As $\mathfrak{h}, \mathfrak{h}'$ are τ–stable $\tau(n)\mathfrak{h} = n\mathfrak{h}$, so $n^{-1}\tau(n) \in N_{G_0}(H) \cap N = \{e\}$. It follows that $\tau(n) = n$ and $n \in N \cap G_0$. Set $g_1 = n^{-1}g_0$. Then $g_1 \in G_0$ and $g_1\mathfrak{h}' = \mathfrak{h}$ and $g_1\mathfrak{b}' = \mathfrak{b}$.

We complete the proof by showing that $g_1 \in K_0$. Write $g_1 = k_0\exp(X) \in K_0\exp(\mathfrak{s}_0)$. But $g_1\mathfrak{h}' = \mathfrak{h} = \theta(\mathfrak{h}) = \theta(g_1)\mathfrak{h}'$ implies that $\exp(2X) \in N_{G_0}(\mathfrak{h}') \subset K_0$, so $X = 0$. Therefore $g_1 = k_0 \in K_0$. □

Let $\tilde{X}$ be the set of all Borel subalgebras which contain a θ–stable Cartan subalgebra defined over **R**. Note that K_0 acts on $\tilde{X}$.

Corollary 1.22. *Each G_0–orbit and each K–orbit meets $\tilde{X}$ in exactly one K_0–orbit. There are one to one correspondences between G_0–orbits in X, K_0–orbits in $\tilde{X}$ and K–orbits in X:*

$$X/G_0 \leftrightarrow \tilde{X}/K_0 \leftrightarrow X/K.$$

Thus if $\mathfrak{b} \in \tilde{X}$, then $(G_0 \cdot \mathfrak{b}) \cap (K \cdot \mathfrak{b}) = K_0 \cdot \mathfrak{b}$. The *Matsuki duality* is then $G_0 \cdot \mathfrak{b} \leftrightarrow K \cdot \mathfrak{b}$. Put slightly differently, given a G_0–orbit $\mathcal{O}$ in X there is a unique K–orbit $\mathcal{O}'$ in X so that $\mathcal{O} \cap \mathcal{O}'$ is K_0–orbit.

Corollary 1.23. *There are a finite number of K–orbits in X.*

Corollary 1.24. *The number of G_0–orbits in X is*

$$\sum_j |W(\mathfrak{g}, \mathfrak{h}^j)/W(G_0, H_0^j)|.$$

This is also the number of K–orbits in X, which is the number of K_0–orbits in $\tilde{X}$.

Proof. As in the proof of Proposition 1.20, any Borel subalgebra is G_0–conjugate to some

$$\mathfrak{b} = \mathfrak{h}^j + \sum_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}^j)} \mathfrak{g}_\alpha \tag{1.25}$$

for some positive system $\Delta^+(\mathfrak{g}, \mathfrak{h}^j)$ and some j. We may assume that the $\mathfrak{h}^j$ are θ–stable as well as defined over $\mathbf{R}$. If two such Borel subalgebras are G_0–conjugate then the Cartan subalgebras are K_0–conjugate by Lemma 1.21. Now if

$$\mathfrak{b} = \mathfrak{h}^j + \sum_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}^j)} \mathfrak{g}_\alpha \text{ and } \mathfrak{b}' = \mathfrak{h}^j + \sum_{\alpha \in \Delta^{+\prime}(\mathfrak{g}, \mathfrak{h}^j)} \mathfrak{g}_\alpha$$

are G_0–conjugate, then there is a $g \in G_0$ (see exercise 1.4(d)) so that $\mathrm{Ad}(g)\mathfrak{h}^j = \mathfrak{h}^j$ and $\mathrm{Ad}(g)\mathfrak{b} = \mathfrak{b}'$. Thus g represents an element of the Weyl group $W(G_0; H_0^j)$. $\square$

Example 1.26. Here is the simplest example. Let $G = SL(2, \mathbf{C})$ and $G_0 = SU(1,1)$ ($\cong SL(2, \mathbf{R})$). Then $K \cong \mathbf{C}^\times$, the diagonal subgroup. Then $X \simeq \mathbf{CP}(1)$ and the G_0-orbits are the upper hemisphere, the lower hemisphere and the equator. The K–orbits are the North pole, the South pole and the dense orbit consisting of everything else. The K_0–orbits in $\tilde{X}$ consist of the North pole, the South pole and the equator. The Matsuki correspondence is:

$$\{\text{upper hemisphere}\} \leftrightarrow \{\text{North pole}\}$$
$$\{\text{lower hemisphere}\} \leftrightarrow \{\text{South pole}\}$$
$$\{\text{equator}\} \leftrightarrow \{\text{dense orbit}\}.$$

Now let Z be an arbitrary complex flag variety for G. Then there is a G–equivariant fibration

$$\begin{aligned} &\pi : X \to Z \\ &\pi(\mathfrak{b}) = \mathfrak{q}, \text{ where } \mathfrak{q} \supset \mathfrak{b}. \end{aligned} \tag{1.27}$$

Note that there is exactly one parabolic in each conjugacy class which contains a given Borel subalgebra. Writing $Z = G/Q$ and $X = G/B$ with $B \subset Q$, the fibration is $\pi(gB) = gQ$. Note that a typical fiber of π is $\pi^{-1}(eB) \simeq L/L \cap B$, where $Q = LU$, a complex flag variety for L. It is immediate that Propositions 1.17, 1.18 and 1.20 and Corollary 1.23 hold for Z in place of X. However, Lemma

1.21 fails. In particular for $Q = LU$, the subgroup L may contain several θ–stable Cartan subalgebras defined over $\mathbf{R}$ which are not G_0–conjugate.

Define, as we did for the full flag variety,

$$\tilde{Z} = \{\text{parabolics in } Z \text{ containing a } \theta\text{-stable Cartan subalgebra defined over } \mathbf{R}\}.$$

Lemma 1.28. *Let $\mathfrak{q}, \mathfrak{q}' \in \tilde{Z}$ be G_0–conjugate (or K–conjugate). Then $\mathfrak{q}$ and $\mathfrak{q}'$ are K_0–conjugate.*

We may conclude the following theorem.

Theorem 1.29. *Let Z be an arbitrary complex flag variety for G. If $\mathfrak{q} \in Z$, then*

(a) *$\mathfrak{q}$ contains a Cartan subalgebra defined over $\mathbf{R}$ and a θ–stable Cartan subalgebra.*
(b) *$\mathfrak{q}$ is G_0-conjugate (respectively, K–conjugate) to a parabolic containing a θ–stable Cartan subalgebra defined over $\mathbf{R}$.*
(c) *There are a finite number of G_0–orbits and K–orbits in Z.*
(d) *There is a bijection between G_0 and K–orbits in Z, with two orbits in duality whenever their intersection is a K_0–orbit.*

Proof. The first three statements follow immediately from the corresponding statements for X since each parabolic contains a Borel subalgebra. Statement (d) follows from the lemma and the fact that the fibration sends orbits to orbits. $\square$

The material in this section is contained in [**32**] and [**24**].

1.4. Exercises.

(a) For types A-D compute stabilizers of convenient base points for all $Z_{\tilde{m}}$. Conclude that all flag varieties occur as some $Z_{\tilde{m}}$. Carry out the proof, using induction and Witt's Theorem, that for types A-D the groups are transitive as stated in Section 1.1.
(b) Show that the $Z_{\tilde{m}}$ defined for types A-D are flag varieties by showing they are projective. Hint: First prove that the product of projective varieties is projective by showing that the map $\mathbf{P}(V) \times \mathbf{P}(W) \to \mathbf{P}(V \otimes W)$ defined by $([v], [w]) \to [v \otimes w]$ is a closed embedding. Apply this to show that the $\mathbf{Z}_{\tilde{m}}$ of type A are projective varieties. Now conclude that the $\mathbf{Z}_{\tilde{m}}$ of types B-D are projective.
(c) Show that if $G_0 \cdot \mathfrak{b}$ and $K \cdot \mathfrak{b}$ correspond by the Matsuki duality, then the two orbits intersect transversely.
(d) Complete the proof of Corollary 1.24. (Hint: Use an argument similar to the proof of Lemma 1.21.)
(e) Compute the $G_0 = SL(3, \mathbf{R})$ and $K = SO(3, \mathbf{C})$–orbits on $\mathbf{CP}(2)$. Which orbits correspond under the Matsuki duality?

LECTURE 2

Open G_0–orbits

We will now concentrate on the open G_0–orbits in a complex flag variety Z. Later we will consider representations associated to these orbits. We fix a real form G_0 of G which corresponds to a conjugation τ.

2.1. Identifying open orbits.

We need to set up some conventions and notation. As mentioned earlier points $z \in Z = G/Q$ may be identified with parabolics $\mathfrak{q} = \mathfrak{q}_z = Lie(Stab_G(z))$. So $g \cdot z \leftrightarrow \mathrm{Ad}(g)\mathfrak{q}_z$. By Proposition 1.17, for any $z \in Z$ we may choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{q}_z$ defined over $\mathbf{R}$. Our root systems will always be roots with respect to such a Cartan subalgebra. We write

$$\begin{aligned} \mathfrak{q} =& \mathfrak{l} + \mathfrak{u}, \text{ with} \\ \mathfrak{l} =& \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{l})} \mathfrak{g}_\alpha \text{ and} \\ \mathfrak{u} =& \sum_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}_\alpha. \end{aligned}$$

Here $\Delta(\mathfrak{l}) = \Delta(\mathfrak{q}) \cap -\Delta(\mathfrak{q})$ is the same as Δ_S in (1.2) and $\Delta(\mathfrak{u}) = \Delta(\mathfrak{q}) \setminus \Delta(\mathfrak{l}) = \{\alpha \in \Delta(\mathfrak{q}) : -\alpha \notin \Delta(\mathfrak{q})\}$.

There is an action of τ on $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ given by

$$\tau(\alpha) : H \to \overline{\alpha(\tau(H))}, H \in \mathfrak{h}.$$

It follows that $\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{\tau(\alpha)}$. Thus if $\mathfrak{g}_1 \subset \mathfrak{g}$ is $\mathfrak{h}$–invariant then $\tau(\Delta(\mathfrak{g}_1)) = \Delta(\tau(\mathfrak{g}_1))$, in particular $\Delta(\mathfrak{g}_1 \cap \tau(\mathfrak{g}_1)) = \Delta(\mathfrak{g}_1) \cap \tau(\Delta(\mathfrak{g}_1))$.

Let $G_0(z) \subset Z$ and let $Q = Stab_G(z)$. Then $Stab_{G_0}(z) = Q \cap G_0$ has Lie algebra $\mathfrak{q} \cap \mathfrak{g}_0 = \mathfrak{q} \cap \tau(\mathfrak{q}) \cap \mathfrak{g}_0$, a real form of $\mathfrak{q} \cap \tau(\mathfrak{q})$. Thus, the (real) dimension of the stabilizer is

$$\dim(\mathfrak{h}) + |\Delta(\mathfrak{q}) \cap \tau(\Delta(\mathfrak{q}))|.$$

Now write

$$\begin{aligned} &\Delta(\mathfrak{q} \cap \tau(\mathfrak{q})) = \Delta_1 \cup \Delta_2, \\ &\Delta_1 = \Delta(\mathfrak{l}) \cap \tau(\Delta(\mathfrak{l})) \text{ and} \\ &\Delta_2 = \big(\Delta(\mathfrak{l}) \cap \tau(\Delta(\mathfrak{u}))\big) \cup \big(\tau(\Delta(\mathfrak{l})) \cap \Delta(\mathfrak{u})\big) \cup \big(\Delta(\mathfrak{u}) \cap \tau(\Delta(\mathfrak{u}))\big). \end{aligned} \tag{2.1}$$

Then

$$\mathfrak{q}\cap\tau(\mathfrak{q}) = \big(\mathfrak{h} + \sum_{\alpha\in\Delta_1}\mathfrak{g}_\alpha\big) + \sum_{\alpha\in\Delta_2}\mathfrak{g}_\alpha \tag{2.2}$$

is a Levi decomposition of the complexification of the stabilizer in $\mathfrak{g}_0$.

Proposition 2.3. *Let $Q = Stab_G(z)$. Then*

$$\begin{aligned}
\dim_{\mathbf{R}}(Stab_{G_0}(z)) &= \dim(\mathfrak{h}) + |\Delta(\mathfrak{l})| + |\Delta(\mathfrak{u})\cap\tau(\Delta(\mathfrak{u}))|,\\
\dim_{\mathbf{R}} G_0(z) &= \dim_{\mathbf{C}}(\mathfrak{u}+\tau(\mathfrak{u})),\\
\operatorname{codim}_{\mathbf{R}}(G_0(z)) &= |\Delta(\mathfrak{u})\cap\tau(\Delta(\mathfrak{u}))|.
\end{aligned}$$

Proof. This is a straightforward calculation.

$$\begin{aligned}
\dim_{\mathbf{R}}(Stab_{G_0}(z)) &= \dim\mathfrak{h} + |\Delta(\mathfrak{q})\cap\tau(\Delta(\mathfrak{q}))|\\
&= \dim\mathfrak{h} + |\Delta(\mathfrak{l})\cap\tau(\Delta(\mathfrak{q}))| + |\Delta(\mathfrak{l})\cap\tau(\Delta(\mathfrak{u}))| + |\Delta(\mathfrak{u})\cap\tau(\Delta(\mathfrak{u}))|\\
&= \dim\mathfrak{h} + |\Delta(\mathfrak{l})| + |\Delta(\mathfrak{u})\cap\tau(\Delta(\mathfrak{u}))|.
\end{aligned}$$

This gives the second equality below.

$$\begin{aligned}
\dim_{\mathbf{R}}(G_0(z)) &= \dim_{\mathbf{C}}(\mathfrak{g}) - \dim_{\mathbf{C}}(\mathfrak{q}\cap\tau(\mathfrak{q}))\\
&= |\Delta(\mathfrak{l})| + 2|\Delta(\mathfrak{u})| - |\Delta(\mathfrak{l})| - |\Delta(\mathfrak{u})\cap\tau(\Delta(\mathfrak{u}))|\\
&= (\dim_{\mathbf{C}}\mathfrak{u} + \dim_{C}\tau(\mathfrak{u})) - \dim_{\mathbf{C}}(\mathfrak{u}\cap\tau(\mathfrak{u}))\\
&= \dim_{\mathbf{C}}(\mathfrak{u}+\tau(\mathfrak{u})).
\end{aligned}$$

The second gives us the codimension formula.

$$\begin{aligned}
\operatorname{codim}_{\mathbf{R}}(G_0(z)) &= 2\dim_{\mathbf{C}}\mathfrak{u} - \dim_{\mathbf{C}}(\mathfrak{u}+\tau(\mathfrak{u}))\\
&= \dim_{\mathbf{C}}(\mathfrak{u}\cap\tau(\mathfrak{u}))\\
&= |\Delta(\mathfrak{u})\cap\tau(\Delta(\mathfrak{u}))|.
\end{aligned}$$

□

Corollary 2.4. *Let $z\in Z$ and $\mathfrak{q}=\mathfrak{q}_z$. Then the following are equivalent:*

(a) *$G_0(z)$ is open in Z;*
(b) *$\Delta(\mathfrak{u})\cap\tau(\Delta(\mathfrak{u}))=\emptyset$;*
(c) *$\mathfrak{q}+\tau(\mathfrak{q})=\mathfrak{g}$.*

Proof. The first two are equivalent by the formula for the codimension. Now $\dim(\mathfrak{q}+\tau(\mathfrak{q}))+\dim(\mathfrak{q}\cap\tau(\mathfrak{q})) = 2\dim(\mathfrak{q})$, so

$$\begin{aligned}
&\dim(\mathfrak{q}+\tau(\mathfrak{q}))\\
&\quad= 2\dim\mathfrak{h} + 2|\Delta(\mathfrak{l})| + 2|\Delta(\mathfrak{u})| - \big(\dim\mathfrak{h} + |\Delta(\mathfrak{l})| + |\Delta(\mathfrak{u})\cap\tau(\Delta(\mathfrak{u}))|\big)\\
&\quad= \dim\mathfrak{g} - |\Delta(\mathfrak{u})\cap(\tau\Delta(\mathfrak{u}))|.
\end{aligned}$$

The equivalence of (b) and (c) follows. □

Corollary 2.5. *Suppose $G_0 = G_u$ is the compact real form of G. Then G_u acts transitively on Z.*

Proof. Since the only Cartan subalgebra is compact, $\tau(\Delta^+) = -\Delta^+$ for any positive system. So the second condition in Corollary 2.4 always holds. But $G_u(z)$ is also compact, so $G_u(z) = Z$. □

There is a useful condition, in terms of Cartan subalgebras contained in $\mathfrak{q} = \mathfrak{q}_z$, for an orbit $G_0(z)$ to be open. Suppose that $\mathfrak{h} = (\mathfrak{h}_0)_{\mathbf{C}}$ is a Cartan subalgebra defined over $\mathbf{R}$. By Remark 1.19 we may choose a Cartan involution θ' so that $\mathfrak{h}_0$ is θ'–stable. Writing $\mathfrak{g}_0 = \mathfrak{k}'_0 + \mathfrak{s}'_0$ as in 1.16 there is a decomposition

$$\mathfrak{h}_0 = \mathfrak{t}'_0 + \mathfrak{a}'_0, \mathfrak{t}'_0 = \mathfrak{h}_0 \cap \mathfrak{k}'_0 \text{ and } \mathfrak{a}'_0 = \mathfrak{h}_0 \cap \mathfrak{s}'_0. \tag{2.6}$$

We say that $\mathfrak{h}_0$ is a *maximally compact* Cartan subalgebra if $\mathfrak{t}'_0$ is a Cartan subalgebra of $\mathfrak{k}'_0$. The following is contained in [**20**], Chapter VI, Section 6.

