HARMONIC SPINORS ON REDUCTIVE HOMOGENEOUS SPACES

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Abstract. *An integral intertwining operator is given from certain principal series representations into spaces of harmonic spinors for Kostant's cubic Dirac operator. This provides an integral representation for harmonic spinors on a large family of reductive homogeneous spaces.*

Introduction

A realization of the discrete series representations of a semisimple Lie group as an L^2-space of harmonic spinors was given in [14] and [1]. More precisely, suppose G is a non-compact connected semisimple real Lie group with finite center and K a maximal compact subgroup of G. Write S for the spin representation of K (after passing to a cover if necessary). For a finite dimensional K-representation E, the tensor product $S \otimes E$ determines a homogeneous vector bundle $\mathcal{S} \otimes \mathcal{E}$ over G/K and a geometric Dirac operator (defined in terms of an invariant connection) acting on smooth sections:

$$\mathcal{D}_{G/K}(\mathcal{E}) : C^\infty(G/K, \mathcal{S} \otimes \mathcal{E}) \to C^\infty(G/K, \mathcal{S} \otimes \mathcal{E}).$$

If G has a non empty discrete series, then the kernel of $D_{G/K}(\mathcal{E})$ on L^2-sections is an irreducible unitary representation in the discrete series of G, and every discrete series representation of G occurs this way for some K-representation E. (See [10] for a thorough discussion.) A similar construction of tempered representations is given by 'partially harmonic' spinors in [18].

It is therefore natural to study the kernels of Dirac operators on other homogeneous spaces. In [11] and [12], we addressed this problem in a more general context where G is a connected real reductive Lie group and K is replaced by connected closed reductive subgroups H for which the complex ranks of H and G are equal, but H is not necessarily compact. The Dirac operator $\mathcal{D}_{G/K}(\mathcal{E})$ is replaced by Kostant's cubic Dirac operator ([8]):

$$D_{G/H}(\mathcal{E}) : C^\infty(G/H, \mathcal{S} \otimes \mathcal{E}) \to C^\infty(G/H, \mathcal{S} \otimes \mathcal{E}).$$

This operator is the sum of a first order term and a zeroth order term, which comes from a degree three element in the Clifford algebra of the orthogonal complement of the complexified Lie algebra of H. (The zeroth order term vanishes when H is any symmetric subgroup.) Integral formulas for harmonic spinors are given in [11] and [12]. The L^2-theory is begun in [2].

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In the present article we consider a larger class of homogeneous spaces by removing the equal rank condition. Suppose that $E = E_\mu$ is a finite dimensional representation of H with highest weight μ (with respect to some positive system). Under certain conditions on H and μ we prove the following theorem. This is Theorem 3.9 of Section 3.

Theorem A. *There is a parabolic subgroup P in G, a representation W of P and a non-zero G-equivariant map*

$$\mathcal{P} : C^\infty(G/P, \mathcal{W}) \to C^\infty(G/H, \mathcal{S} \otimes \mathcal{E}_\mu)$$

with $D_{G/H}(\mathcal{E}_\mu) \circ \mathcal{P} = 0$, where $C^\infty(G/P, \mathcal{W})$ denotes the space of smooth vectors of the principal series representation $\mathrm{Ind}_P^G(W)$. In particular, the kernel of $\mathcal{D}_{G/H}(\mathcal{E}_\mu)$ contains a smooth representation of G.

The intertwining operator $\mathcal{P}$ is an integral operator; the formula for $\mathcal{P}$ is analogous to the classical Poisson integral formula giving harmonic functions on the disk. The condition on H referred to above is stated in §3.3 as Assumption 3.4. It is that H is not too small; it guarantees that certain Dirac cohomology spaces are nonzero. The conditions on μ are very mild regularity conditions.

The representation of $P = MAN$ on W is formed from a fundamental series of M, a character of A and the trivial action of N. As an important ingredient the fundamental series is realized as a space of harmonic spinors on $M/M \cap H$. This is the content of the following proposition (which occurs as Proposition 2.9 in Section 2).

Proposition B. *Every fundamental series representation occurs in the kernel of a Dirac operator $\mathcal{D}_{G/H}(\mathcal{E})$.*

The paper is organized as follows. In Section 1 we fix the notation and give some well-known facts about the spin representations. We also describe a reciprocity for geometric and algebraic Dirac operators. This is an important technique for relating spaces of harmonic spinors and Dirac cohomology. In Section 2, we realize certain cohomologically induced representations as submodules of kernels of geometric Dirac operators and prove Proposition B. Finally, Section 3 is devoted to the construction of the parabolic subgroup P and the proof of Theorem A. A proof of the reciprocity between geometric and algebraic harmonic spinors is provided in an appendix.

1. Preliminaries

1.1. **The groups and homogeneous spaces.** Let G be a connected real reductive Lie group. We will denote the complexification of $Lie(G)$, the Lie algebra of G, by $\mathfrak{g}$ (and similarly for other Lie groups). By a real reductive grup we mean that $\mathfrak{g}$ is reductive, i.e., $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}$, where $\mathfrak{z}$ denotes the center of $\mathfrak{g}$. Fix a G-invariant nondegenerate bilinear symmetric form $\langle\ ,\ \rangle$ on $\mathfrak{g}$. If K/Z is a maximal compact subgroup of G/Z, where Z is the center of G, then K is the fixed point group of a Cartan involution θ of G. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ for the corresponding Cartan decomposition of $\mathfrak{g}$, where

$\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{s} = \mathfrak{k}^{\perp}$. We take H to be a closed subgroup of G, with complexified Lie algebra denoted by $\mathfrak{h}$, satisfying the following conditions:

$$\begin{aligned}&H \text{ is connected and reductive,}\\ &H \text{ is } \theta\text{-stable,} \\ &\langle\ ,\ \rangle \text{ remains nondegenerate when restricted to } \mathfrak{h}.\end{aligned} \tag{1.1}$$

In this situation, there is a decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q}\,,\ \text{ where }\ \mathfrak{q} = \mathfrak{h}^{\perp}.$$

The restriction of $\langle\ ,\ \rangle$ to $\mathfrak{q}$ remains nondegenerate and $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$.

1.2. **Spin representation.** The construction of the spin representation is briefly reviewed here; we follow the discussion of [3, Ch. 9]. Let $\mathcal{C}l(\mathfrak{q})$ be the Clifford algebra of the complexification of $\mathfrak{q}$, i.e., the quotient of the tensor algebra of $\mathfrak{q}$ by the ideal generated by the elements $X\otimes Y + Y\otimes X - \langle X\,, Y\rangle$ with $X, Y \in \mathfrak{q}$. Let

$$\mathfrak{so}(\mathfrak{q}) = \big\{T \in \mathrm{End}(\mathfrak{q}) : \langle T(X)\,, Y\rangle + \langle X\,, T(Y)\rangle = 0,\ \forall X, Y \in \mathfrak{q}\big\}.$$

Then the endomorphisms $R_{X,Y} : W \mapsto \langle Y\,, W\rangle X - \langle X\,, W\rangle Y$ span $\mathfrak{so}(\mathfrak{q})$. The linear extension of the map $R_{X,Y} \mapsto \frac{1}{2}(XY - YX)$ is an injective Lie algebra homomorphism of $\mathfrak{so}(\mathfrak{q})$ into $\mathcal{C}l_2(\mathfrak{q})$, the subalgebra of degree two elements in the Clifford algebra. Let $\mathfrak{q}^+$ be a maximally isotropic subspace of $\mathfrak{q}$ and write $S_{\mathfrak{q}}$ for the exterior algebra $\wedge\mathfrak{q}^+$ of $\mathfrak{q}^+$. The spin representation $(s_{\mathfrak{q}}, S_{\mathfrak{q}})$ of $\mathfrak{h}$ is defined as the composition map

$$\mathfrak{h} \overset{\mathrm{ad}}{\hookrightarrow} \mathfrak{so}(\mathfrak{q}) \hookrightarrow \mathcal{C}l_2(\mathfrak{q}) \subset \mathcal{C}l(\mathfrak{q}) \overset{\gamma_{\mathfrak{q}}}{\to} \mathrm{End}(S_{\mathfrak{q}})$$

where $\gamma_{\mathfrak{q}}$ denotes the Clifford multiplication. Although the construction is independent of maximal isotropic subspace $\mathfrak{q}^+$, the explicit description of a particularly useful $\mathfrak{q}^+$ will be given in section 3.3 below.

There is an hermitian inner product $\langle\ ,\ \rangle_{S_{\mathfrak{q}}}$ on $S_{\mathfrak{q}}$ for which $\gamma_{\mathfrak{q}}(X)$, $X \in \mathfrak{q} \subset \mathcal{C}l(\mathfrak{q})$, is skew hermitian ([17, Lemma 9.2.3]):

$$\langle \gamma_{\mathfrak{q}}(X)u\,, v\rangle_{S_{\mathfrak{q}}} = -\langle u\,, \gamma_{\mathfrak{q}}(X)v\rangle_{S_{\mathfrak{q}}}, \quad \forall X \in \mathfrak{q},\ \forall u, v \in S_{\mathfrak{q}}. \tag{1.2}$$

1.3. **Geometric Dirac operators.** Let E be a finite dimensional representation of $\mathfrak{h}$ such that the tensor product $S_{\mathfrak{q}} \otimes E$ lifts to a representation of the group H. There is an associated smooth homogeneous vector bundle over G/H, which we denote by $\mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}$, whose space of smooth sections is

$$\begin{aligned}C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}) &\simeq \Big\{C^{\infty}(G) \otimes (S_{\mathfrak{q}} \otimes E)\Big\}^{H}\\ &\simeq \{f : G \to S_{\mathfrak{q}} \otimes E \,|\, f \text{ is smooth and } f(gh) = h^{-1}\cdot f(g), h \in H\}.\end{aligned}$$

We remark that G acts by left translations on each of these function spaces.

