PRINCIPAL SERIES REPRESENTATIONS AND HARMONIC SPINORS

S. Mehdi
School of Mathematics
Tata Institute of Fundamental Research
Mumbai 400005, India
e-mail: *mehdi@math.tifr.res.in*
and
R. Zierau
Mathematics Department
Oklahoma State University
Stillwater, Oklahoma 74078
e-mail: *zierau@math.okstate.edu*

Abstract. Let G be a real reductive Lie group and G/H a reductive homogeneous space. We consider Kostant's cubic Dirac operator D on G/H twisted with a finite dimensional representation of H. Under the assumption that G and H have the same complex rank, we construct a nonzero intertwining operator from principal series representations of G into the kernel of D. The Langlands parameters of these principal series are described explicitly. In particular, we obtain an explicit integral formula for certain solutions of the cubic Dirac equation $D = 0$ on G/H. These results generalize our previous results in [12].

1. Introduction.

In this article we study the kernel of Kostant's cubic Dirac operator on reductive homogeneous spaces. For the introduction let G be a connected semisimple linear Lie group and let K be a maximal compact subgroup. Our results are in fact stated and proved for general connected reductive Lie groups. It is well-known that there is a natural G-invariant spin structure and connection on the riemannian symmetric space G/K (or possibly a double cover) giving rise to a Dirac operator. In particular, for a finite dimensional irreducible K-representation E there are corresponding homogeneous vector bundles $\mathcal{E}$ and $\mathcal{S} \otimes \mathcal{E}$ over G/K (where S is the spin representation of K) and a Dirac operator on sections:

$$D_{G/K}(\mathcal{E}) : C^\infty(G/K, \mathcal{S} \otimes \mathcal{E}) \to C^\infty(G/K, \mathcal{S} \otimes \mathcal{E}).$$

In the 1970's it was shown ([13] and [2]) that the kernel of $D_{G/K}(\mathcal{E})$ (on L^2 sections) is an irreducible unitary representation in the discrete series of G, and every discrete series representation occurs this way (for some bundle $\mathcal{E}$). Recently Kostant has defined a G-invariant differential operator in the following more general setting. Suppose that H is a closed connected reductive subgroup of G so that the Killing form of $\mathfrak{g}$ restricts to a nondegenerate form on $\mathfrak{h}$. Write

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q}, \quad \mathfrak{q} = \mathfrak{h}^{\perp}.$$

Then the Killing form is nondegenerate (and possibly indefinite) on $\mathfrak{q}$, so we may construct the corresponding Clifford algebra $C\ell(\mathfrak{q})$ and spin representation $S_{\mathfrak{q}}$ of $\mathfrak{h}$. Then, given a finite dimensional representation E of $\mathfrak{h}$ so that $S_{\mathfrak{q}} \otimes E$ integrates to a representation of H, Kostant ([8]) defined the algebraic analogue of a G-invariant differential operator

$$D = D_{G/H}(\mathcal{E}) : C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}) \to C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E})$$

called the cubic Dirac operator. This operator is the sum of a first order term (identical to the Dirac operator on G/K) and a zeroth order term coming from a degree three element of $C\ell(\mathfrak{q})$ (which vanishes when $H = K$). This article studies the kernel of the cubic Dirac operator D. Interest in the cubic Dirac operator and its kernel comes from several directions. In [11] and [9] a generalization of the Bott-Borel-Weil Theorem is proved. This, along with the discrete series example mentioned above, shows that a complex structure on G/H is not necessary for the construction of large families of interesting representations. Instead, the spin structure and Dirac operator can be used. When G/H has an invariant complex structure then it can be shown that $D = \bar{\partial} + \bar{\partial}^{*}$, therefore D is related to the Dolbeault cohomology representations (the $A_{\mathfrak{q}}(\lambda)$'s). It is reasonable to expect interesting representations in terms of D when there is no complex structure. There are also interesting recent results in terms of (the algebraic version of) Kostant's cubic Dirac operator. See, for example, [7], [10] and [1].

In this article we prove that (under a 'sufficiently regular' condition on the highest weight of E) the kernel of $D = D_{G/H}(\mathcal{E})$ is nonzero. This is done by constructing an intertwining operator from a principal series representation into the kernel of D. There are several interesting byproducts of this construction. One is that the Langlands parameters of a constituent in $Ker(D)$ are specified in a natural way. Comparing with [4] and [3] one sees that, in the case when G/H has a complex structure, this constituent is equivalent to the corresponding Dolbeault cohomology representation. Also, the intertwining operator gives an explicit integral formula for solutions to $Df = 0$ in terms of an integral over a piece of the 'boundary' of G/H. The formula is quite analogous to the classical Poisson integral formula giving harmonic functions on the disk.

The techniques we use are similar to those of [4], [3] and [5]. A special case of the results here are contained in [12]. However the results here apply to a much larger class of groups and homogeneous spaces. A key step is the reduction to the case of G/H where H is compact. In this case the kernel of D contains a discrete series representation. This fact follows from [13], [2] and [16] (with some work); one may also apply [15].

This paper begins with some preliminary material, including a quick review of the key facts we need concerning the spin representations, the definition of the cubic Dirac operator and a technical lemma. The intertwining operator we construct maps from a principal series representation of G into the kernel of D. The principal series is based on a parabolic subgroup which has some special properties with respect to H. This is discussed in Section 3. The intertwining operator is constructed in Section 4, where the main theorem, Theorem 4.12, is proved. We remark that Lemma 4.5 is essentially a very general statement about the existence of intertwining operators. The appendix is somewhat independent of the rest of the paper. We show that discrete series representations occur in the kernel of D when H is compact. Since this result gets applied to the 'M' of the parabolic subgroup, we are forced to consider disconnected groups. The appendix therefore becomes somewhat technical.

2. Preliminaries

2.1. **The groups.** Let G be a connected reductive Lie group. By this we mean that G is connected and has reductive Lie algebra, i.e., $\mathfrak{g} = Lie(G) = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ (with $\mathfrak{z}$ = the center of $\mathfrak{g}$). We will consider a more general class of groups in the appendix.

As is customary, Lie groups will be denoted by G, H, etc. and their Lie algebras by $\mathfrak{g}, \mathfrak{h}$, etc. The complexifications of the Lie algebras will be denoted by $\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}}$, etc.

Let $Z = Z(G)$ denote the center of G. Note that Z need not be finite. For example, we have included groups such as $GL(n, \mathbf{C})$ and the simply connected groups of hermitian type. If K/Z is a maximal compact subgroup of $G/Z \simeq \mathrm{Ad}(G)$ then K is the fixed point group of a Cartan involution θ. We write the Cartan decomposition as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}.$$

Suppose that H is a closed connected reductive subgroup of G. We make two further assumptions on H. Let $\langle\ ,\ \rangle$ be an Ad-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$ which coincides with the Killing form on $\mathfrak{g}_{ss} \overset{def.}{=} [\mathfrak{g}, \mathfrak{g}]$. Then we assume that the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{h}$ is nondegenerate. There is therefore an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \quad \mathfrak{q} \overset{def.}{=} \mathfrak{h}^{\perp}. \tag{2.1}$$

Furthermore, $\langle\ ,\ \rangle_{\mathfrak{q}} \overset{def.}{=} \langle\ ,\ \rangle|_{\mathfrak{q}\times\mathfrak{q}}$ is nondegenerate. We also assume that G and H have the same complex rank:

$$\mathrm{rank}(\mathfrak{h}^{\mathbf{C}}) = \mathrm{rank}(\mathfrak{g}^{\mathbf{C}}). \tag{2.2}$$

2.2. **The spin representations.** For the remainder of this section we do not require G or H to be connected.

The (possibly indefinite) nondegenerate form on $\mathfrak{q}$ defines a Clifford algebra and a spin representation $S_{\mathfrak{q}}$ of $\mathfrak{so}(\mathfrak{q})$. This gives a 'spin' representation of $\mathfrak{h}$ via $ad : \mathfrak{h} \to \mathfrak{so}(\mathfrak{q})$. Note that since $\mathfrak{q} \subset \mathfrak{g}_{ss}$, $\langle\ ,\ \rangle_{\mathfrak{q}}$ is precisely the Killing form restricted to $\mathfrak{q}$, therefore the Clifford algebra and spin representations are independent of the extension $\langle\ ,\ \rangle$ of the Killing form of $\mathfrak{g}_{ss}$.

Let us briefly recall how one defines the 'spin' representation of $\mathfrak{h}$. See Section 3 in [12] for the most relevant facts or Chapter 6 of [6] for a detailed study. Extend $\langle\ ,\ \rangle_{\mathfrak{q}}$ linearly to the complexification $\mathfrak{q}^{\mathbf{C}}$ of $\mathfrak{q}$, and use the same symbol to denote this extension. The Lie group $SO(\mathfrak{q})$ of orthogonal endomorphisms of $\mathfrak{q}$ relative to $\langle\ ,\ \rangle_{\mathfrak{q}}$ has Lie algebra

$$\mathfrak{so}(\mathfrak{q}) = \{A \in End(\mathfrak{q}) \mid \langle AX\,, Y\rangle_{\mathfrak{q}} + \langle X\,, AY\rangle_{\mathfrak{q}} = 0, \quad \text{for all } X, Y \in \mathfrak{q}\}.$$

Observe that, for all X and Y in $\mathfrak{q}$, the endomorphisms $R_{X,Y}$ defined by

$$R_{X,Y}(W) = \langle Y\,, W\rangle_{\mathfrak{q}} X - \langle X\,, W\rangle_{\mathfrak{q}} Y, \ \ W \in \mathfrak{q},$$

span $\mathfrak{so}(\mathfrak{q})$. There is an embedding of $\mathfrak{so}(\mathfrak{q})$ in the Clifford algebra of $\mathfrak{q}^{\mathbf{C}}$. Indeed, the Clifford algebra $Cl(\mathfrak{q})$ of $\mathfrak{q}^{\mathbf{C}}$ is defined as the quotient of the tensor algebra $T(\mathfrak{q})$ of $\mathfrak{q}^{\mathbf{C}}$ by the ideal $\mathcal{I}$ generated by elements $X \otimes Y + Y \otimes X - \langle X\,, Y\rangle_{\mathfrak{q}}$, with X and Y in $\mathfrak{q}^{\mathbf{C}}$:

$$Cl(\mathfrak{q}) = T(\mathfrak{q})/\mathcal{I}.$$

Then the linear extension of

$$R_{X,Y} \mapsto \frac{1}{2}(XY - YX)$$

is an injective Lie algebra homomorphism $\mathfrak{so}(\mathfrak{q}) \to Cl_2(\mathfrak{q})$, where $Cl_2(\mathfrak{q})$ is the Lie algebra defined as the subspace of $Cl(\mathfrak{q})$ generated by the degree 2 elements X_1X_2, with X_1 and X_2 in $\mathfrak{q}$.