Lemma 2.7. *In terms of the decomposition (2.6)*
(1) *The following are equivalent:*
- (a) $\mathfrak{h}_0$ *is maximally compact;*
- (b) $\mathfrak{t}'_0$ *contains a regular element* ξ_0 *of* $\mathfrak{g}_0$ *(i.e.,* $C_{\mathfrak{g}_0}(\xi_0) = \mathfrak{h}_0$*);*
- (c) *There exists a positive system* Δ^+ *so that* $\tau(\Delta^+) = -\Delta^+$.

(2) *If* $\tau(\Delta^+) = -\Delta^+$ *for some positive system* Δ^+*, then* $\mathfrak{h}_0$ *is maximally compact and* $\Delta^+ = \{\alpha \mid \alpha(i\xi_0) > 0\}$ *for some regular* $\xi_0 \in \mathfrak{t}'_0$ *of* $\mathfrak{g}_0$.
(3) *Any two maximally compact Cartan subalgebras are* G_0*–conjugate.*

Proposition 2.8. *Let* $z \in Z$*. The following are equivalent:*
- (a) z *is contained in an open* G_0*–orbit;*
- (b) $\mathfrak{q} = \mathfrak{q}_z$ *contains a maximally compact Cartan subalgebra and there is a positive system* $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ *so that* $\tau(\Delta^+) = -\Delta^+$ *and* $\Delta(\mathfrak{u}) \subset \Delta^+$.

If these hold, Δ^+ *is defined by some* ξ_0 *as in (2) of Lemma 2.7.*

Proof. First suppose $Z = X$ is the full flag variety. Let $z \in Z$ be in an open orbit. Then for $\mathfrak{b} = \mathfrak{b}_z$, $\Delta(\mathfrak{b}) = \Delta^+$ satisfies $\tau(\Delta^+) = -\Delta^+$ by Corollary 2.4. By Lemma 2.7(2), $\mathfrak{h}_0$ is a maximally compact Cartan subalgebra.

Now let $\mathfrak{q}$ be in an open orbit in an arbitrary flag Z. By (1.27) we have that $\pi^{-1}(G_0(z))$, being open and a union of G_0–orbits, contains an open orbit in $X = G/B$. Thus for some $\mathfrak{b} \subset \mathfrak{q}$, $\mathfrak{b}$ contains a maximally compact Cartan subalgebra. But then $\Delta(\mathfrak{u}) \subset \Delta(\mathfrak{b}) = \Delta^+$ with $\tau(\Delta^+) = -\Delta^+$.

The converse follows immediately from Corollary 2.4. □

Now fix a Cartan involution θ, thus determining $K, K_0, \mathfrak{k}, \dots$ as in (1.16). The base points of an open orbit may be chosen in a special way.

Proposition 2.9. *Let* $G_0(z) \subset Z$ *be an open orbit. Then* $\mathfrak{q}_z$ *contains a* θ*–stable Cartan subalgebra defined over* $\mathbf{R}$ *if and only if* $K(z) = K_0(z)$*. Each open orbit contains a base point* z *so that* $K_0(z)$ *is a compact subvariety of* $G_0(z)$*. For this base point* $G_0(z)$ *and* $K(z)$ *correspond under the Matsuki duality.*

Proof. First assume that $Z = X = G/B$, with B a Borel subgroup. Let $G_0(z)$ be an open orbit. If $\mathfrak{b}$ contains a θ–stable Cartan subalgebra $\mathfrak{h}$ defined over $\mathbf{R}$, then $\mathfrak{h}_0$ is maximally compact. (To see this let $\mathfrak{h}'$ be a maximally compact Cartan subalgebra contained in $\mathfrak{b}$. Then some $\mathrm{Ad}(g)\mathfrak{h}'$ is a θ–stable Cartan subalgebra in $\mathrm{Ad}(g)\mathfrak{b}$. Now apply Lemma 1.21.) By Proposition 2.8, the set $\{\alpha|_{\mathfrak{t}} : \alpha \in \Delta^+ \text{ and } \mathfrak{g}_\alpha \subset \mathfrak{k}\}$ is a positive system in $\Delta(\mathfrak{k}, \mathfrak{t})$. Thus $\mathfrak{k} \cap \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{k}$. So $K/K \cap B$ is a complex flag variety for K. The real form K_0 acts transitively by Proposition 2.5, so $K(z) = K_0(z)$.

For an arbitrary Z, assume $\mathfrak{q}_z$ contains a θ–stable Cartan subalgebra $\mathfrak{h}$ defined over $\mathbf{R}$. Then $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{q}$ for some Borel subalgebra $\mathfrak{b} = \mathfrak{b}_x$ contained in an open orbit in X. So $G_0(z) = \pi(G_0(x))$ and

$$K(z) = \pi(K(x)) = \pi(K_0(x)) = K_0(z).$$

For the converse, suppose that $K(z) \subset Z$ is compact (therefore complete). Then $K \cap Q$ is a parabolic subgroup of K. So $\mathfrak{k} \cap \mathfrak{q}$ contains a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ defined over $\mathbf{R}$. As $\mathfrak{t}_0$ contains a regular element ξ of $\mathfrak{g}_0$ (by Lemma 2.7), the centralizer of ξ in $\mathfrak{g}$ is a θ–stable Cartan subalgebra of $\mathfrak{g}$ defined over $\mathbf{R}$ and contained in $\mathfrak{q}$. □

2.2. Measurable orbits.

Among the open orbits, the orbits which arise most naturally in representation theory are the measurable orbits.

Definition 2.10. An open G_0–orbit in Z is called *measurable* if it has a G_0–invariant volume form.

It is a standard fact that G_0/S_0 has a G_0–invariant volume form if and only if S_0 is unimodular. If S_0 is the stabilizer of an orbit, then S_0 is unimodular if and only if it is reductive (by (2.2)). Thus $G_0(z)$ is measurable if it is open and $\mathfrak{q} \cap \tau(\mathfrak{q})$ is reductive. In the notation of (2.1), the subalgebra $\mathfrak{q} \cap \tau(\mathfrak{q})$ is reductive if and only if $\Delta_2 = -\Delta_2$. But, this happens exactly when $\Delta_2 = \emptyset$. We may conclude the first four equivalences below.

Theorem 2.11. *Let $G_0(z)$ be an open orbit in Z and $\mathfrak{q} = \mathfrak{q}_z$. The following are equivalent:*

(a) *$G_0(z)$ is measurable;*
(b) *$\mathfrak{q} \cap \tau(\mathfrak{q})$ is reductive;*
(c) *$\mathfrak{q} \cap \tau(\mathfrak{q}) = \mathfrak{l}$;*
(d) *$\tau(\Delta(\mathfrak{u})) = -\Delta(\mathfrak{u})$;*
(e) *There is $\lambda_0 \in i\mathfrak{t}_0'^*$ so that $\Delta(\mathfrak{q}) = \{\alpha \in \Delta : \langle \lambda_0, \alpha \rangle \geq 0\}$ (with $\mathfrak{t}_0'$ as in (2.6)).*

Proof. We will first prove that (d) implies (e). Note that for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, we have that $\alpha|_{\mathfrak{t}_0'}$ (respectively $\alpha|_{\mathfrak{a}_0}$) takes on imaginary (respectively real) values. Thus, writing $\alpha = \alpha_1 + \alpha_2 \in i\mathfrak{t}_0'^* + \mathfrak{a}_0'^*$, we have $\tau(\alpha) = -\alpha_1 + \alpha_2$. Furthermore, if μ is a sum of roots, then $\tau(\mu) = -\mu$ if and only if $\mu|_{\mathfrak{a}} = 0$, i.e., $\mu \in \mathfrak{t}'^*$. Let $\lambda_0 = \rho(\Delta) - \rho(\Delta(\mathfrak{l})) = \rho(\Delta(\mathfrak{u}))$. Then by (d), we have $\tau(\lambda_0) = -\lambda_0$ so $\lambda_0 \in \mathfrak{t}_0'^*$ and $\Delta(\mathfrak{q}) = \{\alpha \in \Delta : \langle \lambda_0, \alpha \rangle \geq 0\}$.

If (e) holds, then it is clear that $\Delta(\mathfrak{u}) = \{\alpha \in \Delta : \langle \lambda_0, \alpha \rangle > 0\}$. Since $\tau(\lambda_0) = -\lambda_0$, we have that (d) holds. □

Here is an instructive example. Consider $G_0 = SL(3, \mathbf{R})$. Then

$$\mathfrak{h}_0 = \begin{pmatrix} a/3 & 0 & t \\ 0 & -2a/3 & 0 \\ -t & 0 & a/3 \end{pmatrix}$$

is a maximally compact Cartan subalgebra. The roots are $\Delta = \pm\Delta^+ = \pm\{it \pm a, 2it\}$. Let Z be the flag variety consisting of the lines in $\mathbf{C}^3$. By Proposition 2.8 we look for positive systems Δ_1^+ so that $\tau(\Delta_1^+) = -\Delta_1^+$. These are just $\pm\Delta^+$. Since $\pm\Delta^+$ are K_0–conjugate, we may conclude that there is just one open orbit in Z (and just one in $X = G/(Borel)$). Then $\Delta(\mathfrak{l}) = \{\pm(it+a)\}$, so $\Delta(\mathfrak{q}) = \Delta(\mathfrak{l}) \cup \{it-a, 2it\}$.

Observe that $\Delta(\mathfrak{q}) \cap \tau(\Delta(\mathfrak{q})) = \{\pm(it - a)\}$ and $\tau(\Delta(\mathfrak{u})) \neq -\Delta(\mathfrak{u})$. Also, $\mathfrak{q}$ is not defined by a $\lambda_0 \in i\mathfrak{t}_0^*$ as in criterion (e) of the theorem. Thus the orbit of $\mathfrak{q}$ is not measurable.

Corollary 2.12. *For $\mathfrak{q}$ in some measurable orbit, $\tau(\mathfrak{q}) = \mathfrak{q}^{opp}$, the opposite parabolic.*

Thus we use the notation $\overline{\mathfrak{q}}$ for the parabolic which is both the opposite parabolic and the conjugate (with respect to τ) parabolic.

The following corollary shows that there are many measurable orbits.

Corollary 2.13. *All open orbits in Z are measurable in the case when rank$(\mathfrak{k})$ = rank$(\mathfrak{g})$ or in the case where $Z = X = G/(Borel)$.*

Proof. The first statement is clear since $\mathfrak{q}_z$ contains a compact Cartan subalgebra; in this case τ sends any root to its negative. The proof of the second statement is contained in the proof of Proposition 2.8. $\square$

With respect to some fixed Cartan involution we may say the following.

Corollary 2.14. *Suppose that $G_0(z) \subset Z$ is an open measurable orbit. Then we may choose the base point so that $K_0(z) = K(z)$ and $\mathfrak{q} = \mathfrak{q}_z$ is defined by some $\lambda \in i\mathfrak{t}_0^*$; that is, $\Delta(\mathfrak{q}) = \{\alpha \in \Delta : \langle \lambda_0, \alpha \rangle \geq 0\}$. Thus $\mathfrak{q}$ is θ–stable parabolic.*

Since $\mathfrak{l}$ is the centralizer in $\mathfrak{g}$ of an elliptic element of $\mathfrak{g}^*$ (and L is connected) we get the following.

Corollary 2.15. *Each measurable orbit is G_0–equivariantly diffeomorphic to an elliptic coadjoint orbit. Conversely, each elliptic coadjoint orbit is G_0–equivariantly diffeomorphic to a measurable orbit.*

There are several other consequences of Theorem 2.11. We do not include the proofs.

Corollary 2.16. *If one open orbit in Z is measurable, then all open orbits are measurable.*

Corollary 2.17. *Every measurable orbit has a G_0–invariant (possibly indefinite) Kähler metric.*

Since it is the measurable orbits we will consider, we make the following definition.

Definition 2.18. An open orbit which is measurable will be called a *flag domain*.

For a more detailed discussion of the orbit structure see [**32**].

2.3. Exercises.

(a) Show that if the open orbits in Z are measurable, then the number of open orbits is

$$W(G_0; H_0)\backslash W(\mathfrak{g}; \mathfrak{h})/W(\mathfrak{l}; \mathfrak{h}).$$

(b) Determine the number of open orbits of $SL(n, \mathbf{R})$ in $\mathbf{CP}(n-1)$.

(c) How does one count the open orbits in general?

LECTURE 3
Examples, Homogeneous Bundles

3.1. Examples of open orbits.

Example 3.1. Take $G_0 = GL(2n, \mathbf{R}) \subset G = GL(2n, \mathbf{C})$ and

$$\begin{aligned} Z &= \{n\text{–planes in } \mathbf{C}^{2n}\} \\ D &= \{z \in Z | z + \overline{z} = \mathbf{C}^{2n}\} \\ &= \{z \in Z | z \cap \overline{z} = 0\}. \end{aligned}$$

Then D is a G_0–orbit. The point

$$z_0 = \operatorname{span}\{e_1 + ie_{n+1}, \ldots, e_n + ie_{2n}\}$$

lies in D and has stabilizer $L_0 \simeq GL(n, \mathbf{C})$. This may be computed as follows. The complex structure

$$J_0(e_j) = \begin{cases} e_{n+j}, & \text{if } 1 \le j \le n \\ -e_{j-n}, & \text{if } n+1 \le j \le 2n \end{cases}$$

has z_0 (respectively $\overline{z_0}$) as $(-i)$–eigenspace (respectively i–eigenspace). Furthermore, $g(z_0) = z_0$ if and only if $gJ_0 = J_0 g$, for $g \in G_0$. So $Stab_{G_0}(z_0) \simeq GL(n, \mathbf{C})$. Thus $\dim(D) = \dim(Z)$ and D is open.

To compute the stabilizer in G, note that for

$$C = \begin{pmatrix} I & -iI \\ -iI & I \end{pmatrix}$$

we have that $C(z_0) = \operatorname{span}\{e_1, \ldots, e_n\}$ so that

$$Stab_G(z_0) = C^{-1} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} C,$$

a maximal parabolic subgroup of G. Now it is easy to see that Q contains a θ–stable Cartan subalgebra defined over $\mathbf{R}$ and that

$$K(z_0) = K_0(z_0) = \{\text{isotropic } n\text{–planes}\} \simeq SO(2n)/U(n).$$

The flag domain D may also be identified with the space of all complex structures on $\mathbf{R}^{2n}$. The identification is i–eigenspace of $J \leftrightarrow J$; the action is $g \cdot J = gJg^{-1}$.

The sets of roots along with the τ–action may be calculated by letting $\mathfrak{h}_0$ be the Cartan subalgebra

$$\begin{pmatrix} a_1 & & & t_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & t_n \\ -t_1 & & & a_1 & & \\ & \ddots & & & \ddots & \\ & & -t_1 & & & a_n \end{pmatrix},$$

which is a Cayley transform of the Cartan subalgebra

$$\begin{pmatrix} a_1 + it_1 & & & & & \\ & \ddots & & & & \\ & & a_n + it_n & & & \\ & & & a_1 - it_1 & & \\ & & & & \ddots & \\ & & & & & a_n - it_n \end{pmatrix}.$$

So

$$\begin{aligned} \Delta(\mathfrak{q}) &= \Delta(\mathfrak{l}) \cup \Delta(\mathfrak{u}) \\ &= \{(a_j - a_k) + i(t_j - t_k)\} \cup \{(a_j - a_k) + i(t_j + t_k), 2it_j\} \end{aligned}$$

and $\tau(\Delta(\mathfrak{u})) = -\Delta(\mathfrak{u})$. It follows that D is measurable.

Example 3.2. Let $G_0 = U(p,q) \subset G = Gl(n, \mathbf{C})$, $n = p + q$. Let $Z = \{m$–planes in $\mathbf{C}^n\}$. Suppose that $r + s + t = m$ and $0 \le t \le \min\{p - r, q - s\}$ (so that (r, s, t) is the signature of some m–plane). Then the set of m–planes of signature (r, s, t) is a G_0–orbit. The open orbits correspond to $t = 0$.

Consider $D_{r,s}$ with $r + s = m$, the open orbit of m–planes of signature (r, s). Let

$$z_0 = \mathrm{span}\{e_1, \dots, e_r, e_{p+1}, \dots, e_{p+s}\}.$$

Then $K(z_0) = K_0(z_0)$ may be written as

$$\begin{aligned} &\{z \in Z \,|\, \dim(z \cap (\mathbf{C}^p \times \{0\})) = r \text{ and } \dim(z \cap (\{0\} \times \mathbf{C}^q\})) = s\} \\ &\quad \simeq \mathbf{Gr}(r, \mathbf{C}^p) \times \mathbf{Gr}(s, \mathbf{C}^q). \end{aligned} \tag{3.3}$$

Some low dimensional examples:

$$\begin{aligned}
Z = \mathbf{CP}(n),\, G_0 &= U(p,q), p, q \geq 1 \\
D_+ &= \{\text{positive lines}\} \\
D_- &= \{\text{negative lines}\} \\
D_0 &= \{\text{null lines}\} \\
Z = \mathbf{Gr}(2, \mathbf{C}^n),\, G_0 &= U(p,q), p, q \geq 2 \\
D_{++} &= \{\text{planes of signature } (2,0,0)\} \\
D_{+-} &= \{\text{planes of signature } (1,1,0)\} \\
D_{--} &= \{\text{planes of signature } (0,2,0)\} \\
D_{+0} &= \{\text{planes of signature } (1,0,1)\} \\
D_{-0} &= \{\text{planes of signature } (0,1,1)\} \\
D_{00} &= \{\text{planes of signature } (0,0,2)\}
\end{aligned}$$

To see that these are in fact orbits, one may apply Witt's Theorem:

Theorem 3.4. *If V is a vector space with a nondegenerate form (either symmetric, antisymmetric or hermitian) and $T : W_1 \to W_2$ is an isometry, then T extends to an isometry of V.*

For a proof of Witt's Theorem see [**3**], Ch. 6, Section 5.

More generally if $Z = Z_{\tilde{m}}$, as in Lecture 1.1, and $\tilde{m} = (m_1, \dots, m_k)$ with various choices

$$\begin{aligned}
&\tilde{r} = (r_1, \dots, r_k), r_1 \leq \cdots \leq r_k, \\
&\tilde{s} = (s_1, \dots, s_k), s_1 \leq \cdots \leq s_k, \\
&\tilde{t} = (t_1, \dots, t_k), t_1 \leq \cdots \leq t_k, \\
&m_j = r_j + s_j + t_j,
\end{aligned}$$

the $U(p,q)$– orbits correspond to flags for which the m_j–planes have signatures (r_j, s_j, t_j).

Example 3.5. Let $G_0 = U(n,n) \cap Sp(n, \mathbf{C}) \subset Sp(n, \mathbf{C})$. Here we are looking at $\mathbf{C}^{2n}$ with the symplectic and hermitian forms having the following matrices with respect to the standard basis:

$$\omega : \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$
$$\langle\ ,\ \rangle : \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The group G_0 is isomorphic to $Sp(n, \mathbf{R})$. The term isotropic will refer to the symplectic form and the terms signature and null will refer to the hermitian form.

As in Lecture 1.1, let $Z_{\tilde{m}}$ be a space of isotropic flags. Let $\tilde{r}$ and $\tilde{s}$ be k–tuples of non-negative integers with $r_j + s_j = m_j$. Then

$$D_{\tilde{r},\tilde{s}} = \{z \in Z_{\tilde{m}} |\text{ the signature of } z_j \text{ is } (r_j, s_j)\}$$

is an open G_0–orbit. This follows from the following refinement of Witt's Theorem.

3.6. *If $V = \mathbf{C}^{2n}$, with symplectic and hermitian forms as above, and W_1 and W_2 are isotropic and have the same nondegenerate signature, then there is some $g \in G_0$ so that $gW_1 = W_2$.*

It follows that the open orbits are determined by signature. However, the lower dimensional orbits are not determined by signature alone. See exercise (a).

Example 3.7. Let $X_+ = G/KS_+$ when G_0 is an arbitrary group of hermitian type. Since $G_0 \cap KS_+ = K_0$, it follows that X_+ contains $B \cong G_0/K_0$ as an open orbit. (Openness follows from a dimension count or the condition that $\tau(\Delta(\mathfrak{s}_+)) = -\Delta(\mathfrak{s}_+)$.) The orbits of G_0 on X_+ may be described in terms of Cayley transforms.

G_0/K_0 is also an open orbit in $X_- = G/KS_-$. We let *barB* denote this realization of G_0/K_0. It turns out that $\overline{B}$ has the complex structure opposite to that of B.

3.2. Homogeneous vector bundles.

Let A be a Lie group and B a closed subgroup. It is well known that A/B has differentiable structure for which left translations $L_a : A/B \to A/B$ $(a \in A)$ are diffeomorphisms. A general principle is that if one has a B–invariant object, then one can define an A–invariant object on A/B. For example, suppose $\langle\ ,\ \rangle$ is a B–invariant metric on the tangent space $T_{eB}(A/B)$. Denoting the differential of L_a by ℓ_a, a metric on A/B may be defined by $\langle v, w\rangle_{aB} = \langle \ell_a^{-1}v, \ell_a^{-1}w\rangle$ for $v, w \in T_{aB}(A/B) = \ell_a T_{eB}(A/B)$. This is well–defined precisely because $\langle\ ,\ \rangle$ is B–invariant. A very useful application of this principal is to the construction of homogeneous vector bundles.

Definition 3.8. A homogeneous vector bundle is a vector bundle $\pi : \mathcal{V} \to A/B$ along with an action of A on $\mathcal{V}$ so that π is equivariant for A (i.e., sends fiber to fiber) and the restriction to each fiber is linear.

Note that if $\mathcal{V}$ is a homogeneous vector bundle, then the action of B preserves $V \cong \pi^{-1}(eB)$. In other words the fiber over eB is a representation of B. On the other hand if (σ, V) is a representation of B, then a homogeneous vector bundle $\mathcal{V}_\sigma \to A/B$ is constructed as follows. Set

$$\mathcal{V}_\sigma \cong (A \times V)/\sim$$

where the equivalence relation $\sim$ is defined by $(ab, v) \sim (a, \sigma(b)v)$. The map π is just the projection $[a, v] \to aB$. The sections of $\mathcal{V}_\sigma$ are functions of the form $F(aB) = [a, f(a)]$, which are well-defined if and only if $f(ab) = \sigma(b)^{-1}f(a)$ for all $a \in A, b \in B$. We will write the space of smooth sections as

$$C^\infty(A/B, \mathcal{V}_\sigma) = \{f : A \to V | f \in C^\infty(A) \text{ and } f(ab) = \sigma(b)^{-1}f(a), a \in A, b \in B\}.$$

An example of a homogeneous bundle is the tangent bundle. The action of A is given by $a \cdot (a_1B, v) = (L_a a_1 B, \ell_a v), v \in T_{a_1B}(A/B)$. Recall that the tangent space at an arbitrary point may be identified with a quotient of $\mathfrak{a}$ as follows. Define a map $\mathfrak{a} \to T_{aB}(A/B)$ by $\xi \to \tilde{\xi}$ where

$$\tilde{\xi}_a F = \frac{d}{dt} F(\exp(t\xi)a)|_{t=0}.$$

The kernel of this map is $\mathrm{Ad}(a)\mathfrak{b}$, thus $T_{aB}(A/B) \simeq \mathfrak{a}/\mathrm{Ad}(a)\mathfrak{b}$. It is easy to check that $T_{eB}(A/B) \simeq \mathfrak{a}/\mathfrak{b}$ as representations of B. Therefore, the tangent bundle is the homogeneous bundle for the B–representation $\mathfrak{a}/\mathfrak{b}$. Observe that we have given a map of $\mathfrak{a}$ into the vector fields on A/B, the action being the *left* action. These vector fields span the tangent space at each point. There is some connection between this left action and the right action. Functions $F \in C^\infty(A/B)$ correspond to functions $f \in C^\infty(A)$ satisfying $f(ab) = f(a)$ for all $b \in B$. Now, f is the lift of F to A and satisfies $f(a) = F(aB)$. Define

$$(r(\xi)f)(a) = \frac{d}{dt} f(a \exp(t\xi))|_{t=0}. \tag{3.9}$$

Then $dF_a(\tilde{\xi}) = \tilde{\xi}_a F = (r(\mathrm{Ad}(a^{-1})\xi)f)(a)$. So, for example, f is constant if and only if $\tilde{\xi} f = 0$ for all $\xi \in \mathfrak{a}$ if and only if $r(\xi)f = 0$ for all $\xi \in \mathfrak{a}$. Note however that the right translation of a right B–invariant function may not be right B–invariant.

3.3. Complex structure

There are several ways to see that a homogeneous space A/B is a complex manifold with an A–invariant complex structure. An A-invariant almost complex structure comes from a map $J : \mathfrak{a}/\mathfrak{b} \to \mathfrak{a}/\mathfrak{b}$ which is B–invariant and satisfying $J^2 = -I$. There is an integrability condition making the almost complex structure a complex structure. The Cauchy Riemann equations are $df(J(\xi)) = idf(\xi)$, all $\xi \in \mathfrak{a}$, for a smooth function f on A/B. Thus, a function $F : A/B \to \mathbf{C}$ is holomorphic if and only if $r(J(\xi))f = ir(\xi)f$.

A homogeneous space A/B has an A–invariant complex structure if A is a complex Lie group and B is a closed complex subgroup. In this case, the complex structure J is multiplication by i and from the Cauchy Riemann equations we have

$$F \text{ is holomorphic if and only if } r(i\xi)f = ir(\xi)f, \text{ for all } \xi \in \mathfrak{a}. \tag{3.10}$$

Now consider the case of a flag domain $D \subset Z$. As mentioned earlier, D has a G_0-invariant complex structure since it is an open submanifold of the complex manifold Z. We would like to describe this complex structure without reference to Z or G.

Let $p : G \to G/Q$ and $p_0 : G_0 \to G_0/L_0$ be the quotients. Let $\mathcal{O}_D$ be the sheaf of germs of holomorphic functions on D. For an open set $U \subset D$ we will give a description of $\mathcal{O}_D(U)$. If $F \in \mathcal{O}_D(U)$, then there are lifts of F to functions $f \in C^\infty(p^{-1}(U))$ and $f_0 \in C^\infty(p_0^{-1}(U))$. Note that $f_0 = f$ on $p_0^{-1}(U) \subset p^{-1}(U)$. We know that F is holomorphic if and only if f is holomorphic. But we want to give a condition on f_0 for F to be holomorphic.

There are *two* right actions of $\mathfrak{g}$ on $C^\infty(p^{-1}(U))$. First is the action as a *real* vector field as in (3.9). Second is as *complex* vector field:

$$r^c(\xi_1 + i\xi_2)f = r(\xi_1)f + ir(\xi_2)f, \text{ for } \xi_1, \xi_2 \in \mathfrak{g}_0.$$

The Cauchy Riemann equations become

$$F \text{ is holomorphic if and only if } r(\xi)f = r^c(\xi)f, \text{ for } \xi \in \mathfrak{g}. \tag{3.11}$$

Note that the right action as complex vector field is defined on $C^\infty(p_0^{-1}(U))$, but $r(\xi)$ is not unless $\xi \in \mathfrak{g}_0$.

Lemma 3.12. *If $U \subset D$ is an open set then*

$$\mathcal{O}_D(U) = \{\phi \in C^\infty(p_0^{-1}(U)) | r^c(\xi)\phi = 0, \xi \in \mathfrak{u}, \text{ and } \phi(g\ell) = \phi(g), \ell \in L_0\}. \tag{3.13}$$

The anti–holomorphic tangent space may be identified with $\mathfrak{u}$.

Proof. Suppose F is a holomorphic function on U. Then $r^c(\xi)f = r(\xi)f$ for $\xi \in \mathfrak{g}$ by (3.11). But this is 0 for $\xi \in \mathfrak{q}$ since f is right Q–invariant. So $f_0 = f|_{p_0^{-1}(U)}$ is in the r.h.s of (3.13).

Conversely, if ϕ is in the right hand side of (3.13), then ϕ defines a function F on U. Claim: $r(\xi)f = r^c(\xi)f$, for $\xi \in \mathfrak{g}$. Since $\mathfrak{g} = \mathfrak{q} + \tau(\mathfrak{q})$, by (2.4) it is enough to show that $r^c(\xi)f = 0$ for all $\xi \in \mathfrak{q}$. (To see this, for $\xi \in \mathfrak{g}_0$, write $\eta + i\xi \in \mathfrak{q}$. Then $r^c(\eta + i\xi)f = 0 = r(\eta + i\xi)f$, so $ir(\xi)f = r(i\xi)f$.) But $(r^c(\xi)f)(gq) = (r^c(\mathrm{Ad}(q)\xi)f)(g) = r^c(\mathrm{Ad}(q)\xi)\phi(g)$. So $r^c(\xi)f = 0$ for $\xi \in \mathfrak{q}$. □

The preceding discussion extends easily to finite–dimensional homogeneous vector bundles on D. Let E_χ be a finite–dimensional representation of $(L_0, \mathfrak{q})$, that is, a representation χ of L_0 and a representation of the Lie algebra $\mathfrak{q}$ agreeing with $d\chi$ on $\mathfrak{l}_0$. It follows that χ extends to a holomorphic representation of Q. There are corresponding homogeneous bundles on D and Z which we will denote by $\mathcal{E}_\chi$ and $\mathcal{E}_\chi^{\mathbf{C}}$ respectively. Thus there is a diagram

$$\begin{array}{ccc} \mathcal{E}_\chi & \longrightarrow & \mathcal{E}_\chi^{\mathbf{C}} \\ \downarrow & & \downarrow \\ D & \longrightarrow & Z \end{array}$$

Then

$$\begin{aligned} \mathcal{O}(\mathcal{E}_\chi)(U) = \{\phi : & p_0^{-1}(U) \to E_\chi | \phi \text{ is smooth}, \phi(g\ell) = \chi(\ell^{-1})\phi(g) \\ & \text{and } r^c(\xi)\phi + \chi(\xi)\phi = 0 \text{ for } \xi \in \mathfrak{u}\}. \end{aligned} \tag{3.14}$$

For an irreducible representation E_χ, $\chi(\mathfrak{u}) = 0$.

Note that for the bundle $\mathcal{E}_\chi$ we only need a representation of L_0, however, for a *holomorphic* bundle we need a representation of $(L_0, \mathfrak{q})$.

3.4. Holomorphic functions.

Let $D = G_0(z) = G_0/L_0$ be a flag domain in $Z = G/Q$. Assume that the base point z has been chosen as in Proposition 2.9, that is, so that $K(z) = K_0(z)$ is a compact complex subvariety of D. A very natural question to ask is which domains have nonconstant holomorphic functions.

By the maximum principle, compact submanifolds have no nonconstant holomorphic functions. Thus if $G_0 = G_u$, the compact real form of G, then $D = Z$ by Proposition 2.5 so there are no nonconstant holomorphic functions on D. At the other extreme, when $K(z) = \{z\}$, D is the bounded symmetric domain B (or $\overline{B}$)$\simeq G_0/K_0$. By the Harish–Chandra embedding $B \hookrightarrow \mathfrak{s}_+$ we may conclude that the holomorphic functions on $D = B$ separate points (since they include the pullbacks of all polynomials on $\mathfrak{s}_+$). If $\dim_{\mathbf{C}} K(z) \geq 1$, then the holomorphic functions cannot separate points.

In order to describe the situation in a clean way we assume that G is simple. Suppose that

$$\begin{aligned} & L_0 \subset K_0, \\ & G_0 \text{ is of hermitian type, and} \\ & \pi : G_0(z) \to B \text{ (or } \overline{B}\text{) is holomorphic.} \end{aligned} \tag{3.15}$$

Since the choice of $\mathfrak{s}_+$ is arbitrary, we assume $\pi : G_0(z) \to B$ is holomorphic. This means that $\mathfrak{s}_+ \subset \mathfrak{u}$ as π must map antiholomorphic tangent space to antiholomorphic tangent space. In this case, the holomorphic functions on $D = G_0(z)$ are precisely the pullbacks of holomorphic functions on B since all holomorphic functions are constant on $K(z)$.

Proposition 3.16. *Let D be an open orbit in $Z = G/Q$ with G simple. If (3.15) holds, then the holomorphic functions are the pullbacks of the holomorphic functions on B. If (3.15) does not hold, then the only holomorphic functions on D are the constants.*

Proof. The first statement is proved. For the second, it is enough to show that for any holomorphic function f on D

$$r(\xi)f = 0, \tag{3.17}$$

for all $\xi \in \mathfrak{g}$. Since f is right L_0–invariant, (3.17) holds for $\xi \in \mathfrak{l}$. Since f is constant on each compact subvariety $gK(z)$, (3.17) holds for all $\xi \in \mathfrak{k}$. As f is holomorphic, (3.17) holds for $\xi \in \mathfrak{u}$. Let $\mathfrak{m}$ be the set of all ξ so that (3.17) holds. Then $\mathfrak{m}$ is a subalgebra. There are only a few possibilities since $\mathfrak{m}$ contains $\mathfrak{k}$ and $\mathfrak{l} + \mathfrak{u}$. If G_0 is not of hermitian type, then $\mathfrak{k}$ is a maximal subalgebra. Since $\mathfrak{q} \not\subseteq \mathfrak{k}$ we must have $\mathfrak{m} = \mathfrak{g}$ and f is constant. If G_0 is of hermitian type, then $\mathfrak{m}$ is either $\mathfrak{k} + \mathfrak{s}_\pm$ or $\mathfrak{g}$. If $\mathfrak{m} = \mathfrak{k} + \mathfrak{s}_\pm$, then $\mathfrak{q} \subset \mathfrak{k} + \mathfrak{s}_\pm$, so $\mathfrak{l} \subset \mathfrak{k}$ and $\mathfrak{u} \subset \mathfrak{s}_\pm$ and (3.15) holds. $\square$

3.5. $(s+1)$–completeness.

It is well known that the bounded domains B are Stein domains. Stein domains are the domains in $\mathbf{C}^n$ for which the function theory is most reasonable to deal with. One key property is that the space of sections of a coherent sheaf is infinite–dimensional and higher cohomology vanishes ($H^m(B, \mathcal{F}) = 0, m > 0$). One definition of a Stein manifold is that there is an exhaustion function ϕ so that $\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}$ is positive definite at each point.

A generalization of this notion is that of an $(s+1)$–complete complex manifold. The manifold is required to have an exhaustion function so that $\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}$ has at most s nonpositive eigenvalues. Note that M is Stein if and only if it is 1–complete.

Theorem 3.18. *If M is an $(s+1)$–complete complex manifold, then $H^m(M, \mathcal{F}) = 0$ for $m > s$ and $\mathcal{F}$ a coherent sheaf.*

Now let $D = G_0(z)$ be a flag domain and let $s = \dim_{\mathbf{C}} K(z)$ (with base point z chosen as in Proposition 2.9).

Theorem 3.19. *D is $(s+1)$–complete.*

Corollary 3.20. *$K(z) = K_0(z)$ is a maximal compact subvariety of D.*

Proof. Let $Y \subset D$ be a compact subvariety of dimension m. Let $\mathcal{K}$ be the canonical bundle on Y, that is, $\mathcal{K} = \wedge^m (T^{(1,0)}Y)^*$. Therefore $i_*\mathcal{O}(\mathcal{K})$ is a coherent sheaf. Then $H^m(D, i_*(\mathcal{O}(\mathcal{K}))) \simeq H^m(Y, \mathcal{K}) \simeq \mathbf{C}$, so $m \leq s$. $\square$

Theorem 3.18 is proved in [**1**] and Theorem 3.19 is proved in [**31**].

3.6. Exercises.

(a) Determine all orbits of $Sp(2,\mathbf{R})$ on $\mathbf{CP}(3)$ and $\{$isotropic 2–planes in $\mathbf{C}^4\}$.
(b) Determine the orbit structure for $G_0 = SO_e(2,n)$ acting on X_+.
(c) Prove the refinement of Witt's theorem given in Example 3.5.
(d) Let A/B be a homogeneous space and (V,τ) be a representation of B. Show that if τ extends to a representation of A, then the homogeneous bundle $\mathcal{V}$ is the trivial bundle (as a smooth bundle).
(e) $\mathbf{CP}(n)$ is a complex flag variety for $G = SL(n+1,\mathbf{C})$. Write $\mathbf{CP}(n)$ as G/Q. The tautological bundle (fiber over a point (=line) is the line) and the canonical bundle $(= \wedge^{\mathrm{top}}(T^{(0,1)*}\mathbf{CP}(n))$ are homogeneous bundles. To which representations of Q do they correspond?
(f) In Example 3.5, determine $D_{\tilde{r},\tilde{s}}$ as a homogeneous space (i.e., write as G_0/L_0 and find L_0) and find $\xi \in \mathfrak{t}_0^*$ which defines the parabolic $\mathfrak{q}$ as (as in Corollary 2.14).

LECTURE 4
Dolbeault Cohomology, Bott-Borel-Weil Theorem

4.1. The cohomology space.

Let M be a complex manifold. Denote by $A^p(M)$ the space of smooth differential forms of type $(0,p)$, that is, the space of smooth sections of $\wedge^p(T^{(0,1)}M)^*$. Recall that the $\overline{\partial}$ operator is given by

$$\begin{aligned}
&\overline{\partial}_M : A^p(M) \to A^{p+1}(M)\\
&\overline{\partial}_M \omega(X_0,\dots,X_p) = \sum_{k=0}^{p}(-1)^k X_k \omega(X_0,\dots,\hat{X}_k,\dots,X_p)\\
&\qquad\qquad + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p)
\end{aligned}$$

for antiholomorphic vector fields X_j. If $\mathcal{V} \to M$ is a holomorphic vector bundle, then $A^p(M,\mathcal{V})$ denotes the space of smooth $\mathcal{V}$–valued forms of type $(0,p)$, i.e., smooth sections of $\mathcal{V}\otimes\wedge^p(T^{(0,1)}M)^*$. Then $\big(A^\bullet(M,\mathcal{V}), \overline{\partial} = 1_{\mathcal{V}}\otimes\overline{\partial}_M\big)$ is a complex:

$$\begin{aligned}
&\overline{\partial} : A^p(M,\mathcal{V}) \to A^{p+1}(M,\mathcal{V})\\
&\overline{\partial}^2 = 0.
\end{aligned}$$

The Dolbeault cohomology spaces are

$$H^p(M,\mathcal{V}) = ker(\overline{\partial})/im(\overline{\partial}).$$

For a finite–dimensional bundle and a compact manifold, it is known that the Dolbeault cohomology spaces are all finite dimensional. When M is not compact, the cohomology spaces are likely to be infinite–dimensional. One would like the cohomology spaces to have some natural structure of a topological vector space. If we give $A^p(M,\mathcal{V})$ the C^∞–topology, then $\overline{\partial}$ is a continuous operator. Thus $ker(\overline{\partial})$ (and in particular $H^0(M,\mathcal{V})$) is a Frechet space. However the image of $\overline{\partial}$ may not be a closed subspace, so the quotient may not even be Hausdorff. It turns out that the image of $\overline{\partial}$ is closed in the cases we are interested in.

Now let $M = D = G_0/L_0$ be a flag domain and $\mathcal{E}_\chi \to D$ a holomorphic homogeneous vector bundle corresponding to a representation of $(L_0,\mathfrak{q})$. Then the space of smooth forms of type $(0,p)$ is

$$A^p(D,\mathcal{E}_\chi) = \{\omega : G_0 \to \wedge^p\overline{\mathfrak{u}} \otimes E_\chi \,|\, \omega(g\ell) = \ell^{-1}\omega(g) \text{ for } \ell \in L_0\}$$

Here we have made the identification [1] $\overline{\mathfrak{u}} \simeq \mathfrak{u}^*$. It is also convenient to write this as

$$A^p(D,\mathcal{E}_\chi) = \{C^\infty(G_0) \otimes E_\chi \otimes \wedge^p\overline{\mathfrak{u}}\}^{L_0},$$

where $\{\cdots\}^{L_0}$ means L_0–invariants with L_0 acting on the right on $C^\infty(G_0)$.

Remark 4.1. The Dolbeault cohomology coincides with the sheaf cohomology $H^\bullet(D,\mathcal{O}(\mathcal{E}_\chi))$.

Our goal is to study the cohomology spaces $H^p(D,\mathcal{E}_\chi)$. The first step is to see that these spaces are continuous representations of G_0. Since $\overline{\partial}$ is a G_0–invariant operator it is clear that G_0 acts on the cohomology spaces. The problem is that it is not clear that the image of $\overline{\partial}$ is closed (making $H^p(D,\mathcal{E}_\chi)$ a Frechet space). It turns out that under a positivity condition on $\mathcal{E}_\chi$ the representations $H^s(D,\mathcal{E}_\chi)$ are irreducible and are maximal globalizations of cohomologically induced representations. The cohomologically induced representations are also unitarizable. Therefore the unitary realization sits inside $H^s(D,\mathcal{E}_\chi)$. We will give some methods for trying to understand this unitary structure. To give an idea of the type of results we are after we will begin with compact groups.

4.2. Bott-Borel-Weil Theorem

Assume for this section that G_0 is a compact real form of G. Let $Z = G/Q$ be a complex flag manifold. Then G_0 acts transitively on Z. So $Z = G_0/L_0$. The Bott-Borel-Weil Theorem tells us how to compute the Dolbeault cohomology spaces of homogeneous bundles. Let Δ^+ be a positive system of roots containing $\Delta(\overline{\mathfrak{u}}) = -\Delta(\mathfrak{u})$. Suppose that $\chi \in \mathfrak{h}^*$ is dominant for $\Delta^+(\mathfrak{l}) = \Delta^+ \cap \Delta(\mathfrak{l})$. Let E_χ be the irreducible representation of L_0 with highest weight χ. Extending this representation to a representation of Q with U acting trivially defines a holomorphic homogeneous bundle $\mathcal{E}_\chi \to Z$. Define

$$n(\chi) = \big|\{\beta \in \Delta^+ : \langle\chi+\rho,\beta\rangle < 0\}\big|$$

Then the Bott-Borel-Weil Theorem can be stated as follows.

Theorem 4.2. *If $\chi+\rho$ is singular, then $H^p(Z,\mathcal{E}_\chi) = 0$ for all p. Otherwise there is a unique $w \in W(\mathfrak{g},\mathfrak{h})$ so that $w(\chi+\rho)$ is dominant. Then $H^p(Z,\mathcal{E}_\chi) = 0$ for $p \neq n(\chi)$ and $H^{n(\chi)}(Z,\mathcal{E}_\chi)$ is the irreducible representation of highest weight $w(\chi+\rho)-\rho$.*

Here is a sketch of part of the proof. Suppose $Z = X = G/B$, the full flag variety. So $Z = G_0/H_0$, with H_0 a Cartan subgroup. Also $\Delta^+ = -\Delta(\mathfrak{u})$. The sections may be computed as follows. In this case, there is no dominance condition on χ and E_χ is a one dimensional character of H_0. We will denote the corresponding bundle by $\mathcal{L}_\chi \to Z$.

[1] The Killing form identifies $\mathfrak{u}$ with $\mathfrak{u}^{\text{opp}}$. However, since the orbit is measurable, $\mathfrak{u}^{\text{opp}} \simeq \tau(\mathfrak{u})$ which we call $\overline{\mathfrak{u}}$.

Recall the Peter-Weyl Theorem. Denote by F_μ the irreducible representation of G_0 with highest weight μ. Then

$$L^2(G_0) = \bigoplus_\mu F_\mu \otimes F_\mu^* \text{ (Hilbert Space sum).} \tag{4.3}$$

It is a fact from manifold theory that the cohomology spaces for a compact manifold are finite–dimensional. Therefore

$$H^0(X, \mathcal{L}_\chi) \subset \bigoplus_\mu F_\mu \otimes F_\mu^* \text{ (Algebraic direct sum).}$$

By (3.14) the holomorphic sections are

$$\begin{aligned} H^0(X,\mathcal{L}_\chi) &= \{C^\infty(G_0) \otimes \mathbf{C}_\chi\}^{H_0,\mathfrak{u}} \\ &= \bigoplus_\mu F_\mu \otimes \{F_\mu^* \otimes \mathbf{C}_\chi\}^{H_0,\mathfrak{u}} \\ &= \bigoplus_\mu F_\mu \otimes \{(F_\mu/\mathfrak{u}F_\mu)^* \otimes \mathbf{C}_\chi\}^{H_0} \\ &= \bigoplus_\mu F_\mu \otimes \mathrm{Hom}_{H_0}(F_\mu/\mathfrak{u}F_\mu, \mathbf{C}_\chi). \end{aligned}$$

Since $F_\mu/\mathfrak{u}F_\mu$ is the highest weight space of F_μ it has weight μ. So F_μ occurs in $H^0(Z, \mathcal{L}_\chi)$ if and only if $\mathrm{Hom}_{H_0}(\mathbf{C}_\mu, \mathbf{C}_\chi) \neq 0$ if and only if $\mu = \chi$. Thus if χ is dominant, then $H^0(X, \mathcal{L}_\chi) = F_\chi$, the representation with highest weight χ. Otherwise $H^0(X, \mathcal{L}_\chi) = 0$.

The calculation of sections for an arbitrary Z is essentially the same. We will omit the proof for the higher degree cohomology; see [**12**].

Note that the holomorphic sections can be written in terms of matrix coefficients as follows. Let $t \in \mathrm{Hom}_{H_0}(F_\mu/\mathfrak{u}F_\mu, \mathbf{C}_\chi)$ be nonzero. Then for $v \in F_\chi$, $\phi_v(g) = t(g^{-1}v)$ defines a holomorphic section. Furthermore, $v \to \phi_v$ is an isomorphism of F_χ onto $H^0(X, \mathcal{L}_\chi)$.

Another proof of Theorem 4.2 can be given using Lie algebra cohomology. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and V be a representation of $(L_0, \mathfrak{q})$. Then the Lie algebra cohomology $H^\bullet(\mathfrak{u}, V)$ is computed by the following complex.

$$\begin{aligned} &(\mathrm{Hom}_{L_0}(\wedge^\bullet \mathfrak{u}, V), \delta) \text{ with} \\ &\delta(T)(\xi_0, \dots, \xi_p) = \sum_{k=0}^p (-1)^k \xi_k T(\xi_0, \dots, \hat{\xi}_k, \dots, \xi_p) \\ &\qquad + \sum_{i<j} (-1)^{i+j} T([\xi_i, \xi_j], \xi_0, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_p). \end{aligned} \tag{4.4}$$

Taking $V = C^\infty(G_0)$ we get an isomorphism

$$A^p(D, \mathcal{E}_\chi) \simeq \mathrm{Hom}_{L_0}(\wedge^p \mathfrak{u}, C^\infty(G_0) \otimes E_\chi)$$

The isomorphism is given as follows. Let T be in the right hand side. Then a form ω is given by

$$\omega(g)(\xi_1, \dots, \xi_p) = T(\xi_1 \wedge \dots \wedge \xi_p)(g).$$

The operators $\overline{\partial}$ and δ correspond. It follows that

$$H^p(Z,\mathcal{E}_\chi) \simeq \bigoplus_\mu F_\mu \otimes \{H^p(\mathfrak{u},F_\mu^*)\otimes E_\chi\}^{L_0}.$$

This may be viewed as a Frobeneous Reciprocity:

$$\operatorname{Hom}_{G_0}(F,H^p(Z,\mathcal{E}_\chi)) \simeq \operatorname{Hom}_{L_0}(H^p(\mathfrak{u},F^*)^*,E_\chi). \tag{4.5}$$

A theorem of Kostant [**23**] computes the Lie algebra cohomology. Then (4.5) is used to conclude Theorem 4.2.

The Hodge Theorem for compact complex manifolds says that each cohomology class is represented by a unique harmonic form. For this we note that Z has a G_0–invariant positive definite hermitian metric. This metric defines a formal adjoint $\overline{\partial}^*$ of $\overline{\partial}$. The Laplace-Beltrami operator is $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. Then the harmonic forms are the solutions of $\Box\omega = 0$. Since $\Box$ is elliptic, the solutions are smooth forms. Thus the Hodge Theorem says that $H^p(Z,\mathcal{E}_\chi) \simeq ker(\Box)$. As Z is compact the harmonic forms are square integrable. Since the metric is G_0–invariant the L^2–inner product defines the unitary structure on $ker(\Box)$. We conclude that each irreducible *unitary* representation of the compact group G_0 can be realized as some $H^p(X,\mathcal{L}_\chi)$. Note that any given degree p may be used to realize all representations..

Remark 4.6. As G_0 is compact, any representation V has an invariant inner product. One can define the formal adjoint of δ by

$$\begin{aligned} &\delta^* : \operatorname{Hom}_{L_0}(\wedge^p\mathfrak{u},V) \to \operatorname{Hom}_{L_0}(\wedge^{p-1}\mathfrak{u},V)\\ &\langle\delta^*(T),S\rangle = \langle T,\delta(S)\rangle. \end{aligned}$$

Here we are identifying $\operatorname{Hom}_{L_0}(\wedge^p\mathfrak{u},V)$ with $\{\wedge^p\mathfrak{u}^*\otimes V\}^{L_0}$ and giving it the inner product coming from the metric on $\mathfrak{u}$ and the invariant inner product on V. Then the formal harmonic space $\mathcal{H}^p(V)$ is defined as the kernel of $\delta^*\delta+\delta\delta^*$. It is not too difficult to show that $H^p(\mathfrak{u},V)\simeq\mathcal{H}^p(V)$. So one may compute the formal harmonic space instead of the cohomology. Then

$$ker(\Box) = \bigoplus_\mu F_\mu \otimes \{\mathcal{H}^p(F_\mu^*)\otimes E_\chi\}^{L_0}. \tag{4.7}$$

Thus, if one can give an explicit expression for an element of the formal harmonic space, then a harmonic form on Z may be given (in terms of matrix coefficients).

4.3. Holomorphic discrete series.

Another example where one can realize unitary representations in cohomology is the holomorphic discrete series. Here G_0 is a group of hermitian type. So $B = G_0/K_0$ is an open orbit in the flag variety X_+ (as in Example 3.7). Let E_χ be the irreducible representation of K_0 with highest weight χ. The following hold.

(a) B has a G_0–invariant positive hermitian metric.
(b) As B is Stien, cohomology occurs only in degree 0.
(c) If $\langle\chi+\rho,\beta\rangle > 0$ for all $\beta\in\Delta^+$, then $H^0(B,\mathcal{E}_\chi)$ is irreducible and there exist square integrable holomorphic sections.
(d) The space of L^2 holomorphic sections is a unitary representation.

For (c) and (d) see [**16**].

4.4. Exercises.

(a) Compute the holomorphic sections of the homogeneous line bundles on $\mathbf{CP}(n)$. How would you compute the smooth sections?

(b) Use the Bott–Borel–Weil Theorem to compute the holomorphic vector fields on each Z. (The holomorphic tangent space is the homogeneous bundle for $\mathfrak{g}/\mathfrak{q}$.) Hint: In general $\mathfrak{g}/\mathfrak{q}$ is not an irreducible $\mathfrak{q}$–representation, so in order to apply the Bott–Borel–Weil Theorem the standard trick is to filter $\mathfrak{g}/\mathfrak{q}$ by $\mathfrak{q}$–representations with irreducible quotients and apply the corresponding spectral sequence. Some case by case checking will be necessary. Note that differentiating the action of G gives holomorphic vector fields. Thus, the space of holomorphic vector fields contains $\mathfrak{g}$. In most cases all holomorphic vector fields come from the G–action. In the other cases there is in fact a bigger group acting holomorphically on Z.

(c) Use the Bott–Borel–Weil Theorem to derive a formula for the decomposition into irreducibles of the tensor product of two finite–dimensional irreducible representations.

(d) Prove (for G_0 compact) $\mathcal{H}^p(V) \simeq H^p(\mathfrak{u}, V)$.

LECTURE 5
Indefinite Harmonic Theory

For possible generalizations to noncompact groups we would like to focus on the following statement.

5.1. *Suppose G_0 is a compact group and $Z = G/Q = G_0/L_0$ a complex flag variety. Choose a positive system $\Delta^+ \supset \Delta(\overline{\mathfrak{u}})$ and set $\Delta^+(\mathfrak{l}) = \Delta^+ \cap \Delta(\mathfrak{l})$. Let E_χ be the irreducible finite–dimensional representation of L_0 with highest weight χ. Then the cohomology $H^p(Z, \mathcal{E}_\chi)$ vanishes in all but one degree. In that one degree the cohomology is an irreducible representation and each cohomology class is represented by a harmonic form. Note that Z has a positive definite G_0–invariant hermitian metric allowing us to define the formal adjoint $\overline{\partial}^*$ of $\overline{\partial}$, thus defining the Laplace-Beltrami operator $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, an elliptic operator. The L_2–inner product on ker$\{\Box\}$ gives a unitary realization. All irreducible unitary representations are obtained this way.*

The question is how to generalize this to noncompact groups G_0. Let's fix a Cartan involution θ and a flag domain $D \subset Z$. By (2.9) we may choose a base point $z \in D$ so that $\mathfrak{q}_z = \mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ contains a θ–stable Cartan subalgebra defined over $\mathbf{R}$. Then $K(z) = K_0(z)$ is a maximal compact subvariety of D. Attempts to generalize 5.1 encounter numerous obstacles. We will spend some time discussing some of these obstacles. To simplify matters slightly let us assume that χ is a unitary character of L_0. So E_χ is a one dimensional representation of L_0 and we will write the corresponding homogeneous bundle as $\mathcal{L}_\chi$.

5.1. Which $H^p(D, \mathcal{L}_\chi)$ are the interesting representations?

Which degree of cohomology and which parameters χ should we consider? This is fairly well understood. Since our goal is to realize irreducible unitary representations, the following two theorems say we should concentrate on cohomology in degree $s \equiv \dim_{\mathbf{C}}(K(z))$. It is not clear exactly which χ we should use, but $\chi + \rho$ dominant for $\Delta(\mathfrak{u})$ is a good place to start.

In order to state the theorems we will let $\mathcal{R}^p_{\mathfrak{q}}(\mathbf{C}_{\tilde{\chi}})$ be the cohomologically induced representation defined in [**21**], Section 4.11. We will set $\tilde{\chi} = \chi - 2\rho(\mathfrak{u})$; the shift makes things line up a little better with the cohomology representations.

Theorem 5.2. *In the above setting*
(1) $H^p(D, \mathcal{L}_\chi)$ is an admissible representation. In fact, it is a maximal globalization

in the sense of [**29**]. *The underlying Harish-Chandra module is* $\mathcal{R}^p_{\mathfrak{q}}(\mathbf{C}_{\tilde{\chi}})$.
(2) If $\langle \chi + \rho, \beta \rangle > 0$ *for all* $\beta \in \Delta(\mathfrak{u})$*, then* $H^p(D, \mathcal{L}_\chi) = 0$ *if* $p \neq s$ *and is an irreducible representation if* $p = s$.

A few comments are in order. There is a lot known about the cohomologically induced representations. For example, the vanishing and irreducibility statements in part (2) follow directly from part (1) and the corresponding statements about the $\mathcal{R}^p_{\mathfrak{q}}(\mathbf{C}_{\tilde{\chi}})$. A detailed study of the $\mathcal{R}^p_{\mathfrak{q}}(\mathbf{C}_{\tilde{\chi}})$ shows that the condition in (2) may be weakened somewhat. Also, there are formulas for the K_0–types of the $\mathcal{R}^p_{\mathfrak{q}}(\mathbf{C}_{\tilde{\chi}})$. Part (1) of the theorem says that these results hold for the corresponding Dolbeault cohomology spaces.

Part (1) is proved by doing essentially three things. First, for a very $\Delta(\mathfrak{u})$–dominant parameter χ, $H^s(D, \mathcal{L}_\chi)$ is embedded into $C^\infty(G_0/K_0, \mathcal{E}_{\chi'})$ for some homogeneous bundle $\mathcal{E}_{\chi'}$, and is identified with the kernel of a differential operator. It follows that the image of $\overline{\partial}$ is closed and $H^s(D, \mathcal{L}_\chi)$ is a Frechet space. This embedding is denoted by $\mathcal{P} : H^s(D, \mathcal{L}_\chi) \to C^\infty(G_0/K_0, \mathcal{E}_{\chi'})$ and is the topic of [**6**], Lecture 2. The second step is to make a strong connection between the complexes that define $H^\bullet(D, \mathcal{L}_\chi)_{K_0\text{–finite}}$ and $\mathcal{R}^\bullet_{\mathfrak{q}}(\mathbf{C}_{\tilde{\chi}})$. Finally a tensoring argument is given to obtain the result for arbitrary p and χ. (For the tensoring argument one actually needs to work in the context of finite–dimensional bundles instead of line bundles.)

Theorem 5.3. *If* $\langle \chi + \rho, \beta \rangle > 0$ *for all* $\beta \in \Delta(\mathfrak{u})$*, then* $\mathcal{R}^s_{\mathfrak{q}}(\mathbf{C}_{\tilde{\chi}})$ *is unitarizable.*

This theorem, along with the fact that $H^s(D, \mathcal{L}_\chi)$ is a maximal globalization, says that the unitary globalization lies inside $H^s(D, \mathcal{L}_\chi)$. We remark that a maximal globalization has the property that any other globalization embeds continuously into it. Again, the positivity condition can be weakened somewhat.

The cohomologically induced representations account for a large part of the unitary dual of G_0. They play an important role in harmonic analysis on homogeneous spaces. For example they often occur in the L_2–space of semisimple symmetric spaces and G_0/Γ.

Theorem 5.2 was proved in [**35**]. Significant special cases and related work can be found in [**28**] and [**30**]. A proof of Theorem 5.3 can be found in [**21**].

5.2. The Metric.

A flag domain D does not often have a positive definite G_0–invariant metric. The situation is given in the following lemma.

Lemma 5.4. *Let* $D = G_0/L_0$ *be a flag domain. Then*
(1) D *has a* G_0*–invariant (possibly indefinite) hermitian metric. This metric is defined in terms of the Killing form.*
(2) If L_0 *is compact, then* D *has a positive definite* G_0*–invariant metric.*

Proof. Let B denote the Killing form of $\mathfrak{g}$. For part (1) set $\langle \xi, \eta \rangle = B(\xi, \tau(\eta))$, for $\xi, \eta \in \mathfrak{u}$, the holomorphic tangent space. B is nondegenerate on $\mathfrak{g}$ and $\mathfrak{l}$, and $\mathfrak{g}$ is the orthogonal direct sum of $\mathfrak{l}$ and $\mathfrak{l}^\perp = \mathfrak{u} + \overline{\mathfrak{u}}$, so B is nondegenerate on $\mathfrak{u} + \overline{\mathfrak{u}}$. It follows that $\langle\ ,\ \rangle_{\text{inv}}$ is a nondegenerate hermitian form on $\mathfrak{u}$. Since $\langle \xi, \eta \rangle$ is L_0–invariant there is a well-defined metric on each complexified tangent space of D. We denote this metric by $\langle\ ,\ \rangle_{\text{inv}}$. The signature is (a, b) where $a = \dim(\mathfrak{s} \cap \mathfrak{l}^\perp)$ and $b = \dim(\mathfrak{k} \cap \mathfrak{l}^\perp)$. Note that b, the negative part of the signature, is the integer s from Section 5.2.

For part (2) note that $-B(\xi, \theta\tau(\eta))$ is positive definite, but is only invariant under the compact real form. We may define $\langle \xi, \eta \rangle_{\text{pos}} = -B(\xi, \theta\tau(\eta))$ on $\mathfrak{l}^{\perp}$. This is L_0–invariant if and only if L_0 is compact, in which case it defines a positive definite hermitian form on D. $\square$

Remark 5.5. If L_0 is compact, then the representations $H^s(D, \mathcal{L}_\chi)$ are in the discrete series. Our goal is to understand a wider class of representations.

5.3. Strongly harmonic forms.

Consider an arbitrary flag domain D with the G_0–invariant metric $\langle\ ,\ \rangle_{\text{inv}}$. This is ussually indefinite. The metric defines an L_0–invariant metric[1] on $\wedge^p \mathfrak{u}^* \simeq \wedge^p \overline{\mathfrak{u}}$. Since $\mathbf{C}_\chi$ is unitary (so $\mathcal{L}_\chi$ is a hermitian line bundle), there is an L_0–invariant metric on $\wedge^p \overline{\mathfrak{u}} \otimes \mathbf{C}_\chi$, which we again denote by $\langle\ ,\ \rangle_{\text{inv}}$. Note that $D = G_0/L_0$ has a G_0–invariant measure since D is *measurable* orbit.

The formal adjoint of $\overline{\partial}$ is defined by

$$\int_{G_0/L_0} \langle \overline{\partial}\omega_1(g), \omega_2(g) \rangle_{\text{inv}}\, dg = \int_{G_0/L_0} \langle \omega_1(g), \overline{\partial}^* \omega_2(g) \rangle_{\text{inv}}\, dg$$

for compactly supported forms.

The Laplace–Beltrami operator is

$$\square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : A^p(D, \mathcal{L}_\chi) \to A^p(D, \mathcal{L}_\chi)$$

It follows from the invariance of the metric that $\overline{\partial}^*$ and $\square$ are G_0–invariant operators. $\square$ is usually not elliptic.

For us the correct definition of harmonic is the following.

Definition 5.6. A form $\omega \in A^p(D, \mathcal{L}_\chi)$ *strongly harmonic* if $\overline{\partial}\omega = 0$ and $\overline{\partial}^*\omega = 0$. We let $\mathcal{H}^p(D, \mathcal{L}_\chi)$ denote the space of strongly harmonic forms of type $(0, p)$.

Note that $\mathcal{H}^p(D, \mathcal{L}_\chi) \subset \operatorname{Ker}\{\square\}$ and both are Frechet representations of G_0.

Define

$$\langle \omega_1, \omega_2 \rangle_{\text{inv}} = \int_{G_0/L_0} \langle \omega_1(g), \omega_2(g) \rangle_{\text{inv}}\, dg. \tag{5.7}$$

Note that

$$\begin{aligned} \langle \square\omega, \omega \rangle_{\text{inv}} &= \langle \overline{\partial}^*\overline{\partial}\omega, \omega \rangle_{\text{inv}} + \langle \overline{\partial}\,\overline{\partial}^*\omega, \omega \rangle_{\text{inv}} \\ &= \langle \overline{\partial}\omega, \overline{\partial}\omega \rangle_{\text{inv}} + \langle \overline{\partial}^*\omega, \overline{\partial}^*\omega \rangle_{\text{inv}}. \end{aligned}$$

If the invariant form happens to be positive definite (i.e., if $L_0 = K_0$), then a form is strongly harmonic if and only if it is harmonic. An advantage that the notion of strongly harmonic has over the usual notion of harmonic is that there is a well–defined map

$$q : \mathcal{H}^p(D, \mathcal{L}_\chi) \to H^p(D, \mathcal{L}_\chi). \tag{5.8}$$

[1] $\langle\ ,\ \rangle_{\text{inv}}$ is nondegenerate on $\mathfrak{u}$. To define a form on $\wedge^p \overline{\mathfrak{u}}$ one may choose any orthonormal basis of $\overline{\mathfrak{u}}$, say $\{\overline{\xi}_j\}$, and *define* $\{\overline{\xi}_{j_1} \wedge \cdots \wedge \overline{\xi}_{j_p}\}$ to be an orthonormal basis of $\wedge^p \overline{\mathfrak{u}}$ (for $j_1 < \cdots < j_p$).

5.4. The Hilbert Space.

There is not much chance of using an indefinite form to define a Hilbert space, so we will use an auxiliary metric to do this. The auxiliary metric is *not G_0–invariant*, but it is positive definite.

Begin with the positive definite hermitian form $\langle \xi, \eta \rangle_{\text{pos}} = B(\xi, \theta\tau(\eta))$ on $\mathfrak{u} + \overline{\mathfrak{u}} = (T_{eL_0}(G_0/L_0))_{\mathbf{C}}$. As $\langle\ ,\ \rangle_{\text{pos}}$ is not L_0–invariant we cannot simply translate to an arbitrary tangent space by defining $\langle \ell_g \xi, \ell_g \eta \rangle_{\text{pos}} = \langle \xi, \eta \rangle_{\text{pos}}$ at gL_0. However, there is a way to choose coset representatives in a special way which will allow us to translate to an arbitrary coset in a well–defined way. The following decomposition of Mostow ([**25**]) is exactly what we want.

Proposition 5.9. *Writing $\mathfrak{l}_0^\perp = \{\xi \in \mathfrak{g}_0 \mid B(\xi, \eta) = 0$ for all $\eta \in \mathfrak{l}_0\}$ we have the following decomposition of G_0,*

$$G_0 = K_0 \exp(\mathfrak{s}_0 \cap \mathfrak{l}_0^\perp) \exp(\mathfrak{s}_0 \cap \mathfrak{l}_0).$$

We will use the following notation for the decomposition of an element of G_0:

$$\begin{aligned} &g = k(g) \exp(X(g)) \exp(Y(g)), \text{ with} \\ &\qquad k(g) \in K_0, X(g) \in \mathfrak{s}_0 \cap \mathfrak{l}_0^\perp, Y(g) \in \mathfrak{s}_0 \cap \mathfrak{l}_0. \end{aligned}$$

Define the auxiliary positive metric on the tangent space at $g \cdot z$ by

$$\langle \ell_{k(g)\exp(X(g))}(\xi), \ell_{k(g)\exp(X(g))}(\eta) \rangle_{\text{pos}} = \langle \xi, \eta \rangle_{\text{pos}}, \text{ for } \xi, \eta \in \mathfrak{l}^\perp.$$

Lemma 5.10. *The positive metric is well–defined and K_0–invariant.*

Proof. Well–defined follows from the fact that

$$k(g\ell)\exp(X(g\ell)) = k(g)\exp(X(g))\ell_1, \text{ with } \ell_1 \in L_0 \cap K_0.$$

Invariance under K_0 follows from $k(k_1 g) = k_1 k(g)$ for $k_1 \in K_0$ and invariance of $B(\xi, \theta\tau(\eta))$ under K_0. □

For smooth forms of compact support set

$$\langle \omega_1, \omega_2 \rangle_{\text{pos}} = \int_{G_0/L_0} \langle \omega_1(k(g)\exp(X(g))), \omega_2(k(g)\exp(X(g))) \rangle_{\text{pos}}\, dg$$

Now define

$$\begin{aligned} &L_2^{(0,p)}(D, \mathcal{L}_\chi) : \text{completion of compactly supported smooth} \\ &\qquad (0,p)\text{–forms with respect to } \langle\ ,\ \rangle_{\text{pos}}. \end{aligned}$$

Lemma 5.11. *([**27**]) If $\omega \in L_2^{(0,p)}(D, \mathcal{L}_\chi)$, then $L_g \omega \in L_2^{(0,p)}(D, \mathcal{L}_\chi)$. The left action of G_0 defines a continuous Hilbert space representation on $L_2^{(0,p)}(D, \mathcal{L}_\chi)$.*

Lemma 5.12. *If $\omega \in L_2^{(0,p)}(D, \mathcal{L}_\chi)$, then $\langle \omega, \omega \rangle_{\text{inv}} < \infty$.*

Proof. On $\mathfrak{u} + \overline{\mathfrak{u}}$

$$\left|\langle \xi_1, \xi_2 \rangle_{\text{inv}}\right| = \left|\langle \xi_1, \theta(\xi_2) \rangle_{\text{pos}}\right| \le \left|\xi_1\right|_{\text{pos}} \left|\theta(\xi_2)\right|_{\text{pos}} = \left|\xi_1\right|_{\text{pos}} \left|\xi_2\right|_{\text{pos}}.$$

It follows that $\left|\langle \omega_1(g), \omega_2(g) \rangle_{\text{inv}}\right| \le \left|\omega_1(g)\right|_{\text{pos}} \left|\omega_2(g)\right|_{\text{pos}}$ for each $g \in G_0$. □

5.5. Indefinite Harmonic Theory.

The goal is to give a unitary globalization of $H^s(D, \mathcal{L}_\chi)_{K\text{–finite}}$ (under a positivity condition on the bundle) by giving a Hilbert space of L_2–harmonic forms representing cohomology classes.

Definition 5.13. The L_2–harmonic space is

$$\mathcal{H}_2^s = \mathcal{H}_2^s(D, \mathcal{L}_\chi) = \{\omega \in L_2^{(0,s)}(D, \mathcal{L}_\chi) : \overline{\partial}\omega = 0 \text{ and } \overline{\partial}^*\omega = 0 \text{ as distributions}\}.$$

One may calculate Dolbeault cohomology by using the complex of distribution valued $(0,p)$–forms in place of the smooth forms. Therefore, there is a natural map

$$q : \mathcal{H}_2^s \to H^s(D, \mathcal{L}_\chi) \tag{5.14}$$

There is a satisfactory quantization procedure (which we will refer to as *indefinite quantization* or an *indefinite harmonic theory*) if the following statements hold.

(a) $\mathcal{H}_2^s$ is nonzero.
(b) The natural quotient map q of (5.14) is nonzero.
(c) The invariant form (5.7) has null space exactly equal to the image of $\overline{\partial}$.
(d) $\langle\ ,\ \rangle_{\text{inv}}$ is positive semidefinite on $\mathcal{H}_2^s$.

In case (a)-(d) hold, then the invariant form is defined and positive definite on

$$\overline{\mathcal{H}_2^s} = \mathcal{H}_2^s / \{\text{nullspace of } \langle\ ,\ \rangle_{\text{inv}}\}. \tag{5.15}$$

(e) $\overline{\mathcal{H}_2^s}$ is complete with respect to $\langle\ ,\ \rangle_{\text{inv}}$.

If these five conditions hold, then $\overline{\mathcal{H}_2^s}$ is a unitary representation infinitesimally equivalent to $H^s(D, \mathcal{L}_\chi)$.

More details on this setup can be found in [**27**].

LECTURE 6
Intertwining Operators I

We will show that cohomology classes are represented by strongly harmonic forms. This is accomplished by constructing intertwining maps from principal series representations into the spaces of harmonic forms. The intertwining operator is an explicit integral formula which resembles the classical Poisson and Szegö maps. We will begin by giving a general method for constructing intertwining maps and seeing how this method applies to some interesting examples.

6.1. Intertwining maps between induced representations.

Let A be a finite group and B_1 and B_2 two subgroups. For $j = 1, 2$, let σ_j be representations of B_j acting on the vector spaces V_{σ_j}. Then

$$\mathrm{Ind}_{B_j}^{A}(\sigma_j) = \{f : A \to V_{\sigma_j} \mid f(ab) = \sigma(b^{-1})f(a) \text{ for } b \in B_j\}.$$

This is the space of sections of the homogeneous vector bundle $\mathcal{V}_{\sigma_j} \to A/B_j$.

Then intertwining maps

$$\mathrm{Ind}_{B_1}^{A}(\sigma_1) \to \mathrm{Ind}_{B_2}^{A}(\sigma_2)$$

may be constructed as follows. Choose $t \in \mathrm{Hom}_{B_1 \cap B_2}(V_{\sigma_1}, V_{\sigma_2})$ and set

$$(Tf)(a) = \int\limits_{B_2/B_1\cap B_2} \sigma_2(b)t(f(ab))\,db. \tag{6.1}$$

Of course, for finite groups integration is just summation. Thus, we have a map

$$\mathrm{Hom}_{B_1\cap B_2}(V_{\sigma_1}, V_{\sigma_2}) \to \mathrm{Hom}_A(\mathrm{Ind}(V_{\sigma_1}), \mathrm{Ind}(V_{\sigma_2})). \tag{6.2}$$

For Lie groups (or topological groups) this doesn't always work. For instance, there may not be an invariant measure on the homogeneous space $B_2/B_1 \cap B_2$. Even if the invariant measure exists, the integrals may not converge. In any case, we will use (6.1) as a guiding principle.

6.2. Some remarks on the principal series representations.

The applications of (6.1) we have in mind will all map from principal series representations into the space we want to study. We consider the principal series to be fairly well understood. Here is a quick overview.

Let P_0 be a real parabolic subgroup of G_0. Then P_0 has a standard decomposition as

$$P_0 = M_0 A_0 N_0 \tag{6.3}$$

where $A_0 = \exp(\mathfrak{a}_0)$, $\mathfrak{a}_0 \subset \mathfrak{s}_0$ and $N_0 = \exp(\mathfrak{n}_0)$ with $\mathfrak{n}_0 = \sum(\mathfrak{g}_0)_\alpha$ (summing over some set Σ^+ of positive roots of $\mathfrak{a}_0$ in $\mathfrak{g}_0$). The group M_0 is reductive and M_0A_0 is the centralizer in G_0 of $\mathfrak{a}_0$.

For a representation (W, σ) of M_0 and any $\nu \in \mathfrak{a}_0^*$, one obtains a representation $\sigma \otimes \nu \otimes 1$ of P_0 on W defined by $\sigma \otimes \nu \otimes 1(man) = \mathrm{e}^{\nu \log(a)}\sigma(m)$. Denote by ρ_G half the sum of the roots in Σ^+. Then the principal series representation is defined by

$$\begin{aligned} C^\infty(G_0/P_0, W) = \{ & f : G_0 \to W \mid f \text{ is smooth and} \\ & f(gman) = \mathrm{e}^{-(\nu+\rho_G)\log(a)}\sigma(m^{-1})f(g)\}. \end{aligned} \tag{6.4}$$

Note that this is the space of sections of the homogeneous bundle corresponding to the representation $\sigma \otimes (\nu + \rho_G) \otimes 1$.

We have chosen to use *smooth functions* in the definition of a principal series, however other choices are reasonable and sometimes preferable. For example, the space of distributions with the correct right translation property may be a good choice. It follows from the Iwasawa decomposition and (6.3) that K_0 acts transitively on G_0/P_0. Therefore, $G_0/P_0 \simeq K_0/M_0 \cap K_0$. It follows that in (6.4), instead of smooth functions, it is reasonable to require

$$\int_{K_0/M_0\cap K_0} \left|f(k_0)\right|^2 dk_0 < \infty.$$

(Here we are using some $M_0 \cap K_0$–invariant inner product on W. The measure is a K_0–invariant measure.) The shift by ρ_G guarantees that if (σ, W) is unitary then so is the principal series representation. Since G_0/P_0 is compact, the hyperfunctions (the dual of the real analytic functions which satisfy the right transformation property) is a continuous representation. These various choices of function spaces are all globalizations of the same Harish-Chandra module. See [**6**], Lecture 1, for a discussion of globalizations.

Here are some properties of the principal series representations.

(a) The principal series representations are admissible representations. (It follows that they have finite composition series.)
(b) Each irreducible representation occurs as a subrepresentation of some principal series representation.
(c) The Langlands classification gives a precise way to embed irreducibles into principal series representations (given enough information about the irreducible).

Details about principal series representation scan be found in [**19**].

6.3. Poisson and Szegö maps.

As examples of constructions of intertwining maps we will briefly describe generalizations of the classical Poisson and Szegö transforms in terms of intertwining maps. Recall that for the unit ball in $\mathbf{R}^n$ the Poisson transform is an operator which maps functions on the boundary to harmonic functions on the ball. The formula is

$$P_0 f(x) = c_n \int\limits_{S^{n-1}} \frac{1-|x|^2}{|x-t|^{2n}}\, f(t)\, dt. \tag{6.5}$$

This is a kernel operator since it is given by integration over t of the 'kernel' $k(x,t) = \frac{1-|x|^2}{|x-t|^{2n}}$. For the unit ball in $\mathbf{C}^n$ the Szegö transform maps functions on the boundary to holomorphic functions on the ball. This is given by the kernel operator

$$Sf(z) = c'_n \int\limits_{S^{2n-1}} \frac{1}{(1 - z\cdot\bar{\xi})^n}\, f(\xi)\, d\xi.$$

We will describe reasonable generalizations of P_0 and S.

For the Poisson transform consider an arbitrary Riemannian symmetric space G_0/K_0. Note that the transform (6.5) gives functions harmonic with respect to the Euclidean metric. The Laplacian with respect to the G_0–invariant metric on G_0/K_0 is an invariant operator. The generalization we have in mind is the space of all joint eigenfunctions of the space of all invariant differential operators on G_0/K_0. Let $\mathcal{M}_\nu$ denote the system of differential operators defining this joint eigenspace for some parameter ν. Let $C^\infty(G_0/K_0; \mathcal{M}_\nu)$ be the solutions to $\mathcal{M}_\nu$.

The classical Poisson map suggests that the domain of an intertwining operator should be functions on the boundary of G_0/K_0. This boundary can be thought of as G_0/P_0, with P_0 a minimal parabolic. Thus we look for intertwining maps

$$P_\nu : C^\infty(G_0/P_0, W) \to C^\infty(G_0/K_0; \mathcal{M}_\nu).$$

By (6.2) we begin by looking for a representation W of P_0 and a nonzero element of $\mathrm{Hom}_{P_0\cap K_0}(W, \mathbf{C})$. Note that $M_0 = P_0 \cap K_0$. Choose W to be the one dimensional representation $1 \otimes \nu \otimes 1$. Then (6.1) becomes

$$(P_\nu f)(g) = \int\limits_{K_0/M_0} f(gk)\, dk. \tag{6.6}$$

At this point one must do the following:

(a) Choose the parameter $\nu \in \mathfrak{a}^*$ so that P_ν is a joint eigenfunction.
(b) See when P_ν is one-to–one and onto at the K_0–finite level. This holds generically.
(c) See when P_ν is onto. For P_ν to be onto, the maximal globalization of the principal series (i.e., the hyperfunctions) must be used.

The first two are in [**17**]. The third is in [**18**] and [**29**]. It is useful to give a different formula for P_ν. We use the following facts about the Iwasawa decomposition. Recall that $P_0 = M_0 A_0 N_0$ is the standard decomposition of the minimal parabolic and the Iwasawa decomposition of G_0 is

$$\begin{aligned} &G_0 = K_0 A_0 N_0 \\ &g = \kappa(g) e^{H(g)} n_g, \text{ with } \kappa(g) \in K_0, H(g) \in \mathfrak{a}_0 \text{ and } n_g \in N_0. \end{aligned} \tag{6.7}$$

The decomposition is smooth and unique. For a function $F \in C^\infty(K_0/M_0)$ and $g \in G_0$

$$\int_{K_0/M_0} F(k)\,dk = \int_{K_0/M_0} F(\kappa(gk))\mathrm{e}^{-2\rho_G(H(gk))}\,dk. \tag{6.8}$$

A second fact is that since $k = g^{-1}\kappa(gk)\mathrm{e}^{H(gk)}n_{gk}$ we have

$$H(gk) + H(g^{-1}\kappa(gk)) = 0. \tag{6.9}$$

So (6.6) becomes

$$\begin{aligned}
P_\nu f(g) &= \int_{K_0/M_0} f(gk)\,dk \\
&= \int_{K_0/M_0} \mathrm{e}^{-(\nu+\rho_G)H(gk)} f(\kappa(gk))\,dk \\
&= \int_{K_0/M_0} \mathrm{e}^{(\nu-\rho_G)H(g^{-1}\kappa(gk))} f(\kappa(gk))\mathrm{e}^{-2\rho_G(H(gk))}\,dk \\
&= \int_{K_0/M_0} \mathrm{e}^{(\nu-\rho_G)H(g^{-1}k)} f(k)\,dk.
\end{aligned} \tag{6.10}$$

Observe that P_ν is given by integration against a kernel function:

$$\mathrm{e}^{(\nu-\rho_G)(H(g^{-1}k))}. \tag{6.11}$$

Now consider the Szegö transform. A reasonable generalization is the following. The representations $H^s(D,\mathcal{L}_\chi)$ (with χ satisfying a positivity condition) map onto the solution space of some elliptic differential operator $\mathcal{D}^\chi$ on $\mathbf{C}^\infty(G_0/K_0,\mathcal{E}_{\chi'})$. A very special case of this is when the cohomology is in the holomorphic discrete series. In this case $\mathcal{D}^\chi = \overline{\partial}$.

Thus, a Szegö map will be an intertwining operator

$$S : C^\infty(G_0/P_0, W) \to C^\infty(G_0/K_0,\mathcal{E}_{\chi'}) \tag{6.12}$$

with image satisfying $\mathcal{D}^\chi$. Thus we look for a nonzero element of $\mathrm{Hom}_{M_0}(W, E_{\chi'})$ for some W. The representation W is chosen as follows. Let $v_+ \in E_{\chi'}$ and let W be the span of $\{\chi'(m)v_+ \mid m \in M_0\}$ and denote by σ the representation of M_0 on W. The inclusion of W in $E_{\chi'}$ is the M_0–homomorphism we use. Thus

$$(Sf)(g) = \int_{K_0/M_0} \chi'(k)f(gk)\,dk. \tag{6.13}$$

A calculation similar to (6.10) determines a Szegö kernel:

$$(Sf)(g) = \int_{K_0/M_0} \mathrm{e}^{(\nu-\rho_G)H(g^{-1}k)}\chi'(\kappa(g^{-1}k))f(k)\,dk. \tag{6.14}$$

Intertwining maps of this type are studied in [**22**] and [**4**].

6.4. Intertwining maps into cohomology.

An intertwining map from a principal series representation into the $\mathcal{L}_\chi$–valued smooth forms of type $(0, s)$ will be constructed in much the same way that the Poisson and Szegö transforms were constructed. The image should consist of strongly harmonic forms and should be nonzero in cohomology.

The first complication is that (6.1), as stated, requires an invariant measure on $A/B_1 \cap B_2$. We are looking for an intertwining map

$$C^\infty(G_0/P_0, W) \to C^\infty(G_0/L_0, \wedge^s \overline{\mathfrak{u}} \otimes \mathbf{C}_\chi),$$

but the homogeneous space $L_0/P_0 \cap L_0$ does not have an L_0–invariant measure. We make the following adjustment. Begin with some nonzero

$$t \in \mathrm{Hom}_{P_0 \cap L_0}\big((W \otimes \mathrm{e}^{\rho_G}) \otimes \mathrm{e}^{-2\rho_L}, \wedge^s \overline{\mathfrak{u}} \otimes \mathbf{C}_\chi\big).$$

This is a bit confusing. The term e^{ρ_G} occurs because of the built in shift by e^{ρ_G} in the definition of the principal series and the term $\mathrm{e}^{-2\rho_L}$ compensates for the lack of invariant measure. Using (6.8) one can show that

$$(Tf)(g) = \int_{K_0 \cap L_0} \ell \cdot t(f(g\ell))\, d\ell \tag{6.15}$$

is an intertwining map $C^\infty(G_0/P_0, W) \to C^\infty(G_0/L_0, \wedge^s \overline{\mathfrak{u}} \otimes \mathbf{C}_\chi)$.

Assume that the positivity condition of Theorem 5.2 holds. Also assume that K and G have the same rank. This last assumption is not necessary but it makes the notation a bit less cumbersome. We may arrange for the following,

$\mathfrak{t}$, a Cartan subalgebra of $\mathfrak{l}$ and $\mathfrak{g}$,

$\mathfrak{a}_0 \subset \mathfrak{l}_0 \cap \mathfrak{s}_0$, a maximal abelian subalgebra,

$\mathfrak{t}_M = \mathfrak{t} \cap \mathfrak{m}$, a Cartan subalgebra of $\mathfrak{l} \cap \mathfrak{m}$ and $\mathfrak{m}$, and

$\mathfrak{t}_M + \mathfrak{a}$, a Cartan subalgebra of $\mathfrak{l}$ and $\mathfrak{g}$.

We will use the parabolic subgroup $P_0 = M_0 A_0 N_0$ for some choice of $\mathfrak{n}_0$. Let $s_{_M}$ be the dimension of the maximal compact subvarieties in $M_0/M_0 \cap L_0$. Then $M_0 \cap L_0$ is compact and $H^{s_M}(M_0/M_0 \cap L_0, \mathcal{L}_\chi \otimes \wedge^{\mathrm{top}}(\mathfrak{n} \cap \overline{\mathfrak{u}}))$ is a discrete series representation. Suppose that W is a representation of M_0 satisfying the following:

$$\begin{aligned} &\text{There is } A \in \mathrm{Hom}_{M_0}\big(W, A^{s_M}(M_0/M_0 \cap L_0, \mathcal{L}_\chi \otimes \wedge^{\mathrm{top}}(\mathfrak{n} \cap \overline{\mathfrak{u}}))\big) \text{ so that:} \\ &\mathrm{Im}(A) \subset \mathcal{H}^{s_M}(M_0/M_0 \cap L_0, \mathcal{L}_\chi \otimes \wedge^{\mathrm{top}}(\mathfrak{n} \cap \overline{\mathfrak{u}})) \\ &\mathrm{Im}(A) \text{ is nonzero when mapped to cohomology.} \end{aligned} \tag{6.16}$$

Let Ω be defined by

$$\Omega(w) = (Aw)(e). \tag{6.17}$$

Therefore,

$$\Omega \in \mathrm{Hom}_{M_0 \cap L_0}\big(W, \wedge^{s_M}(\mathfrak{m} \cap \overline{\mathfrak{u}}) \otimes (\mathbf{C}_\chi \otimes \wedge^{\mathrm{top}}(\mathfrak{n} \cap \overline{\mathfrak{u}}))\big).$$

We are looking for some nonzero

$$t \in \mathrm{Hom}_{P_0 \cap L_0}\big((W \otimes \mathrm{e}^{\rho_G}) \otimes \mathrm{e}^{-2\rho_L}, \wedge^s \overline{\mathfrak{u}} \otimes \mathbf{C}_\chi\big).$$

The N–action on W is trivial and the $\mathfrak{a}_0$–action will be determined in a moment. It is a short calculation to show that $\Omega(w)$ is invariant under $N \cap L$ and has $\mathfrak{a}_0$–weight $\rho_G - \rho_L$. Therefore, we must choose the $\mathfrak{a}_0$–parameter of W to be $\nu = \rho_L$. With this $\mathfrak{a}_0$ parameter for W take $t = \Omega$.

For Ω as in (6.16) set

$$\mathcal{S}f(g) = \int_{K_0 \cap L_0} \ell \cdot \Omega(f(gl))\, d\ell, \tag{6.18}$$

with $f \in C^\infty(G_0/P_0, W)$. Therefore

$$\mathcal{S} : C^\infty(G_0/P_0, W) \to A^s(D, \mathcal{L}_\chi).$$

Theorem 6.19. *There exists a nonzero Ω satisfying (6.16). Using this choice of Ω we have*

(a) *$\mathcal{S}$ is a continuous G_0–intertwining operator.*
(b) *The image of $\mathcal{S}$ consists of strongly harmonic forms.*
(c) *$\mathcal{S}$ is nonzero in cohomology, i.e., if q is the quotient map to cohomology (5.14), then $q \circ \mathcal{S} \neq 0$.*

The proof will be sketched in the next lecture. Various versions of this are contained in [**9**], [**7**],[**36**], [**4**], [**7**] and [**15**].

6.5. Exercises.

(a) Show that (6.1) is well–defined. In other words, show that the integrand is independent of the $B_1 \cap B_2$ coset and show that Tf satisfies the correct right transformation property to be in $\mathrm{Ind}_{B_2}^{A}(V_{\sigma_2})$.
(b) Let B be the unit ball in $\mathbf{C}^n$. Show that B is biholomorphic to an open orbit in $\mathbf{CP}(n)$. $G_0 = U(n,1)$ acts linearly on $\mathbf{CP}(n)$. What is the corresponding action on B and its boundary S^{2n-1}? Write a formula for the Poisson transform (6.6).
(c) Apply (6.8) to verify that the formula of (6.15) is a well–defined intertwining operator.

LECTURE 7
Intertwining Operators II

7.1. A transform to smooth sections on G_o/K_0.

An important tool for studying representations in cohomology is a transform (i.e., a G_0–intertwining map) $\mathcal{P}$ from $H^s(D,\mathcal{L}_\chi)$ into a space of smooth sections on G_0/K_0. This is sometimes called a 'real' (or C^∞) Penrose transform. It is used to show part (c) of Theorem 6.19.

The transform $\mathcal{P}$ is defined on a differential form by restricting cohomology classes to the maximal compact subvariety $K(z) = K_0(z)$. Since we are viewing forms as functions on G_0 with a right transformation property, the map becomes

$$\mathcal{P}\omega(g) = R(L_{g^{-1}}\omega), \tag{7.1}$$

where R stands for restriction to K_0 and restriction of the forms to $\wedge^s(\mathfrak{k}\cap\mathfrak{u})$. Note that $\mathcal{P}\omega \in E_{\chi'} \cong H^s(K(z),\mathcal{L}_\chi)$. The representation $E_{\chi'}$ is computed by the Bott-Borel-Weil Theorem. If ω is closed (respectively exact), then $\mathcal{P}\omega(g)$ is also closed (respectively exact) since R is the pullback of forms which commutes with $\overline{\partial}$ operators. Thus, $\mathcal{P}$ is well–defined on cohomology. We have $\mathcal{P} : H^s(D,\mathcal{L}_\chi) \to C^\infty(G_0/K_0,\mathcal{E}_{\chi'})$.

Theorem 7.2. *For $\chi+\rho$ very $\Delta(\mathfrak{u})$–dominant we have*

(a) $\mathcal{P} : H^s(D,\mathcal{L}_\chi) \to C^\infty(G_0/K_0,\mathcal{E}_{\chi'})$ *is an injection.*

(b) *The image of $\mathcal{P}$ is the solution space of a differential operator $\mathcal{D}^\chi$.*

This theorem is proved in [**28**] and [**35**].

To show $\mathcal{P}\circ\mathcal{S}$ is nonzero one projects the cohomology class $\mathcal{P}\circ\mathcal{S}(f) \in H^s(K_0/K_0\cap L_0)$ to its harmonic representative. The projection may be written in terms of matrix coefficients as in (4.7) and computing an integral over K_0. The details are contained in [**9**], [**4**] and [**15**]. Also see Lectures 1 and 2 of [**6**] for more details and a proof of (a).

7.2. Partial proof of Theorem (6.19).

Recall the formula for $Sf : C^\infty(G_0/P_0,W) \to A^s(D,\mathcal{L}_\chi)$.

$$\mathcal{S}f(g) = \int_{K_0\cap L_0} \ell\cdot\Omega(f(gl))\,d\ell \tag{7.3}$$

where Ω satisfies (6.16).

The first step is in the proof of Theorem 6.19 is to show that such an Ω exists. This is a fact about the discrete series of M_0. Let us give the appropriate statements in the context of G_0. So $D = G_0/L_0$ with L_0 compact.

For any smooth representation W of G_0 there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{G_0}(W, A^p(D, \mathcal{L}_\chi)) & \longrightarrow & \mathrm{Hom}_{L_0}(W, \wedge^p \overline{\mathfrak{u}} \otimes \mathbf{C}_\chi) \\ \Big\downarrow {\scriptstyle 1\otimes\overline{\partial}} & & \Big\downarrow {\scriptstyle \delta} \\ \mathrm{Hom}_{G_0}(W, A^{p+1}(D, \mathcal{L}_\chi)) & \longrightarrow & \mathrm{Hom}_{L_0}(W, \wedge^{p+1} \overline{\mathfrak{u}} \otimes \mathbf{C}_\chi). \end{array}$$

The horizontal maps are evaluation at e and the map $1 \otimes \overline{\partial}$ is given by $\big((1 \otimes \overline{\partial})(A)\big)(w) = \overline{\partial}(Aw)$; they are isomorphisms. The inverse of the horizontal map is given by $Aw(g) = a(g^{-1}w)$, for a in the right hand side. There is a similar diagram with $\overline{\partial}^*$ and δ^*. We define the formal harmonic space to be $ker(\delta) \cap ker(\delta^*)$ (as in Remark 4.6). It follows that the formal harmonic space of W may be identified with G_0 maps of W into $\mathcal{H}^s(D, \mathcal{L}_\chi)$.

Theorem 7.4. *If L_0 is compact, then there is an intertwining map A from some minimal principal series representation W into $\mathcal{H}^s(D, \mathcal{L}_\chi)$ so that the image is nonzero in cohomology. The representation $H^s(D, \mathcal{L}_\chi)$ is infinitesimally equivalent to a discrete series representation.*

This is the content of ([**5**]). The proof is very much in the spirit of the proof that the Szegö map (6.13) and (6.14) satisfies a certain differential equation $\mathcal{D}^\chi$. In fact the Szegö map is known to be the composition of A, the map to cohomology and $\mathcal{P}$:

$$W \longrightarrow \mathcal{H}^s(D, \mathcal{L}_\chi) \longrightarrow H^s(D, \mathcal{L}_\chi) \longrightarrow C^\infty(G_0/K_0, \mathcal{E}_{\chi'}).$$

Since the image of A consists of harmonic forms, the corresponding $\Omega \in \mathrm{Hom}_{L_0}(W, \wedge^s \overline{\mathfrak{u}} \otimes \mathbf{C}_\chi)$ is formally harmonic. Applying this to $M_0/M_0 \cap L_0$, we see how to choose Ω in (6.17).

We now indicate how to show that the image of $\mathcal{S}$ consists of harmonic forms. First, the formula for $\overline{\partial}$ is given as follows. Let $\{\xi_\alpha \,|\, \alpha \in \Delta(\mathfrak{u})\}$ be a basis for $\mathfrak{u}$ and $\{\omega_\alpha\}$ the dual basis. For a smooth function ϕ on D and some $\omega \in \wedge^s \overline{\mathfrak{u}} \otimes \mathbf{C}_\chi$

$$\overline{\partial}(\phi\omega) = \sum_{\alpha\in\Delta(\mathfrak{u})} r(\xi_\alpha)\phi(\omega_\alpha \wedge \omega) + \frac{1}{2} \sum_{\alpha\in\Delta(\mathfrak{u})} f\, ad(\xi_\alpha)(\omega_\alpha \wedge \omega). \tag{7.5}$$

There is a similar formula for $\overline{\partial}^*$.

We may bring both terms of the $\overline{\partial}$ operator inside the integral in (7.3).

Lemma 7.6. *For $f \in C^\infty(G_0/P_0, W)$,*

$$\overline{\partial} \int_{L_0\cap K_0} \ell \cdot \Omega(f(gl))\, d\ell = \int_{L_0\cap K_0} Ad(\ell) \cdot \overline{\partial}\Omega(f(\cdot))|_{x\ell}\, d\ell.$$

The proof is fairly straightforward and uses the fact that the formula for the $\overline{\partial}$–operator is independent of the basis.

Since

$$\Delta(\mathfrak{u}) = \Delta(\mathfrak{n} \cap \mathfrak{u}) \cup \Delta(\theta\mathfrak{n} \cap \mathfrak{u}) \cup \Delta(\mathfrak{m} \cap \mathfrak{u})$$

we consider the terms in $\overline{\partial}$ with root vectors in each of these three sets separately.

First suppose $\alpha \in \Delta(\theta\mathfrak{n} \cap \mathfrak{u})$. Then $\omega_\alpha \wedge \Omega(f(e)) = 0$ since Ω contains $\wedge^{\mathrm{top}}(\theta\mathfrak{n} \cap \overline{\mathfrak{u}})$. If $\alpha \in \Delta(\mathfrak{n} \cap \mathfrak{u})$, then the right differentiations are zero because of the right transformation property of f in a principal series. The second term is zero because the image of Ω is invariant under N_0. What remains is precisely the $\overline{\partial}$ operator for $M_0/M_0 \cap L_0$. This is zero because $\Omega(f(x\ell m)) = \Omega(m^{-1}f(x\ell))$ is a harmonic form on $M_0/M_0 \cap L_0$.

7.3. Square integrability.

For this section assume that the real ranks of L_0 and G_0 are equal. This means that P_0 is a minimal parabolic subgroup. In particular, M_0 is compact. Then the M_0 representation may be chosen to be the harmonic space for the finite–dimensional representation $\mathcal{H}^{s_M}(M_0/M_0 \cap L_0, \mathcal{L}_\chi \otimes \wedge^{\mathrm{top}}(\mathfrak{n} \cap \overline{\mathfrak{u}}))$.

Theorem 7.7. *If ρ_L is nonsingular and if L is the fixed point set of an involution, then the image of $\mathcal{S}$ lies in $\mathcal{H}^s_2(D, \mathcal{L}_\chi)$, that is, it consists of square integrable harmonic forms. The invariant inner product is nonzero on the image of $\mathcal{S}$ and is positive definite on the Hilbert space $\overline{\mathcal{H}^s_2}$.*

This is contained in [**10**]. Other partial results, including more general square integrability, are also contained in [**10**]. See [**6**], Lecture 4, a proof which is a simplification of the proof in [**10**].

7.4. Holomorphic double fibration transform.

In this section we will show how to construct a different type of intertwining operator. This will be similar in nature to the construction in Section 7.1, however it will be in the holomorphic category. We obtain an intertwining operator from $H^s(D, \mathcal{L}_\chi)$ into the space of holomorphic sections of a vector bundle on a Stein space. The image is characterized by an operator $\mathcal{D}^\chi_{\mathbf{C}}$. In [**2**] it was suggested that the correct space should be a Stein extension of G_0/K_0. More precisely, note that $G_0/K_0 \subset G/K$ is a real analytic submanifold with the property that $T(G_0/K_0) + \overline{T(G_0/K_0)} = T(G/K)$. Then a Stein extension is a Stein neighborhood of G_0/K_0 inside the affine space G/K.

One approach to giving such a realization of $H^s(D, \mathcal{L}_\chi)$ is to take functions in the image of the Szegö map and try to extend them to some Stein domain in G/K. The point is that the G_0/K_0–variable in the Szegö kernel can be extended to some open set in G/K. In ([**8**]) the largest neighborhood of G_0/K_0 to which the Szegö kernel can be extended holomorphically was determined in the case of $U(p,q)$ and open orbits in the full flag variety. It was found that this largest (connected) neighborhood is $G_0/K_0 \times \overline{G_0/K_0}$. Since the kernel extends holomorphically, the functions in the image of the Szegö map also extend holomorphically (and there is an explicit formula for the functions).

Our approach will not involve extending the Szegö map holomorphically however the end result will be the same: we realize the representation as a space of holomorphic sections on a Stein extension of G_0/K_0. Suppose the flag domain

$D \subset Z$ fits into a holomorphic double fibration

$$\begin{tikzcd} & Y \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ D & & M. \end{tikzcd} \tag{7.8}$$

By *holomorphic double fibration* we mean that Y, M are complex manifolds and both μ and ν are holomorphic fibrations. In fact, D may be any complex manifold for now. Then under some conditions there is a transform

$$\mathcal{P}^{\mathbf{C}} : H^s(D, \mathcal{L}_\chi) \to H^0(M, \mathcal{E}_{\chi'}). \tag{7.9}$$

The conditions we have in mind are the following.

(a) μ has contractible fibers.
(b) ν is proper.
(c) M is a Stein manifold.
(d) A vanishing condition for cohomology of the (compact) fibers of ν. For our situation, this is explicitly computable by the Bott-Borel-Weil Theorem and is guaranteed when χ is sufficiently positive.

Suppose that (a)-(d) hold. The construction (7.9) will be the composition of three maps. We will very briefly describe the construction in terms of sheaf operations. First a few comments about the sheaf operations.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_D$–modules on D. The inverse image sheaf is defined by

$$\mu^{-1}(\mathcal{F})(U) = \lim_{V \supset \mu(U)} \mathcal{F}(V).$$

This is not a sheaf of $\mathcal{O}_Y$–modules. To obtain a sheaf of $\mathcal{O}_Y$–modules we define

$$\mu^*(\mathcal{F}) = \mathcal{O}_Y \underset{\mu^{-1}(\mathcal{O}_D)}{\otimes} \mu^{-1}(\mathcal{F}).$$

If $\mathcal{F}$ happens to be the sheaf of germs of holomorphic sections of a holomorphic vector bundle then the pullback sheaf is just the sheaf of germs of holomorphic sections of the pullback bundle. The direct image sheaf is written as $\nu_*^s(\mathcal{F}')$ for a sheaf of $\mathcal{O}_Y$–modules. The direct image map has the property that it sends coherent sheaves to coherent sheaves.

There is a natural map

$$H^s(D, \mathcal{O}(\mathcal{L}_\chi)) \to H^s(Y, \mu^{-1}\mathcal{O}(\mathcal{L}_\chi)). \tag{7.10}$$

A topological result in ([**14**]) guarantees that (7.10) is an isomorphism if (a) holds. There is a natural map

$$H^s(Y, \mu^{-1}\mathcal{O}(\mathcal{L}_\chi)) \to H^s(Y, \mu^*\mathcal{O}(\mathcal{L}_\chi)). \tag{7.11}$$

This is an injection by (d). The image is precisely the kernel of a d–operator for some spectral sequence. An application of the Leray spectral sequence gives a map

$$H^s(Y, \mu^*\mathcal{O}(\mathcal{L}_\chi)) \to H^0(M, \nu_*^s\mu^*\mathcal{O}(\mathcal{L}_\chi)). \tag{7.12}$$

This is an isomorphism since (b) and (c) hold.

Then $\mathcal{P}^{\mathbf{C}}$ is the composition of the three maps (7.10), (7.11) and (7.12) and is injective. This transform is sometimes called a 'complex' (or holomorphic) Penrose transform. There is a treatment of this construction in [**11**].

In order to study $H^s(D, \mathcal{L}_\chi)$ we want must find the proper space M which fits into a holomorphic double fibration with the flag domain D. This is the content of Lecture 8.

LECTURE 8
The Linear Cycle Space

8.1. Holomorphic double fibration.

In the construction of a holomorphic double fibration transform, a double fibration (7.8) must be specified. A general principle is that the right thing to take M to be is some space of maximal compact subvarieties. Then $Y = \{(z, V) \mid z \in V \in M\}$, with the obvious maps to D and M, is the appropriate choice. In fact, there are general theorems about the 'cycle space' M being Stein. However we want more detailed information about the space M. We will define M to be some special family of maximal compact subvarieties.

Assume that G is simple. As usual, let $D = G_0(z)$ be a flag domain in Z. We assume that the base point z has been chosen so that $K(z) = K_0(z)$ in accordance with Proposition 2.9. By Corollary 3.20 we know $K(z)$ is a maximal compact subvariety of D. Set $V_0 = K(z)$, (sort of) a base point in the space of maximal compact subvarieties.

Definition 8.1. Let $M_Z = \{gV_0 \mid g \in G\}$.

The space M_Z is clearly a homogeneous space for G. In order to determine what exactly this space is we make a definition which separates the flag domains into two classes.

Consider the (C^∞) double fibration

(8.2)
$$
\begin{tikzcd}
 & G_0/L_0 \cap K_0 \arrow[dl] \arrow[dr] & \\
D = G_0/L_0 & & G_0/K_0.
\end{tikzcd}
$$

This is of course not necessarily a holomorphic double fibration since G_0/K_0 may not even have an invariant complex structure.

Definition 8.3. The flag domain D is said to be of *holomorphic type* if there are invariant complex structures on G_0/K_0 and $G_0/L_0 \cap K_0$ so that both fibrations in (8.2) are holomorphic. Otherwise, D is of *nonholomorphic type.*

Let $\mathfrak{q} = \mathfrak{l}+\mathfrak{u}$ be the Lie algebra of the stabilizer of z. If an orbit is of holomorphic type, then G_0 must be of hermitian type. In this case $\mathfrak{s} = \mathfrak{s}_+ + \mathfrak{s}_-$ (irreducible K–representations). Recall that $KS_\pm$ are parabolic subgroups so $X_+ = G/KS_+$

and $X_- = G/KS_-$ are complex flag varieties. Then G_0/K_0 is contained as an open orbit in both $X_\pm$. We let B and $\overline{B}$ denote these two orbits; $B \subset X_+$ and $\overline{B} \subset X_-$. The have opposite complex structures. It is easy to verify the following characterization of holomorphic orbits.

Lemma 8.4. *The following are equivalent.*

(a) *D is of holomorphic type.*
(b) *Either $\mathfrak{u} \cap \mathfrak{s}_+ = 0$ or $\mathfrak{u} \cap \mathfrak{s}_- = 0$.*
(c) *Either $\mathfrak{q} \cap (\mathfrak{k} + \mathfrak{s}_+)$ or $\mathfrak{q} \cap (\mathfrak{k} + \mathfrak{s}_-)$ is a parabolic subalgebra of $\mathfrak{g}$.*

Proposition 8.5. *([**33**]) For the structure of M_Z there are two possibilities:*

(a) *If D is of holomorphic type, then $M_Z = X_+$ or $M_Z = X_-$.*
(b) *If D is of nonholomorphic type then, $M_Z = G/\tilde{K}$, where $\tilde{K}$ is some finite extension of K.*

This proposition is used to give M_Z the structure of complex manifold. Note that M_Z doesn't depend too much on D. In the nonholomorphic case, the dimension is twice what it is in the holomorphic case. Also, in the holomorphic case M_Z is projective and in the nonholomorphic case it is affine.

Definition 8.6. The *linear cycle space* M_D is the connected component of

$$\{gV_0 \,|\, gV_0 \subset D, g \in G\}$$

in M_Z which contains V_0

It follows that M_D is open in M_Z, so is a complex manifold.

It is natural to define

$$\begin{aligned} Y_Z &= \{(z, V) \,|\, z \in V \in M_Z\} \\ Y_D &= \{(z, V) \,|\, z \in V \in M_D\}. \end{aligned}$$

Then

$$\begin{tikzcd} & Y_D \arrow[dl] \arrow[dr] & \\ D & & M_D \end{tikzcd} \tag{8.7}$$

is a holomorphic double fibration.

Remark 8.8. It is reasonable to compare the linear cycle space with a 'full' cycle space. The dimension of the component of the full cycle space is the dimension of the space of holomorphic sections of the conormal bundle of V_0. This may be computed and one finds that it is the same as the dimension of M_D, so M_D is open in the full cycle space, *except* in several cases. These exceptional cases occur precisely when a bigger group acts on D. An example occurs for $Z = \mathbf{CP}(2n-1)$ which is a flag variety for both $Sp(n, \mathbf{C})$ and $SL(2n, \mathbf{C})$ (because lines are automatically isotropic). The space of positive lines is an open orbit in $\mathbf{CP}(2n-1)$ under $G_0 = U(n,n) \cap Sp(n, \mathbf{C})$ and also an orbit for $U(n,n)$. Here the spaces of linear cycles have different dimensions.

8.2. Structure of the linear cycle space.

Consider the following example. $G_0 = U(p,q) \subset G = GL(n,\mathbf{C})$, with $p+q=n$. Let Z be the set of m–planes in C^n. We have already seen that the orbits are determined by signature and the open orbits are $D_{r,s}$ consisting of the planes of signature (r,s). The flag domain $D_{r,s}$ is of holomorphic type if and only if $rs=0$ or $(p-r)(q-s)=0$. The choice of base point

$$z_0 = \operatorname{span}\{e_1,\dots,e_r,e_{n+1},\dots,e_{n+s}\}$$

satisfies $K(z_0)=K_0(z_0)$. Furthermore, $V_0 = K(z)$ may be written as

$$\begin{aligned}\{z\in Z|\dim(z\cap(\mathbf{C}^p\times\{0\})) = r \text{ and } \dim(z\cap(\{0\}\times\mathbf{C}^q\})) = s\}\\ \simeq \mathbf{Gr}(r,\mathbf{C}^p)\times\mathbf{Gr}(s,\mathbf{C}^q).\end{aligned}$$

Note that we may identify B with the positive p–planes in $\mathbf{C}^n$ and $\overline{B}$ with the negative q–planes.

Consider the holomorphic type orbit $D_{m,0}$ first. Then $V_0 = \{z\in Z\,|\,z\subset \mathbf{C}^p\times\{0\}\}$ and $gV_0 = \{z\in Z\,|\,z\subset g\cdot(\mathbf{C}^p\times\{0\})\}$. For p dimensional subspaces U define $Y_U = \{z\in Z\,|\,z\subset U\}$. Then $M_Z = \{Y_U\mid \dim(U)=p\}\simeq G/KS_+$. But $Y_U\subset D$ if and only if U is positive, therefore $M_D = B$.

Now consider the nonholomorphic type orbits. Suppose U is a p–plane and W is a q–plane, and U and W are transverse (i.e., $U\cap W=0$). Define

$$V_{U,W} = \{z\in Z|\dim(z\cap U)=r \text{ and } \dim(z\cap W)=s\}.$$

Then $gY_{U,W} = Y_{gU,gW}$. This gives a G–equivariant bijection between M_Z and

$$\{(U,W)\mid \dim(U)=p,\ \dim(W)=q \text{ and } U\cap W=0\}\simeq G/K$$

except in the case $p=q$. For the case $p=q$ and $r=s$ one has $M_Z\simeq G/N_{G_u}(K)$. Then it is clear that $V_{U,W}\subset D_{r,s}$ if $U\in B$ and $W\in\overline{B}$. The converse also holds. Therefore $B\times\overline{B}\simeq M_D$.

More generally we have the following theorem.

Theorem 8.9. *If G_0 is a classical group of hermitian type then*

(a) *if D is of holomorphic type, then $M_D\simeq B$ or $\overline{B}$, and*
(b) *if D is of nonholomorphic type, then $M_D\simeq B\times\overline{B}$.*

Furthermore, the fibers of μ are contractible.

The proofs of (a) and (b) are in [**34**].

Thus there is a holomorphic double fibration transform

$$\mathcal{P}^{\mathbf{C}} : H^s(D,\mathcal{L}_\chi)\to H^0(B\times\overline{B},\mathcal{E}^{\mathbf{C}}_{\chi'}). \tag{8.10}$$

This is injective when χ is dominant enough for condition (d) following (7.9) to hold. Restriction to the diagonal is injective because of the way the G_0/K_0 sits inside G/K.

8.3. The relationship between the various realizations of representations in cohomology.

We have studied a number of intertwining operators between representations. It is interesting that they fit together in a commuting diagram.

$$\begin{array}{ccc} & C^\infty(G_0/P_0, W) & \\ \mathcal{S} & \downarrow & \searrow S \\ & H^s(D, \mathcal{L}_\chi) \xrightarrow{\mathcal{P}} & C^\infty(G_0/K_0, \mathcal{E}_{\chi'}) \\ \mathcal{P}^{\mathbf{C}} & \downarrow & \nearrow \text{rest.} \\ & H^0(M_D, \mathcal{E}^{\mathbf{C}}_{\chi'}) & \end{array} \tag{8.11}$$

It should be pointed out that in most of our constructions of these intertwining operators we assumed that the parameter χ was 'sufficiently positive'. This can be relaxed somewhat but things start to get tricky. For example S may still be defined, but the image is only known to be *contained* in the kernel of $\mathcal{D}^\chi$. Also a less direct proof of injectivity of $\mathcal{P}^{\mathbf{C}}$ is necessary.

8.4. Exercise

(a) Show 8.11 commutes.

8.5. Final notes

For those who want to study the ideas discussed in these lectures in more detail a good place to start is with the following references. For the orbit structure in flag varieties see [**32**], for representations in cohomology see [**28**], for Szegö maps and discrete series see [**22**] and for intertwining maps in cohomology see [**9**].

Here are a few interesting problems to think about.

(a) Give an explicit description of the flag varieties and orbits (G_0 or $K_{\mathbf{C}}$) for the exceptional groups F_4, E_6, E_7 and E_8 in much the same way Witt's theorem was used for the classical groups.

(b) Carry out the quantization procedure outlined at the end of Lecture 5 for the discrete series. This should be done in terms of the indefinite metric. See [**6**] for a discussion of some of the difficulties.

(c) Give a conceptual proof of the positivity of the invariant form on $\overline{\mathcal{H}^s_2}$. Start with the hypothesis of Theorem 7.7.

(d) Determine the structure of M_D in general in terms of the structure of the group G.

BIBLIOGRAPHY

1. A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Math. Soc. France **90** (1962), pp.193–259.
2. D. Akheizer and S. Gindikin, *On the Stein extensions of real symmetric spaces*, Math. Ann. **286** (1990), pp. 1-12.
3. E. Artin, *Geometric Algebra.* Interscience Publishers, New York, 1957.
4. L. Barchini, *Szegö mappings, harmonic forms and Dolbeault cohomology*, J. of Functional Analysis **118** (1993), pp. 351–406.
5. L. Barchini, *Strongly harmonic forms for the discrete series representations*, to appear.
6. L. Barchini, *Unitary representations associated to elliptic orbits: A geometric approach*, this volume.
7. L. Barchini, M. G. Eastwood and R. Gover *An integral operator into Dolbeault cohomology*, J. of Functional Analysis 137 (1996), pp. 364–380.
8. L. Barchini, S. Gindikin and H.-W. Wong *The geometry of flag manifolds and holomorphic extensions of Szegö kernels for $U(p,q)$* , Pacific Journal of Math. **179** (1997), pp. 201–220 .
9. L. Barchini, A. W. Knapp and R. Zierau, *Intertwining operators into Dolbeault cohomology representations*, J. of Functional Analysis **107** (1992), pp. 302–341.
10. L. Barchini and R. Zierau, *Square integrable harmonic forms and representation theory*, Duke Math. J. **92** (1998), pp. 645–664.
11. R. Baston and M. G. Eastwood, *The Penrose Transform: Its Interaction with representation Theory* Oxford University Press, Oxford 1989.
12. R. Bott, *Homogeneous vector bundles*, Annals of Math. **66** (1957), pp. 203–248.
13. A. Borel, *Linear Algebraic Groups.* Springer–Verlag, 1991.
14. N. Buchdahl, *On the relative DeRham sequence*, Proc. Amer. Math. Soc. **87** (1983), pp. 363–366 .
15. R. Donley, *Intertwining operators into cohomology representations for semi–simple Lie groups*, J. of Functional Analysis **151** (1997), pp. 138–165.
16. Harish-Chandra, *Representations of semisimple Lie Groups VI*, American J. of Math. **78** (1956), pp. 564–628.
17. S. Helgason, *Groups and Geometric Analysis.* Academic Press, New York, 1984.
18. M. Kashiwara, A. Kowato, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Annals of Math. **107** (1978), pp. 1-39.

19. A. W. Knapp, *Representation Theory of Real Semisimple Groups: An Overview Based on Examples.* Princeton University Press, Princeton, New Jersey, 1986.
20. A. W. Knapp, *Lie Groups Beyond an Introduction.* Birkhäuser, Boston, 1996.
21. A. W. Knapp and D. A. Vogan, *Cohomological Induction and Unitary Representations.* Princeton University Press, Princeton, NJ, 1995.
22. A. W. Knapp and N. Wallach, *Szegö kernels associated with discrete serie*, Invent. Math. **34** (1976), pp. 163-200.
23. B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Annals of Math. **74** (1961), pp. 329–387.
24. T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), pp. 331–357.
25. G. D. Mostow, *Some new decomposition theorems for semi–simple groups*, Memoirs AMS **14** (1955), pp. 31-54.
26. J. D. Novak, *Explicit Realizations of Certain Representations of $Sp(n;\mathbf{R})$ via the Penrose Transform*, Ph.D. thesis, Oklahoma State University, 1996.
27. J. Rawnsley, W. Schmid and J. A. Wolf, *Singular unitary representations and indefinite harmonic theory*, J. of Functional Analysis **51** (1983), pp. 1–114.
28. W. Schmid, *Homogeneous complex manifolds and representation theory*, Representation Theory and Harmonic Analysis on Semisimple Lie Groups, Math. Surveys and Monographs, American Mathematical Society, Providence, RI, 1989 pp. 223–286.
29. W. Schmid, *Boundary value problems for group invariant differential equations*, Soc. Math. France Asterisque, Numero hors series (1985), pp. 311–321
30. W. Schmid and J. A. Wolf, *Geometric quantization and derived functor modules for semisimple Lie groups*, J. of Functional Analysis **90** (1990), pp. 48–112.
31. W. Schmid and J. A. Wolf, *A vanishing theorem for open orbits on a complex flag manifold*, Proc. Amer. Math. Soc. **92** (1984), pp. 461–464.
32. J. A. Wolf, *The action of a real semisimple Lie group on a complex manifold,* I: Orbit structure and holomorphic arc components, Bull. Amer. Math. Soc. **75** (1969) pp. 1121–1237.
33. J. A. Wolf, *The Stein condition for cycle spaces of open orbits on complex flag manifolds*, Annals of Math. **136** (1992) pp. 541-555.
34. J. A. Wolf and R. Zierau, *Linear cycle spaces in flag domains, I: Flags corresponding to classical groups*, Reports of the Erwin Schrödinger Institute for Mathematical Physics, No. 357, (1996)
35. Hon-Wai Wong, *Dolbeault cohomologies and Zuckerman modules associated with finite rank representations*, J. of Functional Analysis **127** (1995), pp. 428–454.
36. R. Zierau, *Unitarity of certain Dolbeault cohomology representations*, The Penrose Transform and Analytic cohomology in Representation Theory, Contemporary Mathematics Vol. 154, A.M.S., Providence, RI, pp. 239–259.