Let $\{X_j\}$ be a fixed basis of $\mathfrak{q}$ satisfying

$$\langle X_j\,, X_k\rangle = \delta_{jk}. \tag{1.3}$$

Denote the universal enveloping algebra of $\mathfrak{g}$ by $\mathcal{U}(\mathfrak{g})$. Let $\mathbf{c}_\mathfrak{q}$ be the degree three element in $\mathcal{C}l(\mathfrak{q})$ defined as the image under the Chevalley isomorphism of the 3-form

$$(X, Y, Z) \mapsto \langle X\,, [Y, Z]\rangle$$

on $\mathfrak{q}$. The element $\sum X_j \otimes (\gamma(X_j) \otimes 1) - 1 \otimes (\gamma(\mathbf{c}_\mathfrak{q}) \otimes 1)$ in $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(S_\mathfrak{q} \otimes E)$ is H-invariant so defines a G-invariant differential operator

$$\mathcal{D}_{G/H}(\mathcal{E}) : C^\infty(G/H, \mathcal{S}_\mathfrak{q} \otimes \mathcal{E}) \to C^\infty(G/H, \mathcal{S}_\mathfrak{q} \otimes \mathcal{E})$$

acting on $C^\infty(G/H, \mathcal{S}_\mathfrak{q} \otimes \mathcal{E})$. We refer to $\mathcal{D}_{G/H}(\mathcal{E})$ as the geometric (cubic) Dirac operator; it is given by the following formula:

$$\mathcal{D}_{G/H}(\mathcal{E}) = \sum R(X_j) \otimes \gamma(X_j) \otimes 1 - 1 \otimes \gamma(\mathbf{c}_\mathfrak{q}) \otimes 1. \tag{1.4}$$

Note that $\mathcal{D}_{G/H}(\mathcal{E})$ is independent of the basis $\{X_j\}$ satisfying (1.3) (since each of the two terms is, by itself, independent of the basis). We will often write $\mathcal{D}_{G/H}$ for $\mathcal{D}_{G/H}(\mathcal{E})$.

1.4. **Dirac cohomology.** Associated with a $\mathfrak{g}$-module (π, V), there is an *algebraic cubic Dirac operator* $D_V : V \otimes S_\mathfrak{q} \longrightarrow V \otimes S_\mathfrak{q}$ defined by

$$D_V = \sum_j \pi(X_j) \otimes \gamma(X_j) - 1 \otimes \gamma(\mathbf{c}_\mathfrak{q}). \tag{1.5}$$

The following formula[1] for the square of D_V is due to Kostant ([8, Theorem 2.16]):

$$2D_V^2 = \Omega_\mathfrak{g} \otimes 1 - \Omega_{\Delta\mathfrak{h}} + ||\rho(\mathfrak{g})||^2 - ||\rho(\mathfrak{h})||^2, \tag{1.6}$$

where $\Omega_\mathfrak{g}$ is the Casimir element for $\mathfrak{g}$ acting on V and $\Omega_{\Delta\mathfrak{h}}$ is the Casimir element of $\mathfrak{h}$ acting in $V \otimes S_\mathfrak{q}$. In the case when $\mathfrak{h} = \mathfrak{k}$ the cubic term $\mathbf{c}_\mathfrak{q}$ vanishes and formula (1.6) is due to Parthasarathy (see [14]).

The (cubic) Dirac cohomology of the $\mathfrak{g}$-module V is the $\mathfrak{h}$-module defined as the quotient

$$H^{(\mathfrak{h},\mathfrak{g})}(V) = \ker(D_V)/\ker(D_V) \cap \mathrm{Im}(D_V).$$

The Dirac cohomology will also be denoted by $H_D(V)$ when the pair $(\mathfrak{h}, \mathfrak{g})$ is understood.

Finally we note that in the case when V is a unitarizable $(\mathfrak{g}, K)$-module, i.e., V has a nondegenerate $(\mathfrak{g}, K)$-invariant positive definite hermitian form $\langle\ ,\ \rangle_V$, then one gets a nondegenerate hermitian form on $V \otimes S_\mathfrak{q}$ defined by

$$\langle\ ,\ \rangle_{V\otimes S_\mathfrak{q}} = \langle\ ,\ \rangle_V \otimes \langle\ ,\ \rangle_{S_\mathfrak{q}} \tag{1.7}$$

[1]The factor of 2 in this formula does not appear in [8]. This is because we are taking $xy + yx = \langle x\,, y\rangle$ in the definition of the Clifford algebra, while $xy + yx = 2\langle x\,, y\rangle$ is used in [8].

with respect to which D_V is selfadjoint ([17, Lemma 9.3.3]). In the case where $\mathfrak{h} = \mathfrak{k}$, since the hermitian form on $S_\mathfrak{s}$ is positive definite, the form $\langle\ ,\ \rangle_{V\otimes S_\mathfrak{s}}$ is also positive definite and it follows that

$$H_D(V) = \ker(D_V).$$

When $\mathfrak{h} \subset \mathfrak{k}$, the same conclusion holds for $H_D^{(\mathfrak{h},\mathfrak{k})}(V)$ for a finite dimensional representation V.

1.5. **Algebraic Dirac operators vs. geometric Dirac operators.** Let V be a smooth admissible representation of G, V_K the space of K-finite vectors in V and $V_K^\star$ the K-finite dual of V_K. Let E be a finite dimensional representation of $\mathfrak{k}$ such that the tensor product $S_\mathfrak{s} \otimes E$ with the spin representation of $\mathfrak{k}$ lifts to a representation of the group K. Then $S_\mathfrak{s} \otimes E$ induces a homogeneous bundle $\mathcal{S}_\mathfrak{s} \otimes \mathcal{E} \longrightarrow G/K$. There is a vector space isomorphism

$$\mathrm{Hom}_G(V, C^\infty(G/K, S_\mathfrak{s} \otimes \mathcal{E})) \simeq \mathrm{Hom}_K(E^*, S_\mathfrak{s} \otimes V_K^*)$$

given by $T \mapsto T_1$, with $T_1(e^*)(v) = \langle e^*\,, T(v)(1)\rangle$, where $e^* \in E^*, v \in V$ and $1 \in G$ is the identity element. In addition one has $\big(D_{V_K^*} T_1(e^*)\big)(v) = \langle e^*\,, \big(\mathcal{D}_{G/K} T(v)\big)(1)\rangle$, for all $e^* \in E^*$ and $v \in V$. One may conclude the following.

Proposition 1.8. *$Hom_G(V, ker(\mathcal{D}_{G/K}(\mathcal{E}))) \simeq Hom_K(E^*, ker(D_{V_K^*}))$.*

See the appendix for details.

There is also an isomorphism

$$\mathrm{Hom}_K(E^*, V_K^* \otimes S_\mathfrak{s}) \simeq \mathrm{Hom}_K(V_K \otimes S_\mathfrak{s}, E)$$

given by $B \mapsto b$, with $\langle s\,, B(e^*)(v)\rangle = \langle e^*\,, b(v \otimes s)\rangle$. We also have the identity

$$\langle s\,, \big(D_{V_K^*} B(e^*)\big)(v)\rangle = \langle e^*\,, b\big(D_{V_K}(v \otimes s)\big)\rangle.$$

The pairing on the lefthand side is a nondegenerate pairing of the self-dual representation $S_\mathfrak{s}$ with itself. From this we may conclude that

$$\mathrm{Hom}_K(E^*, \ker(D_{V_K^*})) \simeq \mathrm{Hom}_K((V_K \otimes S)/\mathrm{Im}(D_{V_K}), E).$$

Now assume that V_K is unitarizable. Then D_{V_K} is self adjoint and

$$(V_K \otimes S)/\mathrm{Im}(D_{V_K}) \simeq \mathrm{Im}(D_{V_K})^\perp \simeq \ker(D_{V_K}).$$

We may conclude that $\ker(D_{V_K^*}) \simeq \big(\ker(D_{V_K})\big)^*$, as K-modules. Therefore, by §1.4,

$$H_D(V_K^*) \simeq \big(H_D(V_K)\big)^*. \tag{1.9}$$

The above discussion applies to the Dirac operator on the homogeneous space K/H, when $H \subset K$, resulting in the statement that

$$\mathrm{Hom}_K(E, \ker(\mathcal{D}_{K/H}(\mathcal{F}))) \simeq \mathrm{Hom}_H(F, H_D^{(\mathfrak{h},\mathfrak{k})}(E)). \tag{1.10}$$

2. The fundamental series

An important special case of our main result occurs when H is compact. In this section we see that in this case (under certain mild conditions) the kernel of $\mathcal{D}_{G/H}$ is nonzero. In fact the kernel contains certain fundamental series representations. This is analogous to the well-know fact that $\ker(\mathcal{D}_{G/K})$ contains a discrete series representation ([14], [1]) when $\operatorname{rank}_{\mathbf{C}}(\mathfrak{g}) = \operatorname{rank}_{\mathbf{C}}(\mathfrak{k})$.

2.1. **Cartan subalgebras and roots.** We assume that $\mathfrak{h}$ is as in §1.1 and that $\mathfrak{h} \subset \mathfrak{k}$. Let $\mathfrak{t}_{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{h}$. Extend to a Cartan subalgebra $\mathfrak{t} = \mathfrak{t}_{\mathfrak{h}} + \mathfrak{t}_{\mathfrak{q}}$ of $\mathfrak{k}$ by choosing $\mathfrak{t}_{\mathfrak{q}} \subset \mathfrak{k} \cap \mathfrak{q}$. Now extend to a Cartan subalgebra $\mathfrak{t} + \mathfrak{a}$ of $\mathfrak{g}$ by choosing $\mathfrak{a}$ abelian in $\mathfrak{s}$.

Let $\Delta^+ \subset \Delta(\mathfrak{t} + \mathfrak{a}, \mathfrak{g})$ be defined by a lexicographic order with $\mathfrak{t}_{\mathfrak{h}}$ first, then $\mathfrak{t}_{\mathfrak{q}}$, then $\mathfrak{a}$. Such a positive system has the property that

$$\Delta^+(\mathfrak{h}) := \{\alpha|_{\mathfrak{t}_{\mathfrak{h}}} : \mathfrak{g}^{(\alpha)} \subset \mathfrak{h}, \alpha \in \Delta^+ \text{ and } \alpha|_{\mathfrak{t}_{\mathfrak{h}}} \neq 0\}$$

is a positive system of roots in $\mathfrak{h}$. Here we are denoting the α-root space in $\mathfrak{g}$ by $\mathfrak{g}^{(\alpha)}$.

Note that $\mathfrak{t}+\mathfrak{a}$ is a fundamental Cartan subalgebra of $\mathfrak{g}$, i.e., is maximally compact. The positive systems described above may also be described as follows. There is $\Lambda_0 \in (\mathfrak{t} + \mathfrak{a})^*$ with $\Lambda_0|_{\mathfrak{a}} = 0$ so that $\Delta^+ = \{\alpha : \langle \Lambda_0 , \alpha\rangle > 0\}$. The Borel subalgebras that arise from such a positive system are the θ-stable Borel subalgebras containing $\mathfrak{t} + \mathfrak{a}$. This gives a positive system of $\mathfrak{t}$-roots in $\mathfrak{k}$: $\Delta^+(\mathfrak{k}) = \{\beta \in \Delta(\mathfrak{t}, \mathfrak{k}) : \langle \Lambda_0 , \beta\rangle > 0\}$.

Suppose that $\mu \in \mathfrak{t}_{\mathfrak{h}}^*$ and that μ is dominant for an *arbitrary* positive system. Then by choosing μ as the first basis vector defining a lexicographic order as above, we arrive at positive systems Δ^+ and $\Delta^+(\mathfrak{h})$ with the property that μ, extended to be 0 on $\mathfrak{t}_{\mathfrak{q}} + \mathfrak{a}$, is dominant for both Δ^+ and $\Delta^+(\mathfrak{h})$. In §2.3, where we begin with a finite dimensional representation E of $\mathfrak{h}$, we may therefore assume that the highest weight μ of E is Δ^+-dominant.

We make the following assumption on H. This is the assumption on H not being to small mentioned in the introduction; it will be necessary for certain Dirac cohomology spaces to be nonzero. See §2.3.

Assumption 2.1. There is no root $\alpha \in \Delta(\mathfrak{g})$ so that $\alpha|_{\mathfrak{t}_{\mathfrak{h}}} = 0$.

Note that this assumption automatically holds when either $H = K$ or $\mathfrak{h}$ and $\mathfrak{g}$ have equal rank.

Under this assumption we may construct the spin representation $S_{\mathfrak{q}}$ by choosing a maximal isotropic subspace $\mathfrak{q}^+$ of $\mathfrak{q}$ as follows. Choose any maximal isotropic subspace $(\mathfrak{t}_{\mathfrak{q}} + \mathfrak{a})^+$ of $\mathfrak{t}_{\mathfrak{q}} + \mathfrak{a}$, then set

$$\mathfrak{q}^+ = (\mathfrak{t}_{\mathfrak{q}} + \mathfrak{a})^+ + \sum_{\gamma} \mathfrak{q}^{(\gamma)},$$

where the sum is over all $\mathfrak{t}_{\mathfrak{h}}$-weights γ in $\mathfrak{q}$. The assumption tells us that no such γ is zero, so $\mathfrak{q}^+$ is indeed maximally isotropic.

We will use the notation of $\rho(\mathfrak{g})$ for one half the sum of the roots in Δ^+. Similarly $\rho(\mathfrak{h})$ denotes one half the sum of the roots in $\Delta^+(\mathfrak{h})$. We also use the notation $\rho(\mathfrak{q})$ for one half the sum of the $\mathfrak{t}_\mathfrak{h}$-weights in $\mathfrak{q}^+$, and similarly for $\rho(\mathfrak{k}\cap\mathfrak{q})$.

2.2. **The fundamental series and its Dirac cohomology.** The fundamental series representations are cohomologically induced representations. They arise as follows. Let $\mathfrak{b}$ be a θ-stable Borel subalgebra in $\mathfrak{g}$ which contains the fundamental Cartan subalgebra $\mathfrak{t}+\mathfrak{a}$. The positive system associated with $\mathfrak{b}$ is described in the previous subsection. Write $\mathfrak{b}=\mathfrak{t}+\mathfrak{a}+\mathfrak{u}$ for the Levi decomposition of $\mathfrak{b}$. Then a fundamental series representation is a cohomologically induced representation $\mathcal{A}_\mathfrak{b}(\lambda)$, for $\lambda\in(\mathfrak{t}+\mathfrak{a})^\star$, with $\lambda_{|_\mathfrak{a}}=0$ and λ a Δ^+-dominant and analytically integral weight, having the following properties.

(a) The infinitesimal character is $\lambda+\rho(\mathfrak{g})$.

(b) $\lambda_{|_\mathfrak{t}}+2\rho(\mathfrak{s}\cap\mathfrak{u})$ is the highest weight of a lowest K-type with respect to $\Delta^+(\mathfrak{k})$, where $\rho(\mathfrak{s}\cap\mathfrak{u})$ denotes half the sum of the $\mathfrak{t}$-weights in $\mathfrak{s}\cap\mathfrak{u}$.

(c) Each K-type has highest weight of the form $\lambda_{|_\mathfrak{t}}+2\rho(\mathfrak{s}\cap\mathfrak{u})+\sum_{\gamma\in\Delta(\mathfrak{s}\cap\mathfrak{u})}n_\gamma\gamma$, where n_γ are non negative integers.

It is know from Vogan and Zuckerman ([16, Theorem 2.5]) that these properties uniquely determine $\mathcal{A}_\mathfrak{b}(\lambda)$. The fundamental series representations are irreducible and unitarizable ([15], [17, Ch. 6]).

The computation of the Dirac cohomology, with respect to $\mathfrak{k}\subset\mathfrak{g}$, is straightforward using Kostant's formula (1.6) for the square of $D_{\mathcal{A}_\mathfrak{b}(\lambda)}:\mathcal{A}_\mathfrak{b}(\lambda)\otimes S_\mathfrak{s}\to\mathcal{A}_\mathfrak{b}(\lambda)\otimes S_\mathfrak{s}$ and the properties (a)-(c) above. Although this is contained in Theorem 5.2 of [7], we give the short proof here. First, by the unitarizability of $\mathcal{A}_\mathfrak{b}(\lambda)$, the Dirac cohomology is $\ker(D_{\mathcal{A}_\mathfrak{b}(\lambda)})=\ker(D^2_{\mathcal{A}_\mathfrak{b}(\lambda)})$. Let F_τ be the finite dimensional highest weight representation of $\mathfrak{k}$ with highest weight τ with respect to $\Delta^+(\mathfrak{k})$. By Kostant's formula (1.6), F_τ occurs in $\ker(D_{\mathcal{A}_\mathfrak{b}(\lambda)})$ if F_τ occurs in $\mathcal{A}_\mathfrak{b}(\lambda)\otimes S_\mathfrak{s}$ and $||\lambda+\rho(\mathfrak{g})||=||\tau+\rho(\mathfrak{k})||$.

The $\mathfrak{t}$-weights of $S_\mathfrak{s}$ are all weights of the form $\frac{1}{2}(\pm\gamma_1\pm\cdots\pm\gamma_k)$ with $\gamma_i\in\Delta(\mathfrak{s}\cap\mathfrak{u})$. Each weight occurs with multiplicity 2^d, where d is the greatest integer in $\dim(\mathfrak{a})/2$. We use the notation $\langle A\rangle=\sum_{\alpha\in A}\alpha$, for any set A of weights. With this notation the $\mathfrak{t}$-weights in $S_\mathfrak{s}$ are

$$\begin{aligned}\Delta(S_\mathfrak{s})&=\{\langle A\rangle-\rho(\mathfrak{s}\cap\mathfrak{u}):A\subset\Delta(\mathfrak{s}\cap\mathfrak{u})\}\\&=\{\rho(\mathfrak{s}\cap\mathfrak{u})-\langle A\rangle:A\subset\Delta(\mathfrak{s}\cap\mathfrak{u})\}.\end{aligned}$$

Each constituent of $\mathcal{A}_\mathfrak{b}(\lambda)\otimes S_\mathfrak{s}$ has highest weight of the form

$$\begin{aligned}\tau&=\lambda_{|_\mathfrak{t}}+2\rho(\mathfrak{s}\cap\mathfrak{u})+\sum_{\gamma\in\Delta(\mathfrak{s}\cap\mathfrak{u})}n_\gamma\gamma+(\langle A\rangle-\rho(\mathfrak{s}\cap\mathfrak{u}))\\&=\lambda_{|_\mathfrak{t}}+\rho(\mathfrak{s}\cap\mathfrak{u})+\sum_{\gamma\in\Delta(\mathfrak{s}\cap\mathfrak{u})}m_\gamma\gamma,\quad\text{for some nonnegative integers } m_\gamma.\end{aligned}$$

Now

$$\begin{aligned}||\tau+\rho(\mathfrak{k})||^2 &= ||\lambda+\rho(\mathfrak{g})+\sum_{\gamma\in\Delta(\mathfrak{s}\cap\mathfrak{u})} m_\gamma\gamma||^2 \\ &= ||\lambda+\rho(\mathfrak{g})||^2+||\sum_{\gamma\in\Delta(\mathfrak{s}\cap\mathfrak{u})} m_\gamma\gamma||^2+\sum_{\gamma\in\Delta(\mathfrak{s}\cap\mathfrak{u})} m_\gamma\langle\lambda+\rho(\mathfrak{g})\,,\gamma\rangle.\end{aligned}$$

For this to equal $||\lambda+\rho(\mathfrak{g})||^2$ one must have $m_\gamma=0$ (because $\langle\lambda+\rho(\mathfrak{g})\,,\gamma\rangle>0$ and $m_\gamma\geq 0$). Therefore, all $n_\gamma=0$ and $\langle A\rangle=0$. So $\tau=\lambda_{|\mathfrak{t}}+\rho(\mathfrak{s}\cap\mathfrak{u})$ and the multiplicity is 2^d, where d is the greatest integer in $\dim(\mathfrak{a})/2$. This proves the following statement.