Now suppose that $\mathfrak{q}$ is even dimensional, as is the case when the equal rank condition (2.2) holds. Then we may choose two maximal dual isotropic subspaces V and V^* of $\mathfrak{q}^{\mathbf{C}}$ with respect to $\langle\,,\rangle_{\mathfrak{q}}$ such that $\mathfrak{q}^{\mathbf{C}} = V \oplus V^*$. Denote by $\wedge V^*$ the exterior algebra of V^*, equipped with the interior product $\imath$ and exterior multiplication ϵ:

$$\begin{aligned} \imath(v)(v_1^* \wedge \cdots \wedge v_l^*) &= \langle v\,, v_1^*\rangle_{\mathfrak{q}} v_2^* \wedge \cdots \wedge v_l^* - v_1^* \wedge \imath(v)(v_2^* \wedge \cdots \wedge v_l^*) \\ \epsilon(v^*)(v_1^* \wedge \cdots \wedge v_l^*) &= v^* \wedge v_1^* \wedge \cdots \wedge v_l^*, \end{aligned}$$

for all $v \in V, v^* \in V^*$ and $v_1^* \wedge \cdots \wedge v_l^* \in \wedge^l V^*$. Now define a map $\gamma : \mathfrak{q}^{\mathbf{C}} \to End(\wedge V^*)$ by

$$\gamma(v + v^*)(u) = (\imath(v) + \epsilon(v^*))(u),$$

for all element u in $\wedge^l V^*$. Observing that

$$\gamma(X) \circ \gamma(Y) + \gamma(Y) \circ \gamma(X) = \langle X\,, Y\rangle_{\mathfrak{q}}, \quad \text{for all } X, Y \in \mathfrak{q}^{\mathbf{C}},$$

one can extend γ naturally to a map $\tilde{\gamma}$ on the Clifford algebra of $\mathfrak{q}^{\mathbf{C}}$:

$$\tilde{\gamma} : Cl(\mathfrak{q}) \to End(\wedge V^*),\ X_1X_2\cdots X_p \mapsto \gamma(X_1) \circ \gamma(X_1) \circ \cdots \circ \gamma(X_p).$$

Finally, if $S_{\mathfrak{q}} \overset{def.}{=} \wedge V^*$, the spin representation $(\sigma_{\mathfrak{q}}, S_{\mathfrak{q}})$ of $\mathfrak{so}(\mathfrak{q})$ is

$$\sigma_{\mathfrak{q}}(R_{X,Y}) = \frac{1}{2}[\gamma(X), \gamma(Y)], \quad \text{for all } X, Y \in \mathfrak{q}.$$

Now the 'spin' representation of $\mathfrak{h}$ is defined by

$$s_{\mathfrak{q}} = \sigma_{\mathfrak{q}} \circ \mathrm{ad}.$$

Lemma 2.4 below is a slight extension of the discussion in [12]; we include a proof since it plays a crucial role in Subsection 2.4.

Suppose $\{X_j\}$ is a basis of $\mathfrak{q}$ so that

$$\langle X_j\,, X_k\rangle_{\mathfrak{q}} = a_j\delta_{jk}, \quad a_j = \pm 1. \tag{2.3}$$

Note that any $X \in \mathfrak{q}$ may be decomposed as $X = \sum a_j \langle X\,, X_j\rangle_{\mathfrak{q}} X_j$.

Lemma 2.4. *If $T \in \mathfrak{so}(\mathfrak{q})$ then*

$$\sigma_{\mathfrak{q}}(T) = -\sum_{j<k} a_j a_k \langle T(X_j)\,, X_k\rangle_{\mathfrak{q}} \gamma(X_j)\gamma(X_k).$$

Proof. It suffices to check the identity for $T = R_{a,b}$ for arbitrary $a, b \in \mathfrak{q}$. We have

$$\begin{aligned}
-\sum_{j<k} & a_j a_k \langle R_{a,b}(X_j)\,, X_k\rangle_{\mathfrak{q}}\gamma(X_j)\gamma(X_k) \\
&= -\sum_{j<k} a_j a_k \langle b\,, X_j\rangle\langle a\,, X_k\rangle\gamma(X_j)\gamma(X_k) + \sum_{j<k} a_j a_k\langle a\,, X_j\rangle\langle b\,, X_k\rangle\gamma(X_j)\gamma(X_k) \\
&= -\sum_{j\neq k} a_j a_k \langle b\,, X_j\rangle\langle a\,, X_k\rangle\gamma(X_j)\gamma(X_k), \text{ by switching } j \text{ and } k \text{ in the 2nd summation,} \\
&= -\sum_{j,k} a_j a_k \langle b\,, X_j\rangle\langle a\,, X_k\rangle\gamma(X_j)\gamma(X_k) + \sum_j a_j^2\langle b\,, X_j\rangle\langle a\,, X_j\rangle\gamma(X_j)^2 \\
&= -\gamma(b)\gamma(a) + \frac{1}{2}\langle b\,, a\rangle_{\mathfrak{q}} I \\
&= -\frac{1}{2}[\gamma(b), \gamma(a)] \\
&= \sigma_{\mathfrak{q}}(R_{a,b}).
\end{aligned}$$

□

2.3. **The cubic term.** We will need to consider a degree three element in the Clifford algebra $Cl(\mathfrak{q})$ in order to define Kostant's 'cubic' Dirac operator. In order to obtain the most natural results, this degree three element is a necessary ingredient in the Dirac operator (see (2.7)). For example, the cubic term is needed for the following: (i) a simple formula for the square of the Dirac operator ([8]), (ii) a generalization of the Borel-Weil-Bott Theorem ([9]), (iii) a strong connection with infinitesimal character ([7, Section 7], [10] and [1]) and (iv) Lemma 2.12 below.

There is an alternating 3-form on $\mathfrak{q}$ defined by

$$(X, Y, Z) \mapsto \langle X\,, [Y, Z]\rangle. \tag{2.5}$$

Using $\langle\ ,\ \rangle_{\mathfrak{q}}$ to identify $\mathfrak{q}^*$ and $\mathfrak{q}$ we have that $\wedge(\mathfrak{q}^*) \simeq \wedge\mathfrak{q}$ embeds naturally into $Cl(\mathfrak{q})$. An element (of degree 3), which we call c, is determined by (2.5). It will be useful for us to have an explicit formula for c in terms of the basis $\{X_j\}$ of (2.3).

Defining $X_j^* \in \mathfrak{q}^*$ by $X_j^* = \langle X_j\,, \cdot\rangle_{\mathfrak{q}}$ gives

$$\langle \cdot\,, [\cdot, \cdot]\rangle = \sum_{j<k<\ell} a_j a_k a_\ell \langle X_j\,, [X_k, X_\ell]\rangle X_j^* \wedge X_k^* \wedge X_\ell^*.$$

It follows that

$$c = \sum_{j<k<\ell} a_j a_k a_\ell \langle X_j\,, [X_k, X_\ell]\rangle X_j X_k X_\ell. \tag{2.6}$$

2.4. **The cubic Dirac operator for homogeneous vector bundles.** Let E be a finite dimensional representation of $\mathfrak{h}$. We assume that the $\mathfrak{h}$-representation $S_{\mathfrak{q}} \otimes E$ integrates to a representation of H. Then there is an associated smooth homogeneous vector bundle over G/H which we denote by $\mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}$. The space of smooth sections is

$$C^\infty(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}) \overset{def.}{=} \{f : G \to S_{\mathfrak{q}} \otimes E \mid f \text{ is smooth and } f(gh) = h^{-1} \cdot f(g), \text{ for } g \in G, h \in H\}.$$

Let $\{X_j\}$ be a fixed basis satisfying (2.3). Denoting the universal enveloping algebra of $\mathfrak{g}$ by $\mathcal{U}(\mathfrak{g})$, an H-invariant in $\mathcal{U}(\mathfrak{g}) \otimes End(S_{\mathfrak{q}} \otimes E)$ is defined by

$$\sum a_j X_j \otimes (\gamma(X_j) \otimes 1) - 1 \otimes (\tilde{\gamma}(c) \otimes 1).$$

Letting $\mathcal{U}(\mathfrak{g})$ act by left invariant differential operators (i.e., $(R(X)f)(g) \overset{def.}{=} \frac{d}{dt}f(g\exp(tX))|_{t=0}$, for $X \in \mathfrak{g}$) a G-invariant differential operator

$$D : C^\infty(G/H, \mathcal{S}_\mathfrak{q} \otimes \mathcal{E}) \to C^\infty(G/H, \mathcal{S}_\mathfrak{q} \otimes \mathcal{E})$$

is defined by

$$D = \sum a_j R(X_j) \otimes \gamma(X_j) \otimes 1 - 1 \otimes \tilde{\gamma}(c) \otimes 1. \tag{2.7}$$

Definition 2.8. D is the *cubic Dirac operator* on $C^\infty(G/H, \mathcal{S}_\mathfrak{q} \otimes \mathcal{E})$. Often we will denote D by $D_{G/H}$ or $D_{G/H}(\mathcal{E})$.

Remark 2.9. D is independent of the basis $\{X_j\}$ satisfying (2.3). In fact, each of the two terms in (2.7) is, by itself, independent of basis.

Remark 2.10. As mentioned in the introduction, when $\mathfrak{h}$ is the fixed points of an involution then $c = 0$ (as $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$ and $\mathfrak{h} \perp \mathfrak{q}$). Therefore D is the more familiar operator of [13], [2], [16] and [12].

2.5. **A lemma.** For the remainder of this section we prove Lemma 2.12, which may (loosely) be thought of as a statement about 'induction in stages'.

Consider $H \subset L \subset G$, reductive groups so that $\langle\ ,\ \rangle$ is nondegenerate on $\mathfrak{h}$ and $\mathfrak{l}$. Then there are orthogonal decompositions

$$\begin{aligned} &\mathfrak{g} = \mathfrak{l} + \mathfrak{r}, \text{where } \mathfrak{r} = \mathfrak{l}^\perp, \\ &\mathfrak{g} = \mathfrak{h} + \mathfrak{q} \text{ and} \\ &\mathfrak{q} = (\mathfrak{q} \cap \mathfrak{l}) + \mathfrak{r}. \end{aligned}$$

As $\langle\ ,\ \rangle$ is nondegenerate on $\mathfrak{q}$ (respectively, $\mathfrak{r}$ and $\mathfrak{q} \cap \mathfrak{l}$) the spin representation $S_\mathfrak{q}$ (respectively, $S_\mathfrak{r}$ and $S_{\mathfrak{q}\cap\mathfrak{l}}$) of $\mathfrak{h}$ (respectively, $\mathfrak{l}$ and $\mathfrak{h} \cap \mathfrak{l}$) is defined. Note that as $\mathfrak{h}$-representations

$$S_\mathfrak{q} \simeq S_\mathfrak{r} \otimes S_{\mathfrak{q}\cap\mathfrak{l}}.$$

There is a G-equivariant isomorphism

$$C^\infty(G/H, \mathcal{S}_\mathfrak{q} \otimes \mathcal{E}) \simeq C^\infty(G/L, \mathcal{S}_\mathfrak{r} \otimes C^\infty(L/H, \mathcal{S}_{\mathfrak{q}\cap\mathfrak{l}} \otimes \mathcal{E})). \tag{2.11}$$

This isomorphism is given as follows. First identify $S_\mathfrak{q}$ with $S_\mathfrak{r} \otimes S_{\mathfrak{q}\cap\mathfrak{l}}$. Then for $f \in C^\infty(G/H, \mathcal{S}_\mathfrak{q} \otimes \mathcal{E})$ define F_f by

$$F_f(g)(k) = (k \otimes 1) f(gk), \text{ for any } g \in G \text{ and } k \in L.$$

(By $k \otimes 1$ we mean k acting on the first factor of $S_\mathfrak{r} \otimes (S_{\mathfrak{q}\cap\mathfrak{l}} \otimes E)$.) One easily checks that F_f is in the right-hand side of (2.11).

The Dirac operator on the left-hand side of (2.11) may be pushed over to a differential operator on the right-hand side, which we will temporarily denote by $\tilde{D}$. Thus, $\tilde{D}$ is determined by

$$\tilde{D} F_f = F_{Df}.$$

As several Dirac operators will appear below we will index them by the corresponding homogeneous spaces.

Lemma 2.12. $\big(\tilde{D} F_f\big)(g) = \big(D_{G/L} F_f\big)(g) + D_{L/H}\big(F_f(g)\big)$.

Proof. Let $\{X_j\}$ be a basis of $\mathfrak{r}$ and $\{Y_k\}$ a basis of $\mathfrak{q}\cap\mathfrak{l}$ so that

$$\langle X_j\,,X_k\rangle = a_j\delta_{jk} \text{ and } \langle Y_k\,,Y_i\rangle = b_k\delta_{ki}$$

with a_j and b_k equal to ± 1. Together the X_j and Y_k form a basis of $\mathfrak{q}$ satisfying (2.3).
Claim 1: $F_{(\sum a_j R(X_j)\otimes\gamma(X_j)\otimes 1)f}(g) = \big((\sum a_j R(X_j)\otimes\gamma(X_j)\otimes 1)F_f\big)(g)$.
Claim 2: $F_{(\sum b_i R(Y_i)\otimes\gamma(Y_i)\otimes 1)f}(g)(e) = \big((\sum b_i R(Y_i)\otimes\gamma(Y_i)\otimes 1)F_f\big)(g)(e) - \big(\sum b_i s_{\mathfrak{q}\cap\mathfrak{l}}(Y_i)\otimes\gamma(Y_i)\big)(F_f(g))(e)$.
Claim 3: Indexing the cubic term by the appropriate tangent space we have
$c_{\mathfrak{q}} = c_{\mathfrak{r}} + c_{\mathfrak{q}\cap\mathfrak{l}} + \sum_{i,j<k} a_j a_k b_i \langle Y_i\,,[X_j,X_k]\rangle\, Y_i X_j X_k$.

Before proving the three claims we will show how they imply the lemma. It is enough to show the two sides are equal when evaluated at $k=e$.

$$\begin{aligned}
\big(\widetilde{D}F_f\big)(g)(e) &= \big(F_{Df}\big)(g)(e)\\
&= F_{(\sum a_j R(X_j)\otimes\gamma(X_j)\otimes 1)f}(g)(e) + F_{(\sum b_j R(Y_j)\otimes\gamma(Y_j)\otimes 1)f}(g)(e) - F_{(1\otimes\tilde\gamma(c_{\mathfrak{q}})\otimes 1)f}(g)(e)\\
&= \big(D_{G/L}F_f\big)(g)(e) + \big(D_{L/H}F_f(g)\big)(e) - \Big(\sum_{i,j<k} b_i a_j a_k \langle Y_i\,,[X_j,X_k]\rangle \gamma(Y_i)\gamma(X_j)\gamma(X_k)\\
&\qquad + \sum_i b_i s_{\mathfrak{q}}(Y_i)\otimes\gamma(Y_i)\Big)F_f(g)(e).
\end{aligned}$$

To see that the last two terms cancel take $T=\mathrm{ad}(Y_i)$ in Lemma 2.4.

Now we turn to the proofs of the three claims.
Proof of Claim 1. For $X\in\mathfrak{r}$,

$$\begin{aligned}
\Big(F_{(R(X)\otimes\gamma(X)\otimes 1)f}(g)\Big)(k) &= (k\otimes 1)\Big((R(X)\otimes\gamma(X)\otimes 1)f\Big)(gk)\\
&= (k\otimes 1)\gamma(X)\frac{d}{dt}f(gk\exp(tX))|_{t=0}\\
&= \gamma(\mathrm{Ad}(k)X)(k\otimes 1)\frac{d}{dt}f(g\exp(t\mathrm{Ad}(k)X)k)|_{t=0}\\
&= \gamma(\mathrm{Ad}(k)X)\frac{d}{dt}F_f(g\exp(t\mathrm{Ad}(k)X))(k)|_{t=0}\\
&= \Big((R(\mathrm{Ad}(k)X)\otimes\gamma(\mathrm{Ad}(k)X)\otimes 1)F_f\Big)(g)(k).
\end{aligned}$$

Now

$$\begin{aligned}
\Big(F_{(\sum a_j R(X_j)\otimes\gamma(X_j)\otimes 1)f}(g)\Big)(k) &= \Big((\sum a_j R(\mathrm{Ad}(k)X)\otimes\gamma(\mathrm{Ad}(k)X_j)\otimes 1)F_f\Big)(g)(k)\\
&= \Big((\sum a_j R(X_j)\otimes\gamma(X_j)\otimes 1)F_f\Big)(g)(k)
\end{aligned}$$

by Remark 2.9.
Proof of Claim 2. For $Y\in\mathfrak{q}\cap\mathfrak{l}$,

$$\begin{aligned}
\Big(F_{(R(Y)\otimes\gamma(Y)\otimes 1)f}\Big)(g)(k) &= (k\otimes 1)\Big((R(Y)\otimes\gamma(Y)\otimes 1)f\Big)(gk)\\
&= (k\otimes 1)(1\otimes\gamma(Y))\frac{d}{dt}f(gk\exp(tY))|_{t=0}\\
&= \frac{d}{dt}(k\otimes 1)(1\otimes\gamma(Y))(\exp(-tY)k^{-1}\otimes 1)F_f(g)(k\exp(tY))|_{t=0}\\
&= -(1\otimes\gamma(Y))(s_{\mathfrak{q}\cap\mathfrak{l}}(\mathrm{Ad}(k)(Y))\otimes 1)F_f(g)(k) + (1\otimes\gamma(Y))\big(R(Y)F_f(g)\big)(k).
\end{aligned}$$

Proof of Claim 3. Since $[\mathfrak{l}\cap\mathfrak{q},\mathfrak{l}\cap\mathfrak{q}]\subset\mathfrak{l}$ and $\mathfrak{l}\perp\mathfrak{r}$, terms of the form $Y_jY_kX_l$ do not occur in the formula (2.6) for $c_{\mathfrak{q}}$. Therefore

$$\begin{aligned} c_{\mathfrak{q}} &= \sum_{j<k<l} b_jb_kb_l\langle Y_j\,,[Y_k,Y_l]\rangle Y_jY_kY_l + \sum_{j<k<l} a_ja_ka_l\langle X_j\,,[X_k,X_l]\rangle X_jX_kX_l \\ &\qquad + \sum_{k<l,\text{ all } j} b_ja_ka_l\langle Y_j\,,[X_k,X_l]\rangle Y_jX_kX_l \\ &= c_{\mathfrak{l}\cap\mathfrak{q}} + c_{\mathfrak{r}} + \sum_{k<l,\text{ all } j} b_ja_ka_l\langle Y_j\,,[X_k,X_l]\rangle Y_jX_kX_l. \end{aligned}$$

□

3. A special parabolic subgroup

The subgroup H determines a special parabolic subgroup of G as follows. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{s}\cap\mathfrak{h}$ and $\Sigma(\mathfrak{g},\mathfrak{a})$ the set of $\mathfrak{a}$-roots in $\mathfrak{g}$. Fix a positive system $\Sigma^+=\Sigma^+(\mathfrak{g},\mathfrak{a})$. The subalgebra $\mathfrak{a}$ extends to a Cartan subalgebra $\mathfrak{a}+\mathfrak{t}_M$ of $\mathfrak{h}$. We may choose $\mathfrak{t}_M\subset\mathfrak{h}\cap\mathfrak{k}$. By the equal rank condition (2.2) $\mathfrak{a}+\mathfrak{t}_M$ is a Cartan subalgebra of $\mathfrak{g}$, therefore defines a root system $\Delta=\Delta(\mathfrak{g}^{\mathbf{C}},(\mathfrak{a}\oplus\mathfrak{t}_M)^{\mathbf{C}})$. Choose a positive system of roots $\Delta^+\subset\Delta$ with the compatibility property that

$$\beta\in\Delta^+ \text{ and } \beta|_{\mathfrak{a}}\neq 0 \implies \beta|_{\mathfrak{a}}\in\Sigma^+.$$

Now set

$$\mathfrak{n}=\sum_{\alpha\in\Sigma^+}\mathfrak{g}^{\alpha} \text{ and } \overline{\mathfrak{n}}=\sum_{\alpha\in\Sigma^+}\mathfrak{g}^{-\alpha}$$

and

$$\mathfrak{m}=\mathfrak{t}_M+\sum_{\beta\in\Delta,\beta|_{\mathfrak{a}}=0}\mathfrak{g}^{\beta}.$$

Then (since $\mathfrak{a}$ is the split part of a Cartan subalgebra of $\mathfrak{g}$) $\mathfrak{p}=\mathfrak{m}+\mathfrak{a}+\mathfrak{n}$ is a parabolic subalgebra. Denote by M_e, A and N the analytic subgroups of G with Lie algebras $\mathfrak{m},\mathfrak{a}$ and $\mathfrak{n}$. There is a unique θ-stable subgroup M so that $Z_G(\mathfrak{a})=MA$. The identity component of M is M_e. Then $P=MAN$ is the Langlands decomposition of our *special* parabolic subgroup. We remark that without the equal rank condition (2.2) it is not clear how to define a useful parabolic subgroup.

The following lemma contains facts which are easily checked (and are essentially contained in [12]). We use the following notation:

$$\rho(\mathfrak{g})=\frac{1}{2}\sum_{\beta\in\Delta^+}\beta \text{ and } \rho_{\mathfrak{g}}=\frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha.$$

Similar notation is used for other subspaces of $\mathfrak{g}$ invariant under $\mathfrak{a}+\mathfrak{t}_M$ or $\mathfrak{a}$. We use the further notation:

$$\mu_{\mathfrak{a}}\overset{def.}{=}\mu|_{\mathfrak{a}} \text{ and } \mu_{\mathfrak{t}}\overset{def.}{=}\mu|_{\mathfrak{t}_M}, \text{ when } \mu\in(\mathfrak{a}+\mathfrak{t}_M)^*.$$

Lemma 3.1. *For the special parabolic subgroup defined above the following hold.*

(1) *The subgroup $P\cap H$ of H is a minimal parabolic subgroup of H having Langlands decomposition $P\cap H=(M\cap H)A(N\cap H)$.*

(2) *Under the action of H on G/P, $H\cdot eP$ is a closed orbit and*

$$H\cdot eP\simeq H/H\cap P\simeq H\cap K/H\cap M.$$

(3) *The complex ranks of $M, M\cap H$ and $M\cap K$ are all equal. Therefore P is a cuspidal parabolic subgroup in the sense that M has nonempty relative discrete series.*

(4) *The following decompositions hold:*

$$\begin{aligned} \mathfrak{q} &= (\mathfrak{m}\cap\mathfrak{q}) + (\mathfrak{n}\cap\mathfrak{q}) + (\overline{\mathfrak{n}}\cap\mathfrak{q}) \\ \mathfrak{m} &= (\mathfrak{m}\cap\mathfrak{h}) + (\mathfrak{m}\cap\mathfrak{q}) \\ \mathfrak{m}\cap\mathfrak{q} &= \mathfrak{m}\cap\mathfrak{s} + \mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q} \\ \mathfrak{m}\cap\mathfrak{k} &= \mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q} + \mathfrak{m}\cap\mathfrak{h}. \end{aligned}$$

Observe that our choice of Δ^+ determines a positive system of $\mathfrak{t}_M$-roots in $\mathfrak{m}$. Using the same notation of $\rho(\,)$ for $\frac{1}{2}$ the sum of positive $\mathfrak{t}_M$-roots in a $\mathfrak{t}_M$-invariant subspace we have (from part (4))

$$\rho(\mathfrak{m}) = \rho(\mathfrak{m}\cap\mathfrak{h}) + \rho(\mathfrak{m}\cap\mathfrak{q}) \tag{3.2}$$

$$\rho(\mathfrak{m}\cap\mathfrak{q}) = \rho(\mathfrak{m}\cap\mathfrak{s}) + \rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q}), \text{ and} \tag{3.3}$$

$$\rho(\mathfrak{m}\cap\mathfrak{h}) = \rho(\mathfrak{m}\cap\mathfrak{k}) - \rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q}).$$

We will need some technical facts about finite dimensional representations of H and their restrictions to $M\cap H$. We let $\Delta(\mathfrak{q})$ (resp. $\Delta(\mathfrak{h})$) denote the $\mathfrak{a}+\mathfrak{t}_M$-roots in $\mathfrak{q}$ (resp. $\mathfrak{h}$), and set $\Delta^+(\mathfrak{q}) = \Delta^+\cap\Delta(\mathfrak{q})$ and $\Delta^+(\mathfrak{h}) = \Delta^+\cap\Delta(\mathfrak{h})$. Let E_μ be the irreducible finite dimensional representation of $\mathfrak{h}$ with highest weight μ (with respect to the positive system $\Delta^+(\mathfrak{h})$). We assume that $S_\mathfrak{q}\otimes E_\mu$ integrates to a representation of H and give the following decomposition in irreducibles. For any subset $Q\subset\Delta^+(\mathfrak{q})$ set

$$\langle Q\rangle \overset{def.}{=} \sum_{\beta\in Q}\beta.$$

Note that the set of weights of $S_\mathfrak{q}$ is $\{\rho(\mathfrak{q})-\langle Q\rangle : Q\subset\Delta^+(\mathfrak{q})\} = \{\langle Q\rangle-\rho(\mathfrak{q}) : Q\subset\Delta^+(\mathfrak{q})\}$. We impose the condition on μ that for all $\alpha\in\Delta^+(\mathfrak{h})$

$$\begin{aligned} &\langle \mu+\rho(\mathfrak{g})-\langle Q\rangle\,,\alpha\rangle \geq 0, \text{ for all } Q\subset\Delta^+(\mathfrak{q}) \text{ and} \\ &\langle \mu+\rho(\mathfrak{g})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})\,,\alpha\rangle > 0. \end{aligned} \tag{3.4}$$

Let $\mathcal{Q} = \{Q : \langle\mu+\rho(\mathfrak{g})-\langle Q\rangle\,,\alpha\rangle > 0 \text{ for all } \alpha\in\Delta^+(\mathfrak{h})\}$. We therefore have the following decomposition as H-representations:

$$S_\mathfrak{q}\otimes E_\mu \simeq \bigoplus_{Q\in\mathcal{Q}} E_{\mu+\rho(\mathfrak{q})-\langle Q\rangle}. \tag{3.5}$$

By (3.4) $E_{\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})}$ occurs in $S_\mathfrak{q}\otimes E_\mu$. We set

$$V_0 \overset{def.}{=} (E_{\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})})^{\mathfrak{n}\cap\mathfrak{h}},$$

a constituent in the $M\cap H$-representation $(S_\mathfrak{q}\otimes E_\mu)^{\mathfrak{n}\cap\mathfrak{h}}$. This $M\cap H$-representation is described as follows. Some care must be taken since M and $M\cap H$ are in general disconnected. By Cor. A.15, $M\cap H = Z_{M\cap H}(\mathfrak{m}\cap\mathfrak{h})(M\cap H)_e$, where $Z_{M\cap H}(\mathfrak{m}\cap\mathfrak{h})$ is the centralizer of $\mathfrak{m}\cap\mathfrak{h}$ in $M\cap H$ and $(\cdots)_e$ denotes the connected component containing the identity. Therefore, the irreducible finite dimensional representations of $M\cap H$ may be described in terms of

(i) $(\sigma_\gamma, U_\gamma)$, the irreducible finite dim. representation of $(M\cap H)_e$ with some highest weight γ and

(ii) (τ, U_τ), an irreducible representation of $Z_{M\cap H}(\mathfrak{m}\cap\mathfrak{h})$

satisfying

(iii) $Z_{M\cap H}(\mathfrak{m}\cap\mathfrak{h})\cap(M\cap H)_e$ (= the center of $(M\cap H)_e$) acts by the same scalars under τ and σ_γ.

Then any irreducible finite dimensional representation of $M\cap H$ occurs on

$$U_{\tau,\gamma} \overset{def.}{=} U_\tau\otimes U_\gamma$$

with well-defined action given by $(\tau \otimes \sigma_\gamma)(zh) = \tau(z) \otimes \sigma_\gamma(h)$ for some choice of τ and γ. Now, since V_0 is irreducible and $\rho(\mathfrak{q})_{\mathfrak{t}} = \rho(\mathfrak{m} \cap \mathfrak{q})$, we may conclude that

$$V_0 \simeq U_{\tau, \mu_{\mathfrak{t}} + \rho(\mathfrak{m}\cap\mathfrak{q}) - 2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})} \tag{3.6}$$

for some $\tau \in Z_{M\cap H}(\mathfrak{m} \cap \mathfrak{h})\widehat{}$ ($\stackrel{def.}{=}$ the set of irreducible representations of $Z_{M\cap H}(\mathfrak{m}\cap\mathfrak{h})$).

For the following recall that there is a natural embedding $S_{\mathfrak{m}\cap\mathfrak{q}} \subset S_{\mathfrak{q}}$.

Lemma 3.7. *With τ as in (3.6), the following hold.*

(1) $S_{\mathfrak{m}\cap\mathfrak{q}} \subset \big(S_{\mathfrak{q}}\big)^{\mathfrak{n}\cap\mathfrak{h}}$

(2) $V_0 \subset S_{\mathfrak{m}\cap\mathfrak{q}} \otimes U_{\tau,\mu_{\mathfrak{t}}}$

Proof. See [12, Lemma 3.8]. □

4. The intertwining operator

Let $\mu \in (\mathfrak{a} + \mathfrak{t}_M)^*$ satisfy the 'sufficiently regular' condition described in (4.3) below and let (σ_μ, E_μ) be an irreducible finite dimensional representation of $\mathfrak{h}$ with highest weight μ. We assume as in the previous sections that $S_{\mathfrak{q}} \otimes E_\mu$ integrates to a representation of H. Our goal is to give an explicit formula for a nonzero G-intertwining operator

$$C^\infty(G/P, \mathcal{W} \otimes \mathbf{C}_{\rho_{\mathfrak{g}}}) \to C^\infty(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}_\mu) \tag{4.1}$$

so that the image lies in the kernel of $D_{G/H}(\mathcal{E}_\mu)$. It turns out that W will be of the form (relative discrete series)$\otimes \mathrm{e}^\nu \otimes 1$. Here P is the parabolic subgroup of Section 3 and

$$C^\infty(G/P, \mathcal{W} \otimes \mathbf{C}_{\rho_{\mathfrak{g}}}) = \{f : G \to W \mid f(gman) = a^{-\rho_{\mathfrak{g}}} m \cdot f(g)\}$$

is the space of smooth sections of the homogeneous vector bundle $\mathcal{W} \otimes \mathbf{C}_{\rho_{\mathfrak{g}}}$.

Due to the disconnectedness of M it is simpler to construct an intertwining operator with domain $C^\infty(G/P^+, \mathcal{W}\otimes\mathbf{C}_{\rho_{\mathfrak{g}}})$ instead of $C^\infty(G/P, \mathcal{W}\otimes\mathbf{C}_{\rho_{\mathfrak{g}}})$, where P^+ is described as follows. Set $M^+ = Z_M(\mathfrak{m})M_e$, $Z_M(\mathfrak{m}) = \{m \in M : \mathrm{Ad}(m) = I \text{ on } \mathfrak{m}\}$. Then $P^+ \stackrel{def.}{=} M^+AN$.

The relative discrete series representations of M may be described in terms of

(i) a relative discrete series representation $\pi(\lambda; M_e)$ of M_e with Harish-Chandra parameter λ, and

(ii) a finite dimensional representation τ_1 of $Z_M(\mathfrak{m})$

satisfying

(iii) $Z_M(\mathfrak{m}) \cap M_e = Z(M_e)$ acts by the same scalars under τ_1 and $\pi(\lambda; M_e)$.

This gives a relative discrete series representation $\pi(\tau_1, \lambda; M^+) \stackrel{def.}{=} \tau_1 \otimes \pi(\lambda; M_e)$ of M^+. Finally,

$$\pi(\tau_1, \lambda; M) = \mathrm{Ind}_{M^+}^{M}(\pi(\tau_1, \lambda; M^+))$$

is an irreducible representation in the relative discrete series of M (and all representations in the relative discrete series are of this form.) See [17] for a thorough discussion of the relative discrete series for reductive groups. Now by induction in stages

$$C^\infty(G/P, \pi(\tau_1, \lambda; M) \otimes \mathrm{e}^{\nu+\rho_{\mathfrak{g}}} \otimes 1) \simeq C^\infty(G/P^+, \pi(\tau_1, \lambda; M^+) \otimes \mathrm{e}^{\nu+\rho_{\mathfrak{g}}} \otimes 1).$$

We conclude that there is no loss of generality in considering intertwining operators

$$C^\infty(G/P^+, \mathcal{W} \otimes \mathbf{C}_{\rho_{\mathfrak{g}}}) \to C^\infty(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}_\mu) \tag{4.2}$$

in place of (4.1).

Throughout the remainder of this paper we assume the following condition on μ holds:

$$\langle \mu\,, \beta\rangle > C, \text{ for all } \beta \in \Delta^+, \tag{4.3}$$

where C is some sufficiently large positive constant. It suffices for C to be large enough so that (3.4) holds and (A.12), applied to $\mu_{\mathfrak{t}} + \rho(\mathfrak{m} \cap \mathfrak{h})$ and $\mathfrak{m}$ in place of $\lambda + \rho(\mathfrak{k})$ and $\mathfrak{g}$, holds.

The following proposition is contained in Theorem A.23 and Corollary A.24.

Proposition 4.4. *Let μ satisfy (4.3), and let V_0 and τ be as in Lemma 3.7. Then for $\tau_1 \in Z_M(\mathfrak{m})\widehat{}$ satisfying $Hom_{Z_{M\cap H}(\mathfrak{m})}(\tau_1, \tau) \neq 0$, the relative discrete series representation $\pi(\tau_1, \mu_{\mathfrak{t}} + \rho(\mathfrak{m} \cap \mathfrak{h}); M^+)$ may be realized as a subspace of $Ker\big(D_{M^+/M^+\cap H}(\mathcal{U}_{\tau,\mu_{\mathfrak{t}}})\big)$. Furthermore, defining π_0 to be the projection $S_{\mathfrak{m}\cap\mathfrak{q}} \otimes U_{\tau,\mu_{\mathfrak{t}}} \to V_0$ (see Lemma 3.7), we have*

$$f \mapsto \pi_0(f(e))$$
$$\pi(\tau_1, \mu_{\mathfrak{t}} + \rho(\mathfrak{m} \cap \mathfrak{h}); M^+) \to V_0$$

is a nonzero $M^+ \cap H$-homomorphism.

The following lemma contains a (sufficient) condition for the existence of a nonzero intertwining operator as in (4.2). We write $g \in G = K \exp(\mathfrak{m} \cap \mathfrak{s})AN$ as $g = \kappa(g)m(g)\mathrm{e}^{H(g)}n(g)$. Observe that if $h \in H$ then $h = \kappa(h)\mathrm{e}^{H(h)}n(h)$ is the Iwasawa decomposition, as $H \cap P$ is a minimal parabolic subgroup of H.

Lemma 4.5. *Let (σ_μ, E_μ) be as above and let W be a representation of $P^+ = M^+AN$. Let $t \in Hom(W \otimes \mathbf{C}_{\rho_{\mathfrak{g}}}, S_{\mathfrak{q}} \otimes E_\mu)$ be $M^+ \cap H$ and $N \cap H$ equivariant and satisfy the following $\mathfrak{a}-$equivariance condition*

$$t(\exp(X) \cdot (w \otimes 1)) = e^{2\rho_{\mathfrak{h}}(X)}(s_{\mathfrak{q}} \otimes \sigma_\mu)(\exp(X))t(w \otimes 1), \textit{ for all } X \in \mathfrak{a}, w \in W.$$

Suppose further that $t \neq 0$. Then

$$(\mathcal{P}_t\phi)(g) = \int_{H\cap K} (s_{\mathfrak{q}} \otimes \sigma_\mu)(\ell)t(\phi(g\ell))d\ell$$

defines a nonzero G-intertwining operator

$$\mathcal{P}_t : C^\infty(G/P^+, \mathcal{W} \otimes \mathbf{C}_{\rho_{\mathfrak{g}}}) \to C^\infty(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}_\mu).$$

Proof. We must check that $\mathcal{P}_t$ satisfies the correct transformation property under H.

$$\begin{aligned}
(\mathcal{P}_t\phi)(gh) &= \int_{H\cap K} \ell \cdot t\big(\phi(g\kappa(h\ell)\mathrm{e}^{H(h\ell)}n(h\ell))\big)d\ell \\
&= \int_{H\cap K} \ell \cdot t\big((\mathrm{e}^{H(h\ell)}n(h\ell))^{-1} \cdot \phi(g\kappa(h\ell))\big)d\ell \\
&= \int_{H\cap K} \ell(\mathrm{e}^{H(h\ell)}n(h\ell))^{-1}\mathrm{e}^{-2\rho_{\mathfrak{h}}(H(h\ell))} \cdot t(\phi(g\kappa(h\ell)))d\ell \\
&= \int_{H\cap K} h^{-1}\kappa(h\ell)t(\phi(g\kappa(h\ell)))\mathrm{e}^{-2\rho_{\mathfrak{h}}(H(h\ell))}d\ell \\
&= h^{-1} \cdot \int_{H\cap K} \ell_1 \cdot t(\phi(g\ell_1))d\ell_1, \text{ by the standard integration formula for the} \\
&\qquad\qquad \text{change of variables } \ell \mapsto \kappa(h\ell) \text{ in } H \cap K.
\end{aligned}$$

$\mathcal{P}_t$ is clearly intertwining for the G-actions (by left translation of functions).

To see that $\mathcal{P}_t$ is nonzero, let $w_0 \in W$ and $v^* \in (S_{\mathfrak{q}} \otimes E_\mu)^*$ be such that $\langle v^*\,, t(w_0)\rangle = 1$. Since the (complex valued) function $Y \mapsto \langle v^*\,, \exp(Y) \cdot t(w_0)\rangle$ is continuous on the orthogonal complement $(\mathfrak{h} \cap \mathfrak{m})^\perp$

of $\mathfrak{h}\cap\mathfrak{m}$ in $\mathfrak{h}\cap\mathfrak{k}$, there exists a neighborhood $\mathcal{U}$ of 0 in $(\mathfrak{h}\cap\mathfrak{m})^{\perp}$ such that $Re(\langle v^*, \exp(Y)\cdot t(w_0)\rangle) > 0$, $\forall Y \in \mathcal{U}$. Now let ψ be a positive real valued smooth function on $(\mathfrak{h}\cap\mathfrak{m})^{\perp}$ with support on $\mathcal{U}$ such that $\psi(0) = 1$. Define, using the local coordinates $(\exp(Y), m) \in \exp(\mathcal{U}) \times (H\cap M)$ on $H\cap K$, a smooth section $\tilde\phi \in C^\infty(H\cap K/H\cap M, \mathcal{W})$ by

$$\tilde\phi(l) = \begin{cases} \psi(Y)(m^{-1}\cdot w_0), & \text{if } l = \exp(Y)m \in \exp(\mathcal{U})\times(H\cap M)\\ 0, & \text{if } l \notin \exp(\mathcal{U})\times(H\cap M).\end{cases}$$

Now pull $\tilde\phi$ back to some smooth section $\phi \in C^\infty(K/M, \mathcal{W})$. We have:

$$\langle v^*, (\mathcal{P}_t\phi)(e)\rangle = \int_{H\cap K}\langle v^*, l\cdot t(\tilde\phi(l))\rangle dl = \int_{\mathcal{U}}\psi(Y)\langle v^*, \exp(Y)\cdot t(w_0)\rangle dY.$$

So $\langle v^*, (\mathcal{P}_t\phi)(e)\rangle \neq 0$, since its real part is positive. Therefore $\mathcal{P}_t$ is nonzero.

$\square$

Remark 4.6. The lemma holds with M in place of M^+.

We now specify the representation W of P^+ which we will need. Set $W = \delta\otimes \mathrm{e}^{\nu}\otimes 1$ with

$$\begin{aligned}&\delta = \pi(\tau_1, \mu_{\mathfrak{t}} + \rho(\mathfrak{m}\cap\mathfrak{h}); M^+) \text{ and}\\ &\nu = \mu_{\mathfrak{a}} + \rho_{\mathfrak{h}},\end{aligned} \tag{4.7}$$

for some $\tau_1 \in Z_M(\mathfrak{m})\widehat{\ }$ with $\mathrm{Hom}_{Z_{M\cap H}(\mathfrak{m})}(\tau_1, \tau) \neq 0$ and τ as in (3.6).

We now make precise a choice of t as in the lemma. Viewing W as a space of harmonic spinors as in Proposition 4.4 we define $t : W\otimes \mathbf{C}_{\rho_{\mathfrak{g}}} \to S_{\mathfrak{q}}\otimes E_\mu$ to be evaluation at $e \in G$ followed by projection:

$$t : \phi \mapsto \pi_0(\phi(e)). \tag{4.8}$$

Lemma 4.9. *The hypotheses of Lemma 4.5 are satisfied by t.*

Proof. Equivariance under $M^+\cap H$ is clear. Equivariance under $N\cap H$ follows from the fact that $V_0 = \big(E_{\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})}\big)^{\mathfrak{n}\cap\mathfrak{h}}$. The stated equivariance with respect to $\mathfrak{a}$ holds because

$$\big(\mu + \rho(\mathfrak{q}) - 2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})\big)|_{\mathfrak{a}} + 2\rho_{\mathfrak{h}} = \mu_{\mathfrak{a}} + \rho_{\mathfrak{q}} + 2\rho_{\mathfrak{h}} = (\mu_{\mathfrak{a}} + \rho_{\mathfrak{h}}) + \rho_{\mathfrak{g}}.$$

The fact that $t \neq 0$ is contained in Prop. 4.4. $\square$

Definition 4.10. Using t as defined in (4.8) we set $\mathcal{P} = \mathcal{P}_t$. Therefore, $\mathcal{P}$ is the nonzero G-intertwining operator

$$\mathcal{P} : C^\infty(G/P^+, \mathcal{W}\otimes \mathbf{C}_{\rho_{\mathfrak{g}}}) \to C^\infty(G/H, \mathcal{S}_{\mathfrak{q}}\otimes\mathcal{E}_\mu)$$

given by

$$\big(\mathcal{P}\phi\big)(g) = \int_{H\cap K}\ell\cdot\pi_0(\phi(g\ell)(e))d\ell. \tag{4.11}$$

We now come to our main theorem.

Theorem 4.12. *For each $\phi \in C^\infty(G/P^+, \mathcal{W}\otimes\mathbf{C}_{\rho_{\mathfrak{g}}})$, $D_{G/H}\big(\mathcal{P}\phi\big) = 0$.*

Proof. Observe that if $\{X_j\}$ is a basis of $\mathfrak{q}$ satisfying (2.3) then $\sum a_jR(X_j)\otimes\gamma(X_j)$ is H-invariant. Therefore

$$\begin{aligned}&\big(\sum a_jR(X_j)\otimes\gamma(X_j)(\mathcal{P}\phi)\big)(g)\\ &\qquad= \int_{H\cap K}\big(\sum a_jR(X_j)\otimes\gamma(X_j)\big)\ell\cdot\pi_0(\phi(\,\cdot\,\ell)(e))|_g d\ell\\ &\qquad= \int_{H\cap K}\ell\cdot\big(\sum a_jR(X_j)\otimes\gamma(X_j)\big)\pi_0(\phi(\,\cdot\,)(e))|_{g\ell} d\ell.\end{aligned}$$

In particular we need to show that

$$\big(\sum a_j R(X_j)\otimes\gamma(X_j) - 1\otimes\tilde{\gamma}(c)\big)\pi_0(\phi(\,\cdot\,)(e))|_{g\ell} = 0.$$

Since $D_{G/H}$ is independent of basis of $\mathfrak{q}$ (subject to (2.3)) we are free to choose such a basis in a special way. To do this let $\{E_j\}$ be a basis of $\mathfrak{n}\cap\mathfrak{q}$ and $\{\bar{E}_j\}$ a basis of $\bar{\mathfrak{n}}\cap\mathfrak{q}$ so that

$$\langle E_j\,,\bar{E}_k\rangle_{\mathfrak{q}} = \delta_{jk} \text{ and } \langle E_j\,,E_k\rangle_{\mathfrak{q}} = \langle\bar{E}_j\,,\bar{E}_k\rangle_{\mathfrak{q}} = 0.$$

Now let

$$\begin{aligned}&\{Z_i\} \text{ be a basis of } \mathfrak{m}\cap\mathfrak{q} \text{ so that } \langle Z_i\,,Z_{i'}\rangle_{\mathfrak{q}} = a_i\delta_{ii'} \text{ (with } a_i=\pm1)\\ &Y_j^{\pm} \overset{def.}{=} \frac{1}{\sqrt{2}}(E_j \pm \bar{E}_j).\end{aligned}$$

Note that $\langle Y_j^{\pm}\,,Y_k^{\pm}\rangle = \pm\delta_{jk}$. We choose the basis $\{X_j\}$ to be $\{Z_i, Y_j^{\pm}\}$. This is a basis satisfying (2.3) and

$$D_{G/H} = \sum a_i R(Z_i)\otimes\gamma(Z_i) + \sum\big(R(Y_j^+)\otimes\gamma(Y_j^+) - R(Y_j^-)\otimes\gamma(Y_j^-)\big) - 1\otimes\tilde{\gamma}(c)\otimes 1.$$

The following lemma puts c into a more useful form.

Lemma 4.13. *For Z_i, E_j and $\bar{E}_j$ as above*

$$c = c_{\mathfrak{m}\cap\mathfrak{q}} + \sum a_i\langle Z_i\,,[E_j,\bar{E}_k]\rangle Z_i\bar{E}_jE_k + \sum\langle E_j\,,[E_k,\bar{E}_l]\rangle\bar{E}_j\bar{E}_kE_l + \sum\langle E_j\,,[\bar{E}_k,\bar{E}_l]\rangle\bar{E}_jE_kE_l.$$

Proof. This is a straightforward computation from (2.6). □

In the realization of the spin representation given in Subsection 2.2 we take V to be the maximal isotropic subspace which is spanned by root vectors for positive roots in $\mathfrak{q}$. Therefore

$$R(Y_j^+)\otimes\gamma(Y_j^+) - R(Y_j^-)\otimes\gamma(Y_j^-) = R(E_j)\otimes\epsilon(\bar{E}_j) + R(\bar{E}_j)\otimes\imath(E_j)$$

and we may write

$$\begin{aligned}D_{G/H} =& \sum a_iR(Z_i)\otimes\gamma(Z_i) - 1\otimes\tilde{\gamma}(c_{\mathfrak{m}\cap\mathfrak{q}}) + \sum(R(E_j)\otimes\epsilon(\bar{E}_j) + R(\bar{E}_j)\otimes\imath(E_j))\\ &- \sum a_i\langle Z_i\,,[E_j,\bar{E}_k]\rangle\gamma(Z_i)\epsilon(\bar{E}_j)\imath(E_k)\\ &- \sum\big(\langle E_j\,,[E_k,\bar{E}_l]\rangle\epsilon(\bar{E}_j)\epsilon(\bar{E}_k)\imath(E_l) - \langle E_j\,,[\bar{E}_k,\bar{E}_l]\rangle\epsilon(\bar{E}_j)\imath(E_k)\imath(E_l)\big).\end{aligned} \tag{4.14}$$

We apply this to $\pi_0(\phi(\,\cdot\,)(e))$ and see that each term is zero. Indeed, for any $X\in\mathfrak{m}\cap\mathfrak{q}$

$$\begin{aligned}R(X)\pi_0\big(\phi(\,\cdot\,)(e)\big)|_g &= \frac{d}{ds}\pi_0\big(\phi(g\exp(sX))(e)\big)|_{s=0}\\ &= \frac{d}{ds}\pi_0\big(\phi(g)(\exp(sX))\big)|_{s=0}\\ &= R(X)\pi_0\big(\phi(g)(\,\cdot\,)\big)|_e.\end{aligned}$$

Therefore

$$\big(\sum a_iR(Z_i)\otimes\gamma(Z_i) - \tilde{\gamma}(c_{\mathfrak{m}\cap\mathfrak{q}})\big)\pi_0(\phi(\,\cdot\,)(e))|_g = \big(D_{M^+/M^+\cap H}(\pi_0\phi(g))\big)(e) = 0,$$

since $\phi(g)\in W\subset Ker(D_{M^+/M^+\cap H})$.

Since ϕ is in the principal series representation, ϕ is invariant under the right action of N. Therefore,

$$\big(R(E_j)\otimes\epsilon(\bar{E}_j)\big)\pi_0(\phi(\,\cdot\,)(e)) = 0.$$

Now observe that for $v\in V_0\subset S_{\mathfrak{m}\cap\mathfrak{q}}\otimes U_{\tau,\mu_\mathfrak{t}}$ (Lemma 3.7), $\imath(E_j)v = 0$ since $E_j\perp\mathfrak{m}\cap\mathfrak{q}$. As image$(\pi_0) = V_0$ each of the remaining terms in (4.14) annihilates $\pi_0(\phi(\,\cdot\,)(e))$.

□

Appendix A. The relative discrete series

For the appendix G is of a more general type than in the body of the paper. We give a description of relative discrete series representations of G as spaces of harmonic spinors on G/H, where $H \subset K$ is a subgroup of G having the same complex rank as G. Our interest is in a description of discrete series representations of the group M of Section 3. It will therefore suffice to consider groups G satisfying the following conditions. We let $Z_G(\mathfrak{g}) \overset{def.}{=} \{g \in G : \mathrm{Ad}(g) = Id_{\mathfrak{g}}\}$ and G_e be the connected component of G containing the identity. Then we require G to satisfy:

$$\text{(A.1)}\quad \begin{aligned} &(1) \text{ For all } g \in G,\ \mathrm{Ad}(g) \text{ is an inner automorphism of } \mathfrak{g}^{\mathbf{C}},\\ &(2)\ G \text{ is reductive in the sense that } \mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}],\ (\mathfrak{z} \overset{def.}{=} \text{center of } \mathfrak{g}),\\ &(3) \text{ There is a closed normal subgroup } Z \subset Z_G(\mathfrak{g}) \text{ so that:}\\ &\qquad G/ZG_e \text{ is finite and } G_e/Z \cap G_e \text{ is compact.}\end{aligned}$$

This class of groups is studied in detail in [17] and [16]. If MAN is the Langlands decomposition of any cuspidal parabolic subgroup of the group G, then M also satisfies (A.1). See [17, pages 11-13] for a discussion of such hereditary properties of various conditions on G.

It follows that $G/Z_G(\mathfrak{g})$ contains a maximal compact subgroup $K/Z_G(\mathfrak{g})$ and K satisfies the following properties.

$$\text{(A.2)}\quad \begin{aligned} &(1)\ K \text{ is the fixed point group of an involution } \theta,\\ &(2)\ Z_G(\mathfrak{g}) \text{ is a subgroup of } K, \text{ so } Z_G(\mathfrak{g}) = Z_K(\mathfrak{g}),\\ &(3)\ K \text{ meets every connected component of } G \text{ and } K \cap G_e = K_e.\end{aligned}$$

As in Section 2, the Cartan decomposition of $\mathfrak{g}$ under the differential of θ is written as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}, \quad \mathfrak{s} = \mathfrak{k}^{\perp}.$$

We now assume that G, in addition to satisfying (A.1), has a nonempty relative discrete series. Therefore, there exists a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. We fix such a Cartan subalgebra $\mathfrak{t}$.

Define $G^+ \overset{def.}{=} Z_G(\mathfrak{g})G_e$. It follows from (A.2(3)) above that

$$G^+ \cap K = Z_G(\mathfrak{g})K_e.$$

The irreducible finite dimensional representations of $G^+ \cap K$ may therefore be described as follows. Fix a positive system of roots in $\Delta(\mathfrak{k},\mathfrak{t})$. Suppose

$$\text{(A.3)}\quad \begin{aligned} &(1)\ (\sigma_\lambda, F_\lambda) \text{ is the irreducible finite dim. representation of } K_e \text{ with some highest weight } \lambda \in \mathfrak{t}^*,\\ &(2)\ (\tau, F_\tau) \text{ is an irreducible finite dimensional representation of } Z_G(\mathfrak{g}),\\ &(3)\ Z_G(\mathfrak{g}) \cap K_e = Z(G_e), \text{ the center of } G_e, \text{ acts by the same scalars under } \tau \text{ and } \sigma_\lambda.\end{aligned}$$

Then there is a well-defined representation of $G^+ \cap K$ given by $(\tau \otimes \sigma_\lambda)(zk) = \tau(z) \otimes \sigma_\lambda(k)$ acting on $F_{\tau,\lambda} \overset{def.}{=} F_\tau \otimes F_\lambda$. This representation is irreducible and every irreducible representation of $G^+ \cap K$ is of this form.

It is known that the relative discrete series representations of G^+ occur as L^2 spaces of harmonic spinors on $G^+/G^+ \cap K$. See [16], [2] and [13]. The following proposition is a version of these results which contains the precise statements we need. Before stating the proposition we make a few observations. Since $Z_G(\mathfrak{g})$ centralizes $\mathfrak{g}$, $\mathrm{Ad}(z)|_{\mathfrak{s}} = I \in SO(\mathfrak{s})$. Therefore, $Z_G(\mathfrak{g})$ acts on $S_{\mathfrak{s}}$ by some character (with values ± 1), so we may define τ' to be this character times τ. Under this convention it follows that $S_{\mathfrak{s}} \otimes F_{\tau,\lambda} = \tau' \otimes (S_{\mathfrak{s}} \otimes F_\lambda)$, as representations of $G^+ \cap K$. Then, $S_{\mathfrak{s}} \otimes F_{\tau,\lambda}$ contains the irreducible constituent $F_{\tau',\lambda+\rho(\mathfrak{s})}$ with multiplicity one. We may therefore define the projection

$$\pi_1 : S_{\mathfrak{s}} \otimes F_{\tau,\lambda} \to F_{\tau',\lambda+\rho(\mathfrak{s})}. \tag{A.4}$$

Remark A.5. The statements of this appendix hold in a slightly more general setting than described above. Suppose that $(\sigma_\lambda, F_\lambda)$ is an irreducible representation of the Lie algebra $\mathfrak{k}$ with highest weight λ and assume that $S_{\mathfrak{s}} \otimes F_\lambda$ integrates to a representation of K_e. (Thus, we do not assume that $S_{\mathfrak{s}}$ and F_λ each integrates to a representation of K_e.) Now let τ' be a finite dimensional irreducible representation of $Z_G(\mathfrak{g})$ so that on $Z(G_e)$ both τ' and $\sigma_{\mathfrak{s}} \otimes \sigma_\lambda$ act by the same scalars. Then, throughout our discussion, we may replace $S_{\mathfrak{s}} \otimes F_{\tau,\lambda}$ by $\tau' \otimes (S_{\mathfrak{s}} \otimes F_\lambda)$ and the results will still hold.

Proposition A.6. *Suppose $\lambda + \rho(\mathfrak{k})$ is dominant for $\Delta^+(\mathfrak{k},\mathfrak{t})$ and nonsingular for $\Delta(\mathfrak{g},\mathfrak{t})$. Let Δ^+ be a positive system of roots in $\mathfrak{g}$ for which $\lambda+\rho(\mathfrak{k})$ is dominant regular. Assume also that $\lambda+\rho(\mathfrak{s})$ is analytically integral. Then there is a positive constant C_1 so that whenever*

$$\langle \lambda\,, \beta\rangle > C_1, \text{ for all } \beta \in \Delta^+, \tag{A.7}$$

$Ker(D_{G^+/G^+\cap K}(\mathcal{F}_{\tau,\lambda}))$ contains, with multiplicity one, a subrepresentation infinitesimally equivalent to a relative discrete series representation of G^+. Denoting this subrepresentation by $W^{\tau',\lambda+\rho(\mathfrak{k})}$,

$$\begin{gathered} f \mapsto \pi_1(f(e)) \\ W^{\tau',\lambda+\rho(\mathfrak{k})} \to F_{\tau',\lambda+\rho(\mathfrak{s})} \end{gathered}$$

is a nonzero $G^+\cap K$-homomorphism.

Remark A.8. The representation $W^{\tau',\lambda+\rho(\mathfrak{k})}$ is infinitesimally equivalent to a relative discrete series representation $\pi(\tau',\lambda+\rho(\mathfrak{k});G^+)$ having infinitesimal character $\lambda+\rho(\mathfrak{k})$ and containing the $G^+\cap K$-type with highest weight $\lambda+\rho(\mathfrak{s})$.

Proof. By [16], $Ker\big(D_{G^+/G^+\cap K}(\mathcal{F}_{\tau,\lambda})\big)$ contains a relative discrete series representation (the L^2 harmonic spinors). Denoting this space by $W^{\tau',\lambda+\rho(\mathfrak{k})}$ we have

$$\begin{aligned} 1 &\le \dim \mathrm{Hom}_{G^+}\big(W^{\tau',\lambda+\rho(\mathfrak{k})}, Ker(D_{G^+/G^+\cap K}(\mathcal{F}_{\tau,\lambda}))\big) \\ &\le \dim \mathrm{Hom}_{G^+}\big(W^{\tau',\lambda+\rho(\mathfrak{k})}, C^\infty(G^+/G^+\cap K, \mathcal{S}_{\mathfrak{s}}\otimes\mathcal{F}_{\tau,\lambda})\big) \\ &= \dim \mathrm{Hom}_{G^+\cap K}\big(W^{\tau',\lambda+\rho(\mathfrak{k})}, S_{\mathfrak{s}}\otimes F_{\tau,\lambda}\big) \qquad \text{(A.9)} \\ &= 1. \qquad \text{(A.10)} \end{aligned}$$

To see the last equality we check that

$$\text{the only } G^+\cap K\text{-type occurring in both } W^{\tau',\lambda+\rho(\mathfrak{k})} \text{ and } S_{\mathfrak{s}}\otimes F_{\tau,\lambda} \text{ is } F_{\tau',\lambda+\rho(\mathfrak{s})}. \tag{A.11}$$

For this, we apply the Dirac inequality ([14, Proposition 2.6]) along with (A.7). We will use the following nonsingularity condition:

$$\begin{aligned} &\langle \lambda+\rho(\mathfrak{k})\,,\beta\rangle > \langle\langle Q\rangle\,,\beta\rangle, \text{ for all } Q\subset\Delta^+(\mathfrak{s}), \beta\in\Delta^+(\mathfrak{k}) \\ &\langle \lambda\,,\beta\rangle > \langle\langle Q\rangle\,,\beta\rangle, \text{ for all } Q\subset\Delta^+(\mathfrak{s}), \beta\in\Delta^+(\mathfrak{s}). \end{aligned} \tag{A.12}$$

This follows from (A.7) when the constant C_1 is

$$C_1 = \max_{Q\subset\Delta^+(\mathfrak{s}),\beta\in\Delta^+} \langle\langle Q\rangle\,,\beta\rangle.$$

The Dirac inequality states that for any irreducible, unitarizable $(\mathfrak{g}, K)$-module V of infinitesimal character Λ, if δ is the highest weight of a $K-$type occurring in $S_{\mathfrak{s}} \otimes V$, then

$$||\Lambda|| \leq ||\delta + \rho(\mathfrak{k})||. \tag{A.13}$$

Note that the infinitesimal character of $W^{\tau',\lambda+\rho(\mathfrak{k})}$ is $\Lambda = \lambda + \rho(\mathfrak{k})$ and the possible $K-$types of $S_{\mathfrak{s}} \otimes F_{\tau,\lambda}$ have highest weights of the form $\lambda + \rho(\mathfrak{s}) - \langle Q\rangle$, for $Q \subset \Delta^+(\mathfrak{s})$. Now suppose that such a $K-$type occurs in $W^{\tau',\lambda+\rho(\mathfrak{k})}$. By the first condition in (A.12), $\delta = (\lambda + \rho(\mathfrak{s}) - \langle Q\rangle) - \rho(\mathfrak{s}) = \lambda - \langle Q\rangle$ occurs in $S_{\mathfrak{s}} \otimes W^{\tau',\lambda+\rho(\mathfrak{k})}$, therefore, by (A.13)

$$||\lambda + \rho(\mathfrak{k})|| \leq ||\lambda + \rho(\mathfrak{k}) - \langle Q\rangle||.$$

However this inequality can only hold when $Q = \emptyset$ (as λ satisfies the second part of (A.12)). Now (A.11) follows.

Since the isomorphism of hom's which gives (A.9) is evaluation at $e \in G$, we may conclude from (A.11) that evaluation at e followed by projection to $F_{\tau',\lambda+\rho(\mathfrak{s})}$ is nonzero on $W^{\tau',\lambda+\rho(\mathfrak{k})}$.

□

Now suppose that H satisfies (A.1), $H \subset K$ and $\operatorname{rank}(\mathfrak{h}^{\mathbf{C}}) = \operatorname{rank}(\mathfrak{k}^{\mathbf{C}})$. We may assume that our Cartan subalgebra $\mathfrak{t}$ is contained in $\mathfrak{h}$. Since our goal is to realize relative discrete series representations of G as harmonic spinors on G/H we will apply Lemma 2.12 to $H \subset G^+ \cap K \subset G$. Proposition A.6 takes care of one step. The other step is essentially the corresponding statement for compact groups, which will follow from [11, Theorem 4]. The exact statement is contained in Proposition A.16 below. We will first need a few facts related to the disconnectedness of G and H.

Lemma A.14. *Let T_e be the analytic subgroup of H with Lie algebra $\mathfrak{t}$. Any automorphism of $\mathfrak{g}$ fixing $\mathfrak{t}$ (pointwise) is $Ad(t)$ for some $t \in T_e$.*

Proof. Let ζ be such an automorphism. Since ζ fixes $\mathfrak{t}$, ζ must be inner. In $\operatorname{Int}(\mathfrak{g}^{\mathbf{C}})$, ζ lies in the Cartan subgroup containing $\operatorname{Ad}(T_e)$. Since the Cartan subgroups in a connected complex group are connected, $\zeta = \operatorname{Ad}(\exp(H_1 + iH_2))$ for some $H_1, H_2 \in \mathfrak{t}$. By using the fact that $\zeta(\mathfrak{g}) \subset \mathfrak{g}$ and considering the action on root vectors one sees that $\operatorname{ad}(H_2) = 0$. □

We obtain the following corollary.

Corollary A.15. *Let H be as above.*

1. *$H = H^+ = Z_H(\mathfrak{h})H_e$.*
2. *$H = Z_H(\mathfrak{g})H_e$.*
3. *$Z_H(\mathfrak{k})K_e = Z_H(\mathfrak{g})K_e$.*

Proof. The first statement follows from the facts that $\operatorname{Ad}(h)$ is inner for any $h \in H$ and $H \subset K$. The second statement follows from the first and the Lemma. Statement (3) follows from the lemma. □

Since $H = Z_H(\mathfrak{g})H_e$ the finite dimensional irreducible representations of H are of the form $(\tau \otimes \sigma_\mu, E_{\tau,\mu})$ with τ an irreducible representation of $Z_H(\mathfrak{g})$, σ_μ the irreducible finite dimensional representation of H_e with some highest weight μ satisfying the condition that τ and σ_μ act by the same scalars on $Z_H(\mathfrak{g}) \cap H_e$. (See (A.3).) Let $\mathcal{E}_{\tau,\mu} \to K/H$ be the associated homogeneous vector bundle over K/H.

Proposition A.16. *Let $K' = Z_H(\mathfrak{k})K_e$ and let $F_{\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}$ be the irreducible finite dimensional representation of K_e of highest weight $\mu - \rho(\mathfrak{k}\cap\mathfrak{q})$. Then*

$$Ker\big(D_{K/H}(\mathcal{E}_{\tau,\mu})\big) \simeq Ind_{K'}^{K}\big(\tau' \otimes F_{\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}\big).$$

Proof. Our proof is a reduction to the case where K is a compact connected semisimple group and H is a connected subgroup, which is handled in [11, Theorem 4].

The first step is to determine $Ker\big(D_{K_e/H_e}(\mathcal{E}_\mu)\big)$. We may write

$$K_e = Z_1 K_{ss} \text{ and } H_e = Z_1(H_e \cap K_{ss})$$

where K_{ss} is the compact semisimple Lie group with Lie algebra $\mathfrak{k}_{ss} \overset{def.}{=} [\mathfrak{k},\mathfrak{k}]$ and $Z_1 = \exp(\mathfrak{z}(\mathfrak{k}))$, $\mathfrak{z}(\mathfrak{k}) =$ center of $\mathfrak{k}$. Note that $H_e \cap K_{ss}$ and K_{ss} are connected compact groups of equal rank. Decomposing μ as $\mu = \mu_{\mathfrak{z}} + \mu_{ss} \in \mathfrak{z}(\mathfrak{k})^* + (\mathfrak{t}\cap\mathfrak{k}_{ss})^*$ we have

$$C^\infty(K_e/H_e, \mathcal{S}_{\mathfrak{k}\cap\mathfrak{q}} \otimes \mathcal{E}_\mu) \simeq \mathrm{e}^{\mu_{\mathfrak{z}}} \otimes C^\infty(K_{ss}/H_e\cap K_{ss}, \mathcal{S}_{\mathfrak{k}\cap\mathfrak{q}} \otimes \mathcal{E}_{\mu_{ss}})$$

and

$$\begin{aligned} Ker\big(D_{K_e/H_e}(\mathcal{E}_\mu)\big) &\simeq \mathrm{e}^{\mu_{\mathfrak{z}}} \otimes F_{\mu_{ss}-\rho(\mathfrak{k}\cap\mathfrak{q})}, \text{ by [11, Theorem 4]} \\ &= F_{\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}. \end{aligned}$$

Observe that $K'/H \simeq K_e/H_e$, so

$$C^\infty(K'/H, \mathcal{S}_{\mathfrak{k}\cap\mathfrak{q}} \otimes \mathcal{E}_{\tau,\mu}) \simeq \tau' \otimes C^\infty(K_e/H_e, \mathcal{S}_{\mathfrak{k}\cap\mathfrak{q}} \otimes \mathcal{E}_\mu)$$

and

$$Ker\big(D_{K'/H}(\mathcal{E}_{\tau,\mu})\big) \simeq \tau' \otimes Ker\big(D_{K_e/H_e}(\mathcal{E}_\mu)\big)$$

Now apply Lemma 2.12 to $H \subset K' \subset K$ to get

$$Ker\big(D_{K/H}(\mathcal{E}_{\tau,\mu})\big) \simeq \mathrm{Ind}_{K'}^{K}\big(\tau' \otimes F_{\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}\big).$$

□

Remark A.17. If K is replaced by $G^+ \cap K$ in the proposition then

$$\mathrm{Ind}_{K'}^{G^+\cap K}\big(\tau' \otimes F_{\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}\big) \simeq \bigoplus_{\tau_1 \in S(\tau)} \big(\tau_1 \otimes F_{\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}\big), \tag{A.18}$$

for some $S(\tau) \subset Z_K(\mathfrak{g})\widehat{}$. The set $S(\tau)$ is easily described as follows: τ_1 occurs in the induced representation $\mathrm{Ind}_{K'}^{G^+\cap K}(F_{\tau',\mu-\rho(\mathfrak{k}\cap\mathfrak{q})})$ if and only if $\mathrm{Hom}_{Z_H(\mathfrak{g})}(\tau_1,\tau') \neq 0$. Therefore

$$S(\tau) = \{\tau_1 \in Z_K(\mathfrak{g})\widehat{} : \mathrm{Hom}_{Z_H(\mathfrak{g})}(\tau_1,\tau') \neq 0\}. \tag{A.19}$$

Continuing with the case where K is replaced by $G^+ \cap K$, let $F_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}$ be one of the constituents occurring in $Ker\big(D_{G^+\cap K/H}(\mathcal{E}_{\tau,\mu})\big)$. Then

$$\mathrm{Hom}_{G^+\cap K}\big(F_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}, Ker\big(D_{G^+\cap K/H}(\mathcal{E}_{\tau,\mu})\big)\big) \neq 0,$$

which gives (by evaluation at $e \in H$) a nonzero H-homomorphism

$$F_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})} \to S_{\mathfrak{k}\cap\mathfrak{q}} \otimes E_{\tau,\mu}. \tag{A.20}$$

The only H-constituent which $F_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}|_H$ and $S_{\mathfrak{k}\cap\mathfrak{q}} \otimes E_{\tau,\mu}$ have in common is $E_{\tau',\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}$. (To see this note that the H_e-constituents in $F_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}$ have highest weights of the form $\mu - \rho(\mathfrak{k}\cap\mathfrak{q}) - \sum m_\gamma \gamma, m_\gamma \geq$

$0, \gamma \in \Delta^+(\mathfrak{k})$ and the highest weights of the H_e-constituents in $S_{\mathfrak{k}\cap\mathfrak{q}} \otimes E_{\tau,\mu}$ are of the form $\mu + \langle Q\rangle - \rho(\mathfrak{k}\cap\mathfrak{q}), Q \subset \Delta^+(\mathfrak{k}\cap\mathfrak{q})$. The only way these can be equal is if all $m_\gamma = 0$ and $Q = \emptyset$.) Now we may conclude that evaluation at $e \in H$ (map (A.20)) followed by the projection

$$\pi_2 : S_{\mathfrak{k}\cap\mathfrak{q}} \otimes E_{\tau,\mu} \to E_{\tau',\mu-\rho(\mathfrak{k}\cap\mathfrak{q})} \tag{A.21}$$

is nonzero. We summarize as follows.

Corollary A.22. *Let τ and μ be as in Proposition A.16. Let $S(\tau) = \{\tau_1 \in Z_K(\mathfrak{g})\widehat{}: Hom_{Z_H(\mathfrak{g})}(\tau_1, \tau') \neq 0\}$. Then*

$$Ker\big(D_{G^+\cap K/H}(\mathcal{E}_{\tau,\mu})\big) \simeq \bigoplus_{\tau_1 \in S(\tau)} F_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}$$

and on any constituent $F_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}$

$$f \mapsto \pi_2(f(e))$$
$$F_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})} \to E_{\tau',\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}$$

is nonzero.

We are now ready to put Propositions A.6 and A.16 together for our final result. It follows from part (2) of Corollary A.15 that $H \subset G^+ \cap K \subset G^+$. Therefore, $H \subset G^+ \cap K$ and we have (by (2.11)):

$$C^\infty(G^+/H, \mathcal{S}_\mathfrak{q} \otimes \mathcal{E}_{\tau,\mu}) \simeq C^\infty(G^+/G^+ \cap K, \mathcal{S}_\mathfrak{s} \otimes C^\infty(G^+ \cap K/H, \mathcal{S}_{\mathfrak{k}\cap\mathfrak{q}} \otimes \mathcal{E}_{\tau,\mu})).$$

Now

$$\begin{aligned}\bigoplus_{\tau_1 \in S(\tau)} W^{\tau_1,\mu+\rho(\mathfrak{h})} &\subset Ker\Big(D_{G^+/G^+\cap K}\big(\oplus_{\tau_1\in S(\tau)} \mathcal{F}_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}\big)\Big), \text{by Proposition A.6,}\\ &\simeq Ker\Big(D_{G^+/G^+\cap K}\big(Ker\big(D_{G^+\cap K/H}\big)\big)\Big), \text{by Corollary A.22,}\\ &\subset Ker\big(D_{G^+/H}(\mathcal{E}_{\tau,\mu})\big), \text{by Lemma 2.12.}\end{aligned}$$

We have established the following theorem.

Theorem A.23. *Suppose $H \subset K$, G and H satisfy (A.1), $rank(\mathfrak{g}^{\mathbf{C}}) = rank(\mathfrak{h}^{\mathbf{C}})$, $\mu + \rho(\mathfrak{q}) \in \mathfrak{t}^*$ is analytically integral, μ satisfies (3.4) and $\tau \in Z_H(\mathfrak{g})\widehat{}$ is compatible with σ_μ as above. Then for $\tau_1 \in S(\tau)$, $\pi(\tau_1, \mu + \rho(\mathfrak{h}); G^+)$ is equivalent to a subrepresