THE DIRAC COHOMOLOGY OF A FINITE DIMENSIONAL REPRESENTATION

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Abstract. The Dirac cohomology of a finite dimensional representation of a complex semisimple Lie algebra $\mathfrak{g}$, with respect to any quadratic subalgebra $\mathfrak{h}$, is computed. This generalizes a formula obtained by Kostant in the case where $\mathfrak{g}$ and $\mathfrak{h}$ have equal rank, and by Huang, Kang and Pandžić in the case where $\mathfrak{h}$ is the fixed points of an involution.

Introduction

The Dirac operator has played an important role in the representation theory of semisimple Lie groups, dating back to Parthasarathy's work on the discrete series [6]. The late 1990's saw renewed interest in representation theoretic Dirac operators with the definition of a cubic Dirac operator [4], a notion of Dirac cohomology and the proof of a conjecture of Vogan [2]. There has been recent activity in computing Dirac cohomology for various representations of a semisimple Lie algebra $\mathfrak{g}$, usually with respect to a symmetric subalgebra $\mathfrak{k}$. In this article the Dirac cohomology of a finite dimensional representation of $\mathfrak{g}$ with respect to an arbitrary quadratic subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is computed. This generalizes results in [5] and [3], which may be thought of as analogues of Kostant's version of the Borel-Weil Theorem. The case of finite dimensional modules, which has its own interest, plays an important role in the computation of Dirac cohomology of infinite dimensional modules through induction.

Fix a nondegenerate invariant symmetric bilinear form $\langle\ ,\ \rangle$ on a complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a reductive subalgebra of $\mathfrak{g}$ for which the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{h}$ remains nondegenerate. Such a reductive subalgebra is often called a quadratic subalgebra. The Lie algebra $\mathfrak{g}$ splits into an orthogonal sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}.$$

A spin representation of $\mathfrak{so}(\mathfrak{q})$ on a space of spinors S is therefore defined; and the map $\mathrm{ad} : \mathfrak{h} \to \mathfrak{so}(\mathfrak{q})$ induces the spin representation of $\mathfrak{h}$ on S.

Consider now the following element c of degree three in the Clifford algebra $Cl(\mathfrak{q})$ of $\mathfrak{q}$ defined by the Chevalley isomorphism

$$\Big(\mathfrak{q} \times \mathfrak{q} \times \mathfrak{q} \longrightarrow \mathbf{C}\Big) \longrightarrow Cl(\mathfrak{q})$$
$$\Big((X, Y, Z) \mapsto \langle X, [Y, Z]\rangle\Big) \mapsto c.$$

Given a $\mathfrak{g}$-module (π, W), there is a 'first order' operator

$$D_W : W \otimes S \longrightarrow W \otimes S$$

defined by

$$D_W = \sum_j \pi(X_j) \otimes \gamma(X_j) - 1 \otimes \gamma(c)$$

known as the (algebraic) cubic Dirac operator associated with W, where $\{X_j\}$ is an orthonormal basis of $\mathfrak{q}$ and

$$\gamma : Cl(\mathfrak{q}) \longrightarrow \mathrm{End}(S)$$

is the Clifford multiplication. Then the (cubic) Dirac cohomology of the $\mathfrak{g}$-module W is defined as the quotient

$$H(W) = \mathrm{Ker}(D_W)/\mathrm{Ker}(D_W) \cap \mathrm{Im}(D_W).$$

The Dirac cohomology $H(W)$ for a finite dimensional $\mathfrak{g}$-module W was computed by Kostant in [5] when $\mathfrak{g}$ and $\mathfrak{h}$ have equal rank. Later Huang, Kang and Pandžić [3] computed the Dirac cohomology when $\mathfrak{h}$ is the fixed points of an involution (and $\mathrm{rank}(\mathfrak{h}) \leq \mathrm{rank}(\mathfrak{g})$). In this paper, we generalize the formula for the Dirac cohomology of a finite dimensional $\mathfrak{g}$-representation when $\mathfrak{h}$ is an arbitrary quadratic subalgebra.

1. The spin representation

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a quadratic subalgebra. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, as in the introduction. Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{h}$ and extend it to a Cartan subalgebra $\mathfrak{t} + \mathfrak{a}$ of $\mathfrak{g}$, with $\mathfrak{a} \subset \mathfrak{q}$. By choosing a lexicographic order one gets a positive system of roots Δ^+ in $\Delta = \Delta(\mathfrak{g}, \mathfrak{t} + \mathfrak{a})$. Then one may check that the following property holds for this positive system.

$$\begin{aligned}&\text{There exists a regular } \Delta^+\text{-dominant } \xi \in (\mathfrak{t}+\mathfrak{a})^* \text{ so that}\\ &\text{if } \alpha \in \Delta^+ \text{ and } \gamma \overset{def.}{=} \alpha|_{\mathfrak{t}} \neq 0, \text{ then } \langle \xi|_{\mathfrak{t}}, \gamma\rangle > 0.\end{aligned} \tag{C}$$

We will be concerned with positive systems satisfying (C). Note that if $\operatorname{rank}(\mathfrak{g}) = \operatorname{rank}(\mathfrak{h})$, then every positive system satisfies (C). For a positive system Δ^+ we write

$$\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}^{(\alpha)}$$

and ρ for half the sum of the roots in Δ^+. Denote by $\Delta(V)$ the set of $\mathfrak{t}$-weights in a $\mathfrak{t}$-stable vector space V.

Lemma 1.1. *If Δ^+ satisfies (C), then*

(1) *Each $\mathfrak{t}$-weight in $\mathfrak{g}$ is the restriction of a root in Δ.*
(2) $\mathfrak{n} = (\mathfrak{n}\cap\mathfrak{h}) + (\mathfrak{n}\cap\mathfrak{q})$
(3) $\Delta^+(\mathfrak{h}) \overset{def.}{=} \{\beta \in \Delta(\mathfrak{h}) : \langle \xi, \beta\rangle > 0\}$ *is a positive system of* $\Delta(\mathfrak{h})$.
(4) $\Delta(\mathfrak{n}\cap\mathfrak{h}) = \Delta^+(\mathfrak{h})$.

Proof. Suppose $\gamma \neq 0$ is a $\mathfrak{t}$-weight in $\mathfrak{g}$. The γ-weight space is

$$\sum_{\alpha\in\Delta,\,\alpha|_{\mathfrak{t}}=\gamma} \mathfrak{g}^{(\alpha)}. \tag{1.2}$$

The first statement follows from this. Since both $\mathfrak{h}$ and $\mathfrak{q}$ are $\mathfrak{t}$-stable we have

$$\mathfrak{h}^{(\gamma)} = \big(\sum_{\alpha|_{\mathfrak{t}}=\gamma} \mathfrak{g}^{(\alpha)}\big)\cap\mathfrak{h} \quad\text{and}\quad \mathfrak{q}^{(\gamma)} = \big(\sum_{\alpha|_{\mathfrak{t}}=\gamma} \mathfrak{g}^{(\alpha)}\big)\cap\mathfrak{q}$$

and the γ-weight space in $\mathfrak{g}$ is the direct sum $\mathfrak{h}^{(\gamma)} + \mathfrak{q}^{(\gamma)}$.

Now suppose $\alpha \in \Delta^+$ and $X_\alpha \in \mathfrak{g}^{(\alpha)} \subset \mathfrak{n}$. There are two cases. First $\gamma = \alpha|_{\mathfrak{t}} \neq 0$. Then (by the above) $X_\alpha \in \mathfrak{h}^{(\gamma)} + \mathfrak{q}^{(\gamma)}$. By (C) *all* roots restricting to γ are positive, so $\mathfrak{h}^{(\gamma)}, \mathfrak{q}^{(\gamma)} \subset \mathfrak{n}$. Therefore, $X_\alpha \in (\mathfrak{n}\cap\mathfrak{h}) + (\mathfrak{n}\cap\mathfrak{q})$. When $\gamma = 0$, $X_\alpha \in \mathfrak{q}$, so $X_\alpha \in \mathfrak{n}\cap\mathfrak{q}$. This proves (2).

Part (3) follows from (1.2). The last statement is now clear. □

Now let us turn to the spin representation built from $\mathfrak{q}$ and the invariant form on $\mathfrak{q}$. For this discussion we consider any positive system Δ^+ satisfying (C). This determines $\Delta^+(\mathfrak{h})$ as in the preceding lemma.

We use the construction of the spin representation as given in [1]. For this we need a maximally isotropic subspace of $\mathfrak{q}$. Observe that

$$\mathfrak{q} = (\mathfrak{n} \cap \mathfrak{q}) + \mathfrak{a} + (\mathfrak{n}^- \cap \mathfrak{q}),$$

where

$$\mathfrak{n}^- = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{(-\alpha)}.$$

Therefore, if we choose a maximal isotropic subspace $\mathfrak{a}^+$ in $\mathfrak{a}$, then

$$\mathfrak{q}^+ = (\mathfrak{n} \cap \mathfrak{q}) + \mathfrak{a}^+$$

is a maximally isotropic subspace of $\mathfrak{q}$. Then, by [1, Proposition 6.2.4], the weights of the spin representation may be written as follows. List the weights of $\mathfrak{q}^+$ as

$$\gamma_1, \gamma_2, \dots, \gamma_N.$$

Here the weights are to be listed with the multiplicity with which they occur in $\mathfrak{q}^+$. In particular, $N = [\dim(\mathfrak{q})/2]$, the greatest integer in $\dim(\mathfrak{q})/2$. The weights of the spin representation S are

$$\frac{1}{2}(\pm\gamma_1 \pm \gamma_2 \pm \dots \pm \gamma_N).$$

It follows immediately that if m is the number of weights γ_i that are zero, then the multiplicity of each weight is a multiple of 2^m. Therefore, $S \simeq 2^m \cdot S_0$, for some $\mathfrak{h}$-representation S_0.

Alternatively, using the notation $\langle A \rangle = \sum_{\gamma \in A} \gamma$, we have

$$\Delta(S) = \{\rho(\mathfrak{n} \cap \mathfrak{q}) - \langle A \rangle : A \subset \Delta(\mathfrak{n} \cap \mathfrak{q})\}.$$

Note that, by Lemma 1.1, $\rho(\mathfrak{n} \cap \mathfrak{q}) = \rho - \rho(\mathfrak{h})$, where $\rho(\mathfrak{h}) = \rho(\Delta^+(\mathfrak{h}))$.

Lemma 1.3. *For any Δ^+ satisfying (C), the spin representation S has a highest weight vector, with respect to $\Delta^+(\mathfrak{h})$, of weight $\rho(\mathfrak{n} \cap \mathfrak{q})$. This weight occurs in S with multiplicity exactly 2^m.*

Proof. For the first statement it suffices to show that $\rho(\mathfrak{n} \cap \mathfrak{q}) + \beta$ is not a weight of S when $\beta \in \Delta^+(\mathfrak{h})$. Suppose otherwise. Then $\rho(\mathfrak{n} \cap \mathfrak{q}) + \beta = \rho(\mathfrak{n} \cap \mathfrak{q}) - \langle A \rangle$, for some $A \subset \Delta(\mathfrak{n} \cap \mathfrak{q})$. Thus $\beta = -\langle A \rangle$. Taking the inner product with ξ gives $\langle \xi, \beta \rangle = -\sum_{\gamma \in A} \langle \xi, \gamma \rangle$. But the lefthand side is positive while the righthand side is nonpositive by (C).

Verifying the multiplicity statement is similar. Suppose that $\rho(\mathfrak{n} \cap \mathfrak{q}) = \rho(\mathfrak{n}\cap\mathfrak{q}) - \langle A \rangle$. Then $0 = \langle A \rangle$. Taking the inner product with ξ gives $\langle \xi, \gamma \rangle = 0$, for each $\gamma \in A$ (by (C)). But (again by (C)), this means that $\gamma = 0$. □

We mention that if we were to take a different Δ^+ satisfying (C) that determines the *same* $\Delta^+(\mathfrak{h})$, then we would have another highest weight (with respect to the same $\Delta^+(\mathfrak{h})$) of multiplicity 2^m in S. This follows from the fact that the spin representation of $\mathfrak{so}(\mathfrak{q})$ is independent of the maximal isotropic subspace used in the construction.

2. Dirac cohomology

Let E be an irreducible finite dimensional representation of $\mathfrak{g}$. We now fix a positive system $\Delta^+(\mathfrak{h})$.

Definition 2.1. Let $\mathbb{P}$ be the set of positive systems Δ^+ so that Δ^+ satisfies (C) and so that the corresponding $\xi|_\mathfrak{t}$ is dominant for $\Delta^+(\mathfrak{h})$. In other words $\Delta^+(\mathfrak{h})$ is the positive system in $\Delta(\mathfrak{h})$ associated to Δ^+ (as in Lemma 1.1(3)).

Lemma 2.2. *Let $\Delta^+ \in \mathbb{P}$. If λ is the highest weight of E with respect to Δ^+, then $\lambda|_\mathfrak{t}$ is a highest weight (with respect to $\Delta^+(\mathfrak{h})$) for a constituent of the restriction of E to $\mathfrak{h}$.*

Proof. Let e_λ be the highest weight vector of the $\mathfrak{g}$-representation E. This vector is annihilated by $\mathfrak{n}$. Therefore is annihilated by $\mathfrak{n} \cap \mathfrak{h}$. But $\Delta(\mathfrak{n} \cap \mathfrak{h}) = \Delta^+(\mathfrak{h})$ by Lemma 1.1(4). □

Now we consider the Dirac cohomology of the representation E. As explained in [2, Remark 3.2.4], since E is finite dimensional,

$$H(E) = Ker(D) = Ker(D^2). \tag{2.3}$$

It follows from [5] (see also [2]) that any constituent F of the $\mathfrak{h}$-representation $H(E)$ must have infinitesimal character described as follows. Choose a positive system $\Delta^+ \in \mathbb{P}$ and let λ be the highest weight of E. Write W (resp. $W(\mathfrak{h})$) for the Weyl group for $\mathfrak{g}$ (resp. $\mathfrak{h}$). Then the infinitesimal character of F is the $W(\mathfrak{h})$-orbit of $w(\lambda+\rho)|_\mathfrak{t}$, for some $w \in W$ with $w(\lambda+\rho)|_\mathfrak{a} = 0$. In particular, the highest weight of F (with respect to $\Delta^+(\mathfrak{h})$) is $\mu = w(\lambda+\rho)|_\mathfrak{t} - \rho(\mathfrak{h})$, where

w is in

$$W_\lambda^1 \stackrel{def.}{=} \{w \in W : w(\lambda+\rho)|_{\mathfrak{a}} = 0 \text{ and } w(\lambda+\rho)|_{\mathfrak{t}} \text{ is } \Delta^+(\mathfrak{h})\text{-dominant}\}.$$

If $E \otimes S$ contains an $\mathfrak{h}$-constituent with such a highest weight, then this constituent lies in $Ker(D^2)$ by Kostant's formula for the square of D ([4, Theorem 2.16]). Recall that Kostant's formula is

$$2D^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\Delta\mathfrak{h}} + (||\rho||^2 - ||\rho(\mathfrak{h})||^2),$$

where $\Omega_{\mathfrak{g}}$ is the Casimir element for $\mathfrak{g}$ acting on E and $\Omega_{\Delta\mathfrak{h}}$ is the Casimir element of $\mathfrak{h}$ acting in $E \otimes S$. It follows from (2.3) that this constituent occurs in $H(E)$. The following lemma is a corollary of this discussion.

Lemma 2.4. *The following are equivalent.*

(a) There is a root $\alpha \in \Delta$ so that $\alpha|_{\mathfrak{t}} = 0$.

(b) No element of $\mathfrak{t}^*$ ($\subset (\mathfrak{t}+\mathfrak{a})^*$) is Δ-regular.

Each implies that $H(E) = 0$ for any irreducible finite dimensional representation of $\mathfrak{g}$.

Proof. Equivalence of (a) and (b) is straightforward. Let E_λ be the irreducible highest weight representation of $\mathfrak{g}$ with highest weight λ with respect to some Δ^+ in $\mathbb{P}$. If $H(E_\lambda) \neq 0$, then above discussion tells us that $W_\lambda^1 \neq \emptyset$. So there is a $w \in W$ so that $w(\lambda+\rho)|_{\mathfrak{a}} = 0$. Therefore, $w(\lambda+\rho)$ is a regular element in $\mathfrak{t}^*$, contradicting (b). □

Note that if $W_\lambda^1 = \emptyset$, then $H(E_\lambda) = 0$.

Theorem 2.5. *Let E_λ be the irreducible highest weight representation of $\mathfrak{g}$ with highest weight λ with respect to some Δ^+ in $\mathbb{P}$. Let W_λ^1 be as above. Then*

$$H(E_\lambda) = \bigoplus_{w \in W_\lambda^1} 2^m \cdot F_{w(\lambda+\rho)|_{\mathfrak{t}} - \rho_{\mathfrak{h}}}$$

where $m = [(\mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{h}))/2]$.

Proof. When $W_\lambda^1 = \emptyset$ the statement of the theorem holds since both sides are zero. So we assume that $W_\lambda^1 \neq \emptyset$. Then there is a Δ-regular element (namely $w(\lambda+\rho)$) in $\mathfrak{t}^*$, so by the lemma no root restricts to 0 on $\mathfrak{t}$. Therefore, the multiplicities of weights in S are 2^m, $m = [\dim(\mathfrak{a})/2] = [(\mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{h}))/2]$. It follows, as in §1, that $S \simeq 2^m \cdot S_0$ and

$$\Delta(S_0) = \{\rho(\mathfrak{n} \cap \mathfrak{q}) - \langle A \rangle : A \subset \Delta(\mathfrak{n} \cap \mathfrak{q})\},$$

each weight occurring exactly once.

By the discussion preceding the above lemma, the theorem will be proved once we show that each $F_{w(\lambda+\rho)|_{\mathfrak{t}}-\rho(\mathfrak{h})}$ occurs in $E \otimes S_0$ exactly once.

Observe that if $w \in W^1_\lambda$, then $w\Delta^+ \in \mathbb{P}$ (as $w\Delta^+$ is defined by $\xi = w(\lambda+\rho)$). With respect to $w\Delta^+$, E_λ has highest weight $w\lambda$. Since Δ^+ may be replaced by $w\Delta^+$ in our discussion of the $\mathfrak{h}$-representation S in §1, we may conclude that

$$\Delta(S_0) = \{\rho((w \cdot \mathfrak{n}) \cap \mathfrak{q}) - \langle A \rangle : A \subset \Delta((w \cdot \mathfrak{n}) \cap \mathfrak{q})\},$$

and $\rho((w \cdot \mathfrak{n}) \cap \mathfrak{q})$ is the highest weight of a constituent of S_0.

Now it is immediate, from Lemma 2.2, that $w(\lambda)|_{\mathfrak{t}} + \rho((w \cdot \mathfrak{n}) \cap \mathfrak{q})$ is the highest weight of a constituent of $E \otimes S_0$. Note that $w(\lambda + \rho)|_{\mathfrak{t}} - \rho(\mathfrak{h}) = w(\lambda)|_{\mathfrak{t}} + \rho((w \cdot \mathfrak{n}) \cap \mathfrak{q})$. We need to check that this constituent occurs with multiplicity one. Writing this weight as an arbitrary sum of weights in E_λ and S_0, we have

$$w(\lambda)|_{\mathfrak{t}} + \rho((w \cdot \mathfrak{n}) \cap \mathfrak{q}) = (w(\lambda) - \langle B \rangle)|_{\mathfrak{t}} + (\rho((w \cdot \mathfrak{n}) \cap \mathfrak{q}) - \langle A \rangle),$$

where $A \subset \Delta((w \cdot \mathfrak{n}) \cap \mathfrak{q})$ and $B \subset w\Delta^+$. It follows that

$$w(\lambda + \rho)|_{\mathfrak{t}} = w(\lambda + \rho)|_{\mathfrak{t}} - \langle B \rangle - \langle A \rangle,$$

so $\langle B \rangle + \langle A \rangle = 0$. Taking the inner product with the $w\Delta^+$-regular element $\xi = w(\lambda + \rho)$ gives

$$\sum_{\alpha \in B} \langle \xi, \alpha|_{\mathfrak{t}} \rangle + \sum_{\gamma \in A} \langle \xi, \gamma \rangle = 0.$$

All $\langle \xi, \gamma \rangle$ and $\langle \xi, \alpha \rangle$ that occur are nonnegative, so zero. Therefore, by (C), no α's can occur. Since no roots restrict to 0, no γ's occur. Therefore, the weight $w(\lambda + \rho)|_{\mathfrak{t}} - \rho_{\mathfrak{h}}$ occurs just once in $E_\lambda \otimes S_0$. □

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