Proposition 2.2. *Let $\mathcal{A}_\mathfrak{b}(\lambda)$ be the cohomologically induced representation described in (a)-(c) above. Then*

$$H^{(\mathfrak{k},\mathfrak{g})}(\mathcal{A}_\mathfrak{b}(\lambda))=2^dF_{\lambda_{|\mathfrak{t}}+\rho(\mathfrak{s}\cap\mathfrak{u})}. \tag{2.3}$$

Now let $\mu\in\mathfrak{t}^*$ be $\Delta^+(\mathfrak{k})$-dominant and integral. Let E_μ be the irreducible finite dimensional representation of $\mathfrak{k}$ with highest weight μ. Define $\lambda_\mu\in(\mathfrak{t}+\mathfrak{a})^*$ by $\lambda_\mu|_\mathfrak{t}=\mu-\rho(\mathfrak{s}\cap\mathfrak{u})$ and $\lambda_\mu|_\mathfrak{a}=0$. Then, as described at the end of §2.1, the positive system Δ^+ can be chosen so that λ_μ is Δ^+-dominant. With such a choice of Δ^+ we have the following corollary, which is a consequence of Prop. 1.8 and equation (1.9).

Corollary 2.4. *Suppose μ is $\Delta^+(\mathfrak{k})$-dominant and $S_\mathfrak{s}\otimes E_\mu$ lifts to a representation of K. Then the kernel of $\mathcal{D}_{G/K}:C^\infty(G/K,\mathcal{S}\otimes\mathcal{E}_\mu)\to C^\infty(G/K,\mathcal{S}\otimes\mathcal{E}_\mu)$ contains a smooth G-representation infinitesimally equivalent to $\mathcal{A}_\mathfrak{b}(\lambda_\mu)$.*

Proof. Since $\lambda_\mu+\rho$ is Δ^+-dominant, Prop. 2.2 gives $\mathrm{Hom}_K\big(E_\mu,H^{(\mathfrak{g},\mathfrak{k})}(\mathcal{A}_\mathfrak{b}(\lambda_\mu))\big)\neq 0$. Now Prop. 1.8 (along with (1.9)) gives

$$\mathrm{Hom}_{(\mathfrak{g},K)}\big(\mathcal{A}_\mathfrak{b}(\lambda_\mu),\ker(\mathcal{D}_{G/K}(\mathcal{E}_\mu))\big)\neq 0.$$

□

2.3. **Induction in stages.** Now assume that $H\subset K$. Let F_μ be the irreducible finite dimensional representation of H having highest weight $\mu\in\mathfrak{t}_\mathfrak{h}^*$. As described at the end of §2.1, we may assume that the extension of μ (by 0 on $\mathfrak{t}_\mathfrak{q}+\mathfrak{a}$) is Δ^+-dominant, and therefore its restriction to $\mathfrak{t}$ is $\Delta^+(\mathfrak{k})$-dominant. Let E_ξ be the irreducible finite dimensional representation of $\mathfrak{k}$ of highest weight $\xi\in\mathfrak{t}^*$.

In [13] it is shown that $F_\mu\subset H_D^{(\mathfrak{h},\mathfrak{k})}(E_\xi)$ if and only if $\mu=w(\xi+\rho(\mathfrak{k}))-\rho(\mathfrak{h})$ for some $w\in W(\mathfrak{k})$ having the property that $w(\xi+\rho(\mathfrak{k}))|_{\mathfrak{t}_\mathfrak{q}}=0$.

Define

$$\begin{aligned}\xi|_{\mathfrak{t}_\mathfrak{h}}&=\mu-(\rho(\mathfrak{k})-\rho(\mathfrak{h}))|_{\mathfrak{t}_\mathfrak{h}}\\ \xi|_{\mathfrak{t}_\mathfrak{q}}&=-\rho(\mathfrak{k})|_{\mathfrak{t}_\mathfrak{q}}.\end{aligned} \tag{2.5}$$

Then $\xi+\rho(\mathfrak{k})$ is $\Delta^+(\mathfrak{k})$-dominant. By taking $w=e$ above, we have $F_\mu\subset H_D^{(\mathfrak{h},\mathfrak{k})}(E_\xi)$. We may conclude from (1.10) that $E_\xi\subset\ker\big(\mathcal{D}_{K/H}(\mathcal{F}_\mu)\big)$.

Our goal now is to realize a fundamental series representation in the kernel of $\mathcal{D}_{G/H}(\mathcal{F}_\mu)$. The induction in stages argument begins with the identification

$$\begin{aligned} C^\infty(G/H, S_{\mathfrak{q}}\otimes\mathcal{F}_\mu) &\simeq C^\infty(G/K, S_{\mathfrak{s}}\otimes C^\infty(K/H, S_{\mathfrak{k}\cap\mathfrak{q}}\otimes\mathcal{F}_\mu)), \\ f &\longleftrightarrow F_f, \end{aligned} \tag{2.6}$$

with $(F_f(g))(k) = (k\otimes 1)\cdot f(gk)$. In [12] it is shown that

$$\big(F_{\mathcal{D}_{G/H}f}\big)(g) = \big(\mathcal{D}_{G/K}F_f\big)(g) + \mathcal{D}_{K/H}\big(F_f(g)\big). \tag{2.7}$$

With ξ defined as in (2.5), we have seen that E_ξ may be realized as a subspace of $\ker\big(\mathcal{D}_{K/H}(\mathcal{F}_\mu)\big)\subset C^\infty(K/H, S_{\mathfrak{k}\cap\mathfrak{q}}\otimes\mathcal{F}_\mu)$. Therefore, when restricted to $C^\infty(G/K,\mathcal{E}_\xi)$, under the identification of (2.6),

$$\mathcal{D}_{G/K}(F_f) = F_{\mathcal{D}_{G/H}(f)}. \tag{2.8}$$

Proposition 2.9. *The kernel of $\mathcal{D}_{G/H}(\mathcal{F}_\mu)$ contains a smooth representation infinitesimally equivalent to a cohomologically induced representation $\mathcal{A}_{\mathfrak{b}}(\lambda_\mu)$ with*

$$\lambda_\mu|_{\mathfrak{t}_{\mathfrak{h}}} = \mu-\rho(\mathfrak{q}),\ \ \lambda|_{\mathfrak{a}} = 0 \ \ \textit{and} \ \ \lambda|_{\mathfrak{t}_{\mathfrak{q}}} = -\rho(\mathfrak{k})|_{\mathfrak{t}_{\mathfrak{q}}}.$$

Proof. Let ξ be as in (2.5) and realize $E_\xi\subset\ker(\mathcal{D}_{K/H}(\mathcal{F}_\mu))$. So we may consider

$$\begin{aligned} C^\infty(G/K, S_{\mathfrak{s}}\otimes\mathcal{E}_\xi) &\subset C^\infty(G/K, S_{\mathfrak{s}}\otimes C^\infty(K/H, S_{\mathfrak{k}\cap\mathfrak{q}}\otimes\mathcal{F}_\mu)) \\ &\simeq C^\infty(G/H, S_{\mathfrak{q}}\otimes\mathcal{F}_\mu). \end{aligned}$$

By (2.8)

$$\ker\big(\mathcal{D}_{G/K}(\mathcal{E}_\xi)\big)\subset\ker\big(\mathcal{D}_{G/H}(\mathcal{F}_\mu)\big).$$

Applying (2.4),

$$\mathcal{A}_{\mathfrak{b}}(\lambda_\mu)\subset\ker(\mathcal{D}_{G/K}(\mathcal{E}_\xi))\subset\ker(\mathcal{D}_{G/H}(\mathcal{F}_\mu)),$$

since $\lambda_\mu|_{\mathfrak{t}} = \xi|_{\mathfrak{t}}-\rho(\mathfrak{s})$. □

Note that the λ_μ appearing in the proposition, is such that $\lambda_\mu+\rho(\mathfrak{g})$ is dominant, but need not be regular. Therefore, $\mathcal{A}_{\mathfrak{b}}(\lambda_\mu)$ may equal 0. The further condition that λ_μ be dominant will ensure that $\mathcal{A}_{\mathfrak{b}}(\lambda_\mu)$ is nonzero.

The following lemma will be important in §3.4. Note that the map $eval_e : C^\infty(G/H, S_{\mathfrak{q}}\otimes\mathcal{F}_\mu)\to S_{\mathfrak{q}}\otimes F_\mu$ given by $eval_e(f) = f(e)$ (with e the identity element of G) is an H-homomorphism. Assuming

$$\text{(i)}\ \langle\mu+\rho(\mathfrak{h})-\langle B\rangle,\beta\rangle\geq 0,\ \text{for all}\ \beta\in\Delta^+(\mathfrak{h})\ \text{and}\ B\subset\Delta(\mathfrak{q}^+);$$

$$\text{(ii)}\ \langle\mu+\rho(\mathfrak{h})-2\rho(\mathfrak{k}\cap\mathfrak{q}),\beta\rangle > 0,\ \text{for all}\ \beta\in\Delta^+(\mathfrak{h}),$$

Steinberg's formula for the decomposition of the tensor product of two finite dimensional representations of H tells us that $S_{\mathfrak{q}}\otimes F_\mu$ contains the irreducible representation of H having highest weight $\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{k}\cap\mathfrak{q})$. Let V_0 be the corresponding isotypic subspace of $S_{\mathfrak{q}}\otimes F_\mu$.

Lemma 2.10. *Realizing $A_{\mathfrak{b}}(\lambda_\mu)\subset ker(\mathcal{D}_{G/H}(\mathcal{F}_\mu))$, we have $eval_e(A_{\mathfrak{b}}(\lambda_\mu))\subset V_0$.*

Proof. We first give the proof for the case $H = K$. In this case $\mathfrak{q} = \mathfrak{s}$ and $\mathfrak{k} \cap \mathfrak{q} = 0$, so $\lambda_\mu = \mu - \rho(\mathfrak{s})$. The possible K-types in $A_\mathfrak{b}(\lambda_\mu)$ have the form $\lambda_\mu + 2\rho(\mathfrak{s}) + \sum n_\gamma \gamma = \mu + \rho(\mathfrak{s}) + \sum n_\gamma \gamma$, with $\gamma \in \Delta^+(\mathfrak{s}), n_\gamma \geq 0$. On the other hand, the K-constituents of $S_\mathfrak{q} \otimes F_\mu = S_\mathfrak{s} \otimes F_\mu$ all have highest weights of the form $\mu + \rho(\mathfrak{s}) - \langle B \rangle, B \subset \Delta^+(\mathfrak{s})$. Since the image of $eval_e$ must consist of K-types common to $A_\mathfrak{b}(\lambda_\mu)$ and $S_\mathfrak{q} \otimes F_\mu$, the only possibility is that all $n_\gamma = 0$ and $\langle B \rangle = 0$. Therefore, $eval_e(A_\mathfrak{b}(\lambda_\mu))$ is contained in the isotypic subspace with highest weight $\mu + \rho(\mathfrak{s}) = \mu + \rho(\mathfrak{q}) - 2\rho(\mathfrak{k} \cap \mathfrak{q})$, i.e., $eval_e(A_\mathfrak{b}(\lambda_\mu)) \subset V_0$.

Now consider arbitrary $H \subset K$. Since $f(e) = (F_f(e))(e)$, we first consider $F_f \mapsto F_f(e)$. By the $H = K$ case, $F_f(e)$ is in the isotypic subspace of $S_\mathfrak{s} \otimes E_\xi$ of type $E_{\xi+\rho(\mathfrak{s})}$ (with ξ as in (2.5)) . Now, evaluation at e gives an H-homomorphism $E_{\xi+\rho(\mathfrak{s})} \to S_\mathfrak{q} \otimes F_\mu$. Again we compare the H-types. The highest weights of H-constituents of $E_{\xi+\rho(\mathfrak{s})}$ are of the form $\xi + \rho(\mathfrak{s}) - \langle A \rangle, A \subset \Delta^+(\mathfrak{h})$. But $\xi + \rho(\mathfrak{s}) - \langle A \rangle = \mu - \rho(\mathfrak{k} \cap \mathfrak{q}) + \rho(\mathfrak{s}) - \langle A \rangle = \mu + \rho(\mathfrak{q}) - 2\rho(\mathfrak{k} \cap \mathfrak{q}) - \langle A \rangle$. The H-types in $S_\mathfrak{q} \otimes F_\mu$ are of the form $\mu + \rho(\mathfrak{q}) - \langle B \rangle, B \subset \Delta(\mathfrak{k} \cap \mathfrak{q}^+)$. The only way for us to have $\mu + \rho(\mathfrak{q}) - 2\rho(\mathfrak{k} \cap \mathfrak{q}) - \langle A \rangle = \mu + \rho(\mathfrak{q}) - \langle B \rangle$ is for $\langle B \rangle = 2\rho(\mathfrak{k} \cap \mathfrak{q}) + \langle A \rangle$. As $B \subset \Delta(\mathfrak{k} \cap \mathfrak{q}^+)$, this means that $B = \Delta(\mathfrak{k} \cap \mathfrak{q}^+)$ and $\langle A \rangle = 0$. We conclude that the image of $eval_e|_{A_\mathfrak{b}(\lambda_\mu)}$ is contained in V_0. □

3. The main theorem

Let H be an arbitrary subgroup of G satisfying (1.1). We associate to H a parabolic subgroup P in G. The main result is the construction of an integral intertwining map $\mathcal{P} : C^\infty(G/P, \mathcal{W}) \to \ker(\mathcal{D}_{G/H})$. The precise statement is contained in Theorem 3.9.

3.1. **Roots and positive systems.** We need to make some choices of Cartan subalgebras and positive root systems, which will be used in the construction of our intertwining operator. These choices are compatible with those made in §2.1, where the special case of $\mathfrak{h} \subset \mathfrak{k}$ was considered.

Consider the complexified Lie algebra $\mathfrak{h}$ of the reductive group H and the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$. Choose a maximal abelian subspace $\mathfrak{a}_\mathfrak{h}$ of $\mathfrak{h} \cap \mathfrak{s}$. Define $\mathfrak{l} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}_\mathfrak{h})$, the centralizer of $\mathfrak{a}_\mathfrak{h}$ in $\mathfrak{g}$. A Cartan subalgebra of $\mathfrak{g}$ is chosen as follows.

- Let $\mathfrak{t}_\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{h} \cap \mathfrak{k} \cap \mathfrak{l}$. Note that $\mathfrak{a}_\mathfrak{h} + \mathfrak{t}_\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$, since $\mathfrak{a}_\mathfrak{h}$ is maximal abelian in $\mathfrak{h} \cap \mathfrak{s}$.
- Extend $\mathfrak{t}_\mathfrak{h}$ to a Cartan subalgebra $\mathfrak{t} = \mathfrak{t}_\mathfrak{h} + \mathfrak{t}_\mathfrak{q}$ of $\mathfrak{k} \cap \mathfrak{l}$ with $\mathfrak{t}_\mathfrak{q} \subset \mathfrak{q} \cap \mathfrak{k} \cap \mathfrak{l}$. Note that $\mathfrak{t}$ is not necessarily a Cartan subalgebra of $\mathfrak{k}$.
- Finally, choose $\mathfrak{a}_\mathfrak{q} \subset \mathfrak{q} \cap \mathfrak{s} \cap \mathfrak{l}$ so that $\mathfrak{a}_\mathfrak{h} + \mathfrak{t} + \mathfrak{a}_\mathfrak{q}$ is a Cartan subalgebra of $\mathfrak{l}$. Write $\mathfrak{a} := \mathfrak{a}_\mathfrak{h} + \mathfrak{a}_\mathfrak{q}$. Since $\mathrm{rank}_\mathbf{C}(\mathfrak{g}) = \mathrm{rank}_\mathbf{C}(\mathfrak{l})$, we see that $\mathfrak{a} + \mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{g}$.

Remarks. (1) When $\mathfrak{h} \subset \mathfrak{k}$, $\mathfrak{a}_\mathfrak{h} = \{0\}$. Therefore $\mathfrak{l} = \mathfrak{g}$ and $\mathfrak{a} + \mathfrak{t}$ is a fundamental Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{l}$, as in §2.1.

(2) When $\mathrm{rank}_\mathbf{C}(\mathfrak{h}) = \mathrm{rank}_\mathbf{C}(\mathfrak{g})$ (as in [12]) $\mathfrak{t} = \mathfrak{t}_\mathfrak{h}$ and $\mathfrak{a} = \mathfrak{a}_\mathfrak{h}$.

Let $\Delta := \Delta(\mathfrak{a}+\mathfrak{t},\mathfrak{g})$ be the set of $\mathfrak{a}+\mathfrak{t}$-roots in $\mathfrak{g}$. For any $\alpha \in \Delta$ we will write $\mathfrak{g}^{(\alpha)}$ for the corresponding root space.

Let Δ^+ be any positive system of roots in $\Delta := \Delta(\mathfrak{a}+\mathfrak{t},\mathfrak{g})$ given by a lexicographic order with a basis of $\mathfrak{a}_\mathfrak{h}$ first, then (in order) bases of $\mathfrak{t}_\mathfrak{h}, \mathfrak{t}_\mathfrak{q}$ and $\mathfrak{a}_\mathfrak{q}$.

A positive system of $(\mathfrak{a}_\mathfrak{h}+\mathfrak{t}_\mathfrak{h})$-roots in $\mathfrak{h}$ is chosen using the lexicographic order with the same basis of $\mathfrak{a}_\mathfrak{h}$ as above, followed by the basis of $\mathfrak{t}_\mathfrak{h}$. Call this positive system $\Delta^+(\mathfrak{h})$.

3.2. **The parabolic subgroup.** Having fixed a positive system of roots Δ^+ in $\mathfrak{g}$ we may define a parabolic subalgebra of $\mathfrak{g}$ as follows. Set

$$\Sigma^+ := \{\alpha \in \Delta^+ : \alpha|_{\mathfrak{a}_\mathfrak{h}} \neq 0\}.$$

Then

$$\mathfrak{p} := \mathfrak{l} + \mathfrak{n}, \text{ where } \mathfrak{n} := \sum_{\alpha \in \Sigma^+} \mathfrak{g}^{(\alpha)},$$

is a parabolic subalgebra of $\mathfrak{g}$. Thanks to the choice of Δ^+, $\mathfrak{p}$ is the complexification of a (real) subalgebra of $Lie(G)$. We define P to be the *connected* subgroup of G corresponding to this real parabolic Lie algebra.

It will be convenient for us to write $\mathfrak{l} = \mathfrak{m} + \mathfrak{a}_\mathfrak{h}$ with

$$\mathfrak{m} = \sum_{\alpha \in \Delta, \alpha|_{\mathfrak{a}_\mathfrak{h}}=0} \mathfrak{g}^{(\alpha)} + (\mathfrak{a}_\mathfrak{q} + \mathfrak{t}).$$

Therefore,

$$\mathfrak{p} = \mathfrak{m} + \mathfrak{a}_\mathfrak{h} + \mathfrak{n}. \tag{3.1}$$

However, this is *not* (typically) the Langlands decomposition of $\mathfrak{p}$. Note that $\mathfrak{a}_\mathfrak{q} \subset \mathfrak{l} \cap \mathfrak{s}$, but it can happen that some (but not all) of $\mathfrak{a}_\mathfrak{q}$ lies in the center of $\mathfrak{l}$. The decomposition (3.1) gives a corresponding decomposition of P, which we write as $P = MA_\mathfrak{h}N$.

Lemma 3.2. *$\mathfrak{p} \cap \mathfrak{h}$ is a minimal parabolic subalgebra of $\mathfrak{h}$. In particular, $\mathfrak{m} \cap \mathfrak{h} \subset \mathfrak{k}$ and $\mathfrak{l} \cap \mathfrak{h} = \mathfrak{l} \cap \mathfrak{h} \cap \mathfrak{k} + \mathfrak{a}_\mathfrak{h}$.*

Proof. The Borel subalgebra of $\mathfrak{h}$ defined by $\mathfrak{a}_\mathfrak{h} + \mathfrak{t}_\mathfrak{h}$ and $\Delta^+(\mathfrak{h})$ is contained in $(\mathfrak{p} \cap \mathfrak{h})$, so $\mathfrak{p} \cap \mathfrak{h}$ is a parabolic subalgebra. Since $\mathfrak{a}_\mathfrak{h}$ is maximal abelian in $\mathfrak{h} \cap \mathfrak{s}$, $\mathfrak{l} \cap \mathfrak{h} = \mathfrak{z}_\mathfrak{h}(\mathfrak{a}_\mathfrak{h}) = \mathfrak{l} \cap \mathfrak{h} \cap \mathfrak{k} + \mathfrak{a}_\mathfrak{h}$. Therefore, $\mathfrak{p} \cap \mathfrak{h}$ is minimal. □

We set $\overline{\mathfrak{n}} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}^{(-\alpha)}$, so $\mathfrak{l} + \overline{\mathfrak{n}}$ is the parabolic subalgebra opposite to $\mathfrak{p}$.

Lemma 3.3. *The following hold.*

(a) $\mathfrak{l} = \mathfrak{l} \cap \mathfrak{h} + \mathfrak{l} \cap \mathfrak{q}$.
(b) $\mathfrak{m} = \mathfrak{m} \cap \mathfrak{h} + \mathfrak{m} \cap \mathfrak{q}$.
(c) $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{l} + \mathfrak{q} \cap \mathfrak{n} + \mathfrak{q} \cap \overline{\mathfrak{n}}$.

(d) $\mathfrak{m}\cap\mathfrak{q}=\mathfrak{m}\cap\mathfrak{q}\cap\mathfrak{k}+\mathfrak{m}\cap\mathfrak{s}$.

Proof. (a) This is clear since $\mathfrak{h}$ and $\mathfrak{q}$ are $\mathfrak{a}_{\mathfrak{h}}$-stable and $\mathfrak{l}$ is the 0-weight space of $\mathfrak{a}_{\mathfrak{h}}$.
(b) This follows from (a) since

$$\begin{aligned}\mathfrak{m}\subset\mathfrak{l}&=\mathfrak{l}\cap\mathfrak{h}+\mathfrak{l}\cap\mathfrak{q}\\&=\mathfrak{m}\cap\mathfrak{h}+\mathfrak{a}_{\mathfrak{q}}+\mathfrak{m}\cap\mathfrak{q}.\end{aligned}$$

(c) Since $\mathfrak{a}_{\mathfrak{h}}$ acts on $\mathfrak{q}$, $\mathfrak{q}$ is a sum of $\mathfrak{a}_{\mathfrak{h}}$-weight spaces. If $X\in\mathfrak{q}$ is a weight vector, then the weight is $\alpha|_{\mathfrak{a}_{\mathfrak{h}}}$, for some $\alpha\in\Delta\cup\{0\}$. Therefore, $X\in\mathfrak{l},\mathfrak{n}$ or $\overline{\mathfrak{n}}$.
(d) Since $\mathfrak{a}_{\mathfrak{h}}$ is θ-stable, so is $\mathfrak{m}$. Therefore, $\mathfrak{m}\cap\mathfrak{q}$ is also θ-stable. It follows that $\mathfrak{m}\cap\mathfrak{q}\cap\mathfrak{k}+\mathfrak{m}\cap\mathfrak{q}\cap\mathfrak{s}$. By Lem. 3.2, $\mathfrak{m}\cap\mathfrak{h}\cap\mathfrak{s}=0$. Therefore, $\mathfrak{m}\cap\mathfrak{q}\cap\mathfrak{s}=\mathfrak{m}\cap\mathfrak{s}$ and (d) follows. □

We need to fix a positive system in $\Delta(\mathfrak{t}+\mathfrak{a}_{\mathfrak{q}},\mathfrak{m})$. Since we will be applying §2 to $H\cap M\subset M$ in place of $H\subset G$, we will need a choice of $\Delta^+(\mathfrak{m})$ as in §2.1. We therefore define $\Delta^+(\mathfrak{m})$ using the lexicographic order with the same bases of $\mathfrak{t}_{\mathfrak{h}},\mathfrak{t}_{\mathfrak{q}}$ and $\mathfrak{a}_{\mathfrak{q}}$ (in that order) that were used in the lexicographic order defining Δ^+ earlier. Observe that $\mathfrak{t}+\mathfrak{a}_{\mathfrak{q}}$ is a fundamental Cartan subalgebra of $\mathfrak{m}$. Using the basis of $\mathfrak{t}_{\mathfrak{h}}$ gives a lexicographic order that in turn gives a positive system $\Delta^+(\mathfrak{m}\cap\mathfrak{h})$.

We make the following assumption on H, which is analogous to and consistent with Assumption 2.1.

Assumption 3.4. There is no root $\beta\in\Delta(\mathfrak{t}_{\mathfrak{h}}+\mathfrak{t}_{\mathfrak{q}}+\mathfrak{a}_{\mathfrak{q}},\mathfrak{m})$ so that $\beta|_{\mathfrak{t}_{\mathfrak{h}}}=0$.

3.3. **The representation $S_{\mathfrak{q}}\otimes E_\mu$.** Let $\mu\in(\mathfrak{a}_{\mathfrak{h}}+\mathfrak{t}_{\mathfrak{h}})^*$ be $\Delta^+(\mathfrak{h})$-dominant and integral. Let E_μ be the irreducible finite dimensional $\mathfrak{h}$-representation with highest weight μ. We consider the tensor product $S_{\mathfrak{q}}\otimes E_\mu$, a representation of $\mathfrak{h}$.

The construction of the spin representation in §1.2 requires a choice of maximal isotropic subspace of $\mathfrak{q}$. This is done as follows. Choose some maximal isotropic subspace $(\mathfrak{a}_{\mathfrak{q}}+\mathfrak{t}_{\mathfrak{q}})^+$ of $\mathfrak{a}_{\mathfrak{q}}+\mathfrak{t}_{\mathfrak{q}}$ and set

$$\mathfrak{q}^+=(\mathfrak{a}_{\mathfrak{q}}+\mathfrak{t}_{\mathfrak{q}})^++\sum_{\beta\in\Delta^+(\mathfrak{m}\cap\mathfrak{q})}\mathfrak{m}^{(\gamma)}+\mathfrak{q}\cap\mathfrak{n}.$$

Then $\mathfrak{q}^+$ is maximally isotropic in $\mathfrak{q}$ by Lemma 3.3(c) and the fact that $\mathfrak{l}\cap\mathfrak{q}=\mathfrak{m}\cap\mathfrak{q}$.

Note that the $(\mathfrak{a}_{\mathfrak{h}}+\mathfrak{t}_{\mathfrak{h}})$-weights in $\mathfrak{q}^+$ are the weights in $\mathfrak{q}$ that are positive with respect to the lexicographic order for the same bases of $\mathfrak{a}_h$ and $\mathfrak{t}_{\mathfrak{h}}$ used in the definition of Δ^+ above. It follows that

$$\rho(\mathfrak{m}\cap\mathfrak{q}):=\rho(\mathfrak{m})|_{\mathfrak{t}_{\mathfrak{h}}}-\rho(\mathfrak{m}\cap\mathfrak{h})$$

is 1/2 the sum of the $\mathfrak{t}_{\mathfrak{h}}$-weights in $\mathfrak{m}\cap\mathfrak{q}^+$.

The weight

$$\rho(\mathfrak{q}):=\frac{1}{2}\sum_{\gamma\in\Delta(\mathfrak{q}^+)}\gamma,$$

with each weight occurring as many times as its multiplicity in $\mathfrak{q}^+$, is $\Delta^+(\mathfrak{h})$-dominant. The set of weights of $S_\mathfrak{q}$ is

$$\begin{aligned}\Delta(S_\mathfrak{q}) &= \{\rho(\mathfrak{q}) - \langle A\rangle : A \subset \Delta(\mathfrak{q}^+)\}\\ &= \{\langle A\rangle - \rho(\mathfrak{q}) : A \subset \Delta(\mathfrak{q}^+)\}.\end{aligned}$$

The subgroup $H \cap M \subset M$ satisfies the conditions of (1.1). Therefore, there is a spin representation $S_{\mathfrak{m}\cap\mathfrak{q}}$. Using the maximal isotropic subspace $\mathfrak{m}\cap\mathfrak{q}^+$ of $\mathfrak{m}\cap\mathfrak{q}$ one easily sees that $S_{\mathfrak{q}\cap\mathfrak{m}}$ is naturally contained in $S_\mathfrak{q}$ as $\mathfrak{h}\cap\mathfrak{m}$-representation. The set of weights of $S_{\mathfrak{m}\cap\mathfrak{q}}$ with respect to the Cartan subalgebra $\mathfrak{t}_\mathfrak{h}$ of $\mathfrak{m}\cap\mathfrak{h}$ is

$$\Delta(S_{\mathfrak{m}\cap\mathfrak{q}}) = \{\rho(\mathfrak{m}\cap\mathfrak{q}) - \langle B\rangle : B \subset \Delta(\mathfrak{m}\cap\mathfrak{q}^+)\}.$$

Lemma 3.5. *The following hold.*

(a) $S_{\mathfrak{q}\cap\mathfrak{m}} \subset (S_\mathfrak{q})^{\mathfrak{h}\cap\mathfrak{n}}$, *the* $\mathfrak{n}\cap\mathfrak{h}$*-invariants in* $S_\mathfrak{q}$.
(b) *As a subspace of* $S_\mathfrak{q}$, $\mathfrak{a}_\mathfrak{h}$ *acts on* $S_{\mathfrak{q}\cap\mathfrak{m}}$ *by* $\rho(\mathfrak{q})|_{\mathfrak{a}_\mathfrak{h}}$.
(c) $\rho(\mathfrak{q})|_{\mathfrak{t}_\mathfrak{h}} = \rho(\mathfrak{m}\cap\mathfrak{q})$.

Proof. The proof of the first statement is as in [11, Lemma 3.8]. The second follows since the weights of $S_{\mathfrak{m}\cap\mathfrak{q}} \subset S_\mathfrak{q}$ are of the form $\rho(\mathfrak{q}) - \langle B\rangle$ with $B \subset \Delta(\mathfrak{m}\cap\mathfrak{q}^+)$. But $\langle B\rangle|_{\mathfrak{a}_\mathfrak{h}} = 0$, since the weights in $\mathfrak{m}$ vanish on $\mathfrak{a}_\mathfrak{h}$. The last statement follows from $\rho(\mathfrak{q})|_{\mathfrak{t}_\mathfrak{h}} - \rho(\mathfrak{m}\cap\mathfrak{q}) = \rho(\mathfrak{n}\cap\mathfrak{q})|_{\mathfrak{t}_\mathfrak{h}} = 0$, which follows from the fact that $\Delta(\mathfrak{n}\cap\mathfrak{q})$ is stable under $-\theta$. □

Now consider E_μ (with $\Delta^+(\mathfrak{h})$-dominant integral $\mu \in (\mathfrak{a}_\mathfrak{h} + \mathfrak{t}_\mathfrak{h})^*$). Set $F_{\mu|_{\mathfrak{t}_\mathfrak{h}}} := \big(E_\mu\big)^{\mathfrak{n}\cap\mathfrak{h}}$, an irreducible representation of $\mathfrak{m}\cap\mathfrak{h}$ of highest weight $\mu|_{\mathfrak{t}_\mathfrak{h}}$.

For the remainder of this article we make the following assumptions on μ.

$$\text{(3.6)}\qquad \begin{aligned}&\text{(i) } \langle \mu + \rho(\mathfrak{m}\cap\mathfrak{h}) - \langle B\rangle, \beta\rangle \geq 0, \text{ for all } \beta\in\Delta^+(\mathfrak{m}\cap\mathfrak{h}) \text{ and } B \subset \Delta^+(\mathfrak{m}\cap\mathfrak{q}).\\ &\text{(ii) } \langle \mu + \rho(\mathfrak{m}\cap\mathfrak{h}) - 2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q}), \beta\rangle > 0, \text{ for all } \beta\in\Delta^+(\mathfrak{m}\cap\mathfrak{h}).\end{aligned}$$

By Steinberg's formula for the decomposition of a tensor product of finite dimensional representations, we see that $S_{\mathfrak{m}\cap\mathfrak{q}} \otimes F_{\mu|_{\mathfrak{t}_\mathfrak{h}}}$ contains the irreducible highest weight representation of $\mathfrak{m}\cap\mathfrak{h}$ having highest weight $\mu + \rho(\mathfrak{m}\cap\mathfrak{q}) - 2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})$. Let V_0 be the isotypic subspace of type $F_{\mu+\rho(\mathfrak{m}\cap\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})}$. Note that the assumption of (2.1) and the definition of V_0 are consistent with §2.3.

Observe that

$$V_0 \subset S_{\mathfrak{m}\cap\mathfrak{q}} \otimes F_{\mu|_{\mathfrak{t}_\mathfrak{h}}} \subset (S_\mathfrak{q}\otimes E_\mu)^{\mathfrak{n}\cap\mathfrak{h}},$$

by Lemma 3.5. It also follows from Lem. 3.5 parts (a) and (c), that $\mathfrak{a}_\mathfrak{h}$ acts on V_0 (as a subspace of $S_\mathfrak{q}\otimes E_\mu$) by the weight $(\mu+\rho(\mathfrak{q}))|_{\mathfrak{a}_\mathfrak{h}}$.

3.4. **Harmonic spinors.** Let P be the parabolic subgroup of G associated to H as in §3.2. Fix $\mu \in (\mathfrak{a}_\mathfrak{h} + \mathfrak{t}_\mathfrak{h})^*$ and assume that $S_\mathfrak{q} \otimes E_\mu$ lifts to a representation of the group H. Therefore, we have a smooth homogeneous vector bundle $S_\mathfrak{q} \otimes \mathcal{E}_\mu \to G/H$ and a cubic Dirac operator

$$\mathcal{D}_{G/H}(\mathcal{E}_\mu) : C^\infty(G/H, S_\mathfrak{q} \otimes \mathcal{E}_\mu) \to C^\infty(G/H, S_\mathfrak{q} \otimes \mathcal{E}_\mu).$$

Our goal is to construct an intertwining operator from a representation induced from P to $\ker\left(\mathcal{D}_{G/H}(\mathcal{E}_\mu)\right)$. Let W be a representation of P. Write the induced representation as

$$C^\infty(G/P, \mathcal{W}) = \left\{\varphi : G \to W : \varphi \text{ is smooth and } \varphi(gp) = p^{-1}\varphi(g), p \in P, g \in G\right\}.$$

The action of $g \in G$ is by left translation on functions: $(g \cdot f)(g_1) = f(g^{-1}g_1)$. The following is essentially Lemma 4.2 of [12], it is easily proved using a standard change of variables formula (e.g., [4, Lem. 5.19]).

Lemma 3.7. *Let W be some representation of P. If $t \in Hom_{P\cap H}(W \otimes \mathbf{C}_{-2\rho(\mathfrak{h})|_{\mathfrak{a}_\mathfrak{h}}}, S_\mathfrak{q} \otimes E_\mu)$ is nonzero, then*

$$(\mathcal{P}_t\varphi)(g) = \int_{H\cap K} \ell \cdot t(\varphi(g\ell))\, d\ell$$

is a nonzero G-intertwining map

$$\mathcal{P}_t : C^\infty(G/P, \mathcal{W}) \to C^\infty(G/H, S_\mathfrak{q} \otimes \mathcal{E}_\mu).$$

Given the bundle $S_\mathfrak{q} \otimes \mathcal{E}_\mu \to G/H$ we now make our choice of P-representation W and homomorphism t.

Let $F_{\mu|_{\mathfrak{t}_\mathfrak{h}}}$ be the irreducible representation of $H \cap M$ of highest weight $\mu|_{\mathfrak{t}_\mathfrak{h}}$. Suppose that μ satisfies the assumptions of (3.6). Then, by §2.3 applied to $H \cap M \subset M$, $\ker(\mathcal{D}_{M/H\cap M}(\mathcal{F}_\mu))$ contains a representation W infinitesimally equivalent to $\mathcal{A}_{\mathfrak{b}\cap\mathfrak{m}}(\lambda_\mu)$ with

$$\lambda_\mu|_{\mathfrak{t}_\mathfrak{h}} = \mu|_{\mathfrak{t}_\mathfrak{h}} - \rho(\mathfrak{m}\cap\mathfrak{q}), \quad \lambda_\mu|_{\mathfrak{a}_\mathfrak{q}} = 0 \quad \text{and} \quad \lambda_\mu|_{\mathfrak{t}_\mathfrak{q}} = -\rho(\mathfrak{m}\cap\mathfrak{k}).$$

By Lemma 2.10, evaluation at the identity is nonzero on W and has image in V_0. Give W the trivial N-action and define $\mathfrak{a}_\mathfrak{h}$ to act by $(\mu+\rho(\mathfrak{q})+2\rho(\mathfrak{h}))|_{\mathfrak{a}_\mathfrak{h}}$. Take t to be evaluation at the identity: $t(w \otimes 1) = w(e)$.

Lemma 3.8. *$t \in Hom_{P\cap H}(W \otimes \mathbf{C}_{-2\rho(\mathfrak{h})|_{\mathfrak{a}_\mathfrak{h}}}, S_\mathfrak{q} \otimes E_\mu)$.*

Proof. Evaluation is an $M\cap H$-homomorphism. The action of $\mathfrak{a}_\mathfrak{h}$ on $W \otimes \mathbf{C}_{-2\rho(\mathfrak{h})}$ is by $(\mu+\rho(\mathfrak{q}))|_{\mathfrak{a}_\mathfrak{h}}$. The action on the image of t is by $(\mu + \rho(\mathfrak{q}))|_{\mathfrak{a}_\mathfrak{h}}$, as pointed out at the end of §3.3. The action of $N \cap H$ on both $W \otimes \mathbf{C}_{-2\rho(\mathfrak{h})}$ and the image of t is trivial. □

When these conditions are satisfied and t is chosen as above, our main theorem, stated as Theorem A in the introduction, is the following.

Theorem 3.9. *The intertwining map $\mathcal{P}_t$ has image in the kernel of $\mathcal{D}_{G/H}(\mathcal{E}_\mu)$. In particular, $ker(\mathcal{D}_{G/H}(\mathcal{E}_\mu)) \neq 0$.*

Proof. The first observation is that

$$\big(\mathcal{D}_{G/H}(\mathcal{P}_t\varphi)\big)(g) = \int_{H\cap K}\left(\sum_i a_i R(X_i)\otimes\gamma(X_i)\right)\ell\cdot t(\varphi(\,\cdot\,\ell))|_g\,d\ell$$
$$\int_{H\cap K}\ell\cdot\left(\sum_i a_i R(X_i)\otimes\gamma(X_i)\right)t(\varphi(\cdot))|_{g\ell}\,d\ell.$$

As t is evaluation at e, it suffices to show that

$$\sum_i \left(a_i R(X_i)\otimes\gamma(X_i)\right)\varphi(\cdot)(e) = 0 \tag{3.10}$$

for $\varphi\in C^\infty(G/P,\mathcal{W})$.

We now choose the basis X_i in a suitable way. Let $\{E_j\}$ be a basis of $\mathfrak{q}\cap\mathfrak{n}$ and $\{\overline{E}_j\}$ a basis of $\mathfrak{q}\cap\overline{\mathfrak{n}}$ such that

$$\langle E_j\,,\overline{E}_k\rangle = \delta_{jk}$$
$$\langle E_j\,,E_k\rangle = \langle\overline{E}_j\,,\overline{E}_k\rangle = 0,$$

and let $\{Z_j\}$ be a basis of $\mathfrak{m}\cap\mathfrak{q}$ so that

$$\langle Z_j\,,Z_k\rangle = \delta_{jk}.$$

Setting

$$Y_j^+ = \frac{1}{\sqrt{2}}(E_j+\overline{E}_j)\text{ and }Y_j^- = \frac{1}{\sqrt{2}}(E_j-\overline{E}_j),$$

we get an orthogonal basis $\{Z_j\}\cup\{Y_j^\pm\}$ of $\mathfrak{q}$ as required in (1.3). The (geometric) Dirac operator $\mathcal{D}_{G/H}(\mathcal{E}_\mu)$ may be written as follows (equation (4.7) in [12]):

$$\begin{aligned}\mathcal{D}_{G/H}(\mathcal{E}_\mu) &= \sum_i\Big(R(Z_i)\otimes\gamma(Z_i)\otimes 1-\gamma(\mathbf{c}_{\mathfrak{m}\cap\mathfrak{q}})\Big)\\ &+\sum_j\Big(R(E_j)\otimes\epsilon(\overline{E}_j)\otimes 1+R(\overline{E}_j)\otimes\imath(E_j)\otimes 1\Big)\\ &-1\otimes\Big(\sum_j\langle Z_i\,,Z_i\rangle\langle Z_i\,,[E_j,\overline{E}_k]\rangle\gamma(Z_i)\epsilon(\overline{E}_j)\imath(E_k)\\ &+\sum\langle E_j\,,[E_k,\overline{E}_\ell]\rangle\epsilon(\overline{E}_j)\epsilon(\overline{E}_k)\imath(E_\ell)+\sum\langle E_j\,,[\overline{E}_k,\overline{E}_\ell]\rangle\epsilon(\overline{E}_j)\imath(E_k)\imath(E_\ell)\Big)\otimes 1,\end{aligned}\tag{3.11}$$

where $\imath$ (resp. ϵ) stands for the interior (resp. exterior) product (resp. multiplication) and $\mathbf{c}_{\mathfrak{m}\cap\mathfrak{q}}$ is the cubic term for $H\cap M\subset M$.

Now insert (3.11) into (3.10). The first term vanishes as follows. First note that

$$\begin{aligned}\big(R(Z_i)\varphi(\cdot)\big)(e)|_g &= \frac{d}{ds}\varphi(g\exp(sZ_i))(e)|_{s=0}\\ &= \frac{d}{ds}\big(\exp(sZ_i)\big)^{-1}\varphi(g)\big)(e)|_{s=0},\ \text{by } M\text{-equivariance of }\varphi,\\ &= \frac{d}{ds}\varphi(g)(\exp(sZ_i))|_{s=0},\ \text{by the definition of } M \text{ action on } W,\\ &= R(Z_i)\big(\varphi(g)(\cdot)\big)|_e.\end{aligned}$$

Now,

$$
\begin{aligned}
\big(\sum_i R(Z_i)\otimes\gamma(Z_i)-\gamma(\mathbf{c}_{\mathfrak{m}\cap\mathfrak{q}})\big)\varphi(\cdot)(e)|_g \\
&= \sum_i a_i\big(R(Z_i)\varphi(g)\big)|_e-\gamma(\mathbf{c}_{\mathfrak{m}\cap\mathfrak{q}})\varphi(g)(e)\\
&= \big(\mathcal{D}_{M/M\cap H}\varphi(g)\big)\,(e)\\
&= 0, \quad \text{since } \varphi(g)\in W\subset \ker(\mathcal{D}_{M/M\cap H}).
\end{aligned}
$$

By the right $P=MA_{\mathfrak{h}}N$-equivariance defining $C^\infty(G/P,\mathcal{W})$, $R(E_j)\varphi=0$, so the next terms in (3.11) vanish. Since the image of t is contained in $S_{\mathfrak{m}\cap\mathfrak{q}}\otimes F_{\mu|_{\mathfrak{t}_{\mathfrak{h}}}}$ and each E_j is orthogonal to $\mathfrak{m}\cap\mathfrak{q}$ (so that $\imath(E_j)S_{\mathfrak{m}\cap\mathfrak{q}}=0$), the remaining terms are 0. □

Appendix: Geometric vs. algebraic Dirac operators

For the convenience of the reader we provide here a proof of Proposition 1.8. Recall that V denotes a smooth admissible representation of G, V_K the space of K-finite vectors in V, $V_K^\star$ the K-finite dual of V_K and $S_{\mathfrak{s}}$ is the spin representation for $\mathfrak{k}$. Let E be a finite dimensional representation of $\mathfrak{k}$ such that the tensor product $E\otimes S_{\mathfrak{s}}$ lifts to a representation of the group K, and denote by $E^\star$ the dual of E. The K-representation $E\otimes S_{\mathfrak{s}}$ induces a homogeneous bundle $\mathcal{S}_{\mathfrak{s}}\otimes\mathcal{E}\longrightarrow G/K$ over G/K whose space of smooth sections, on which G acts by left translations, is denoted by $C^\infty(G/K,\mathcal{S}_{\mathfrak{s}}\otimes\mathcal{E})$. The map

$$\mathrm{Hom}_G(V,C^\infty(G/K,\mathcal{S}_{\mathfrak{s}}\otimes\mathcal{E}))\xrightarrow{\Psi}\mathrm{Hom}_K(E^\star,S_{\mathfrak{s}}\otimes V_K^\star)$$

defined by $\Psi(T)(e^\star)(v)=1\otimes e^\star T(v)(1)$, is an isomorphism, where 1 denotes the identity G.

Next, as in Section 1, consider the (cubic) Dirac operators

$$\mathcal{D}_{G/K}(\mathcal{E}):C^\infty(G/K,\mathcal{S}_{\mathfrak{s}}\otimes\mathcal{E})\longrightarrow C^\infty(G/K,\mathcal{S}_{\mathfrak{s}}\otimes\mathcal{E}) \text{ and } D_{V_K^\star}:S_{\mathfrak{s}}\otimes V_K^\star\longrightarrow S_{\mathfrak{s}}\otimes V_K^\star,$$

and define the maps

$$\mathcal{D}_*:\mathrm{Hom}_G(V,C^\infty(G/K,\mathcal{S}_{\mathfrak{s}}\otimes\mathcal{E}))\longrightarrow\mathrm{Hom}_G(V,C^\infty(G/K,\mathcal{S}_{\mathfrak{s}}\otimes\mathcal{E}))$$

and

$$D_*:\mathrm{Hom}_K(E^\star,S_{\mathfrak{s}}\otimes V_K^\star)\longrightarrow\mathrm{Hom}_K(E^\star,S_{\mathfrak{s}}\otimes V_K^\star)$$

by

$$
\begin{aligned}
(\mathcal{D}_*(T))(v)&=\mathcal{D}_{G/K}(\mathcal{E})(T(v)),\\
(D_*(A))(e^\star)&=D_{V_K^\star}(A(e^\star)).
\end{aligned}
$$

We claim that the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Hom}_G(V, C^\infty(G/K, \mathcal{S}_\mathfrak{s} \otimes \mathcal{E})) \arrow[r, "\Psi"] \arrow[d, "\mathcal{D}_*"'] & \mathrm{Hom}_K(E^\star, S_\mathfrak{s} \otimes V_K^\star) \arrow[d, "D_*"] \\
\mathrm{Hom}_G(V, C^\infty(G/K, \mathcal{S}_\mathfrak{s} \otimes \mathcal{E})) \arrow[r, "\Psi"] & \mathrm{Hom}_K(E^\star, S_\mathfrak{s} \otimes V_K^\star)
\end{tikzcd}$$

Indeed one has

$$\Psi(\mathcal{D}_*(T))(e^\star)(v) = (1 \otimes e^\star)\big((\mathcal{D}_*(T))(v)(1)\big) = (1 \otimes e^\star)\big(\mathcal{D}_{G/K}(T(v))(1)\big)$$

and

$$\begin{aligned}
\mathcal{D}_{G/K}(T(v))(1) &= \sum_i \frac{d}{dt}|_{t=0}(\gamma(X_i) \otimes 1)(T(v)(\exp(tX_i)(1)) - (\gamma(\mathbf{c}_\mathfrak{s}) \otimes 1)(T(v)(1)) \\
&= \sum_i \frac{d}{dt}\ |_{t=0}\ (\gamma(X_i) \otimes 1)(T(\exp(-tX_i)v)(1)) - (\gamma(\mathbf{c}_\mathfrak{s}) \otimes 1)(T(v)(1)) \\
&= -\sum_i (\gamma(X_i) \otimes 1)(T(X_i v)(1)) - (1 \otimes \gamma(\mathbf{c}_\mathfrak{s}))(T(v)(1))
\end{aligned}$$

which means that

$$\Psi(\mathcal{D}_*(T))(e^\star)(v) = -\sum_i (\gamma(X_i) \otimes e^\star)(T(X_i v)(1)) - (\gamma(\mathbf{c}_\mathfrak{s}) \otimes e^\star)(T(v)(1)).$$

On the other hand, one has:

$$\begin{aligned}
\Big((D_*(\Psi(T))(e^\star)\Big)(v) &= \Big(D_{V_K^\star}(\Psi(T)(e^\star))\Big)(v) \\
&= -\sum_i \big((\gamma(X_i) \otimes X_i)(\Psi(T)(e^\star)\big)(v) - (\gamma(\mathbf{c}_\mathfrak{s}) \otimes 1)(\Psi(T)(e^\star)(v)) \\
&= -\sum_i (\gamma(X_i) \otimes 1)(\Psi(T)(e^\star)(X_i v)) - (\gamma(\mathbf{c}_\mathfrak{s}) \otimes 1)(\Psi(T)(e^\star)(v)) \\
&= -\sum_i (\gamma(X_i) \otimes e^\star)(T(X_i v)(1)) - (\gamma(\mathbf{c}_\mathfrak{s}) \otimes e^\star)(T(v)(1)).
\end{aligned}$$

We deduce the following isomorphism relating algebraic and geometric harmonic spinors:

$$\mathrm{Hom}_G(V, \ker(\mathcal{D}_{G/K}(\mathcal{E}))) \overset{\Psi}{\simeq} \mathrm{Hom}_K(E^\star, \ker(D_{V_K^\star})),$$

therefore proving Proposition 1.8.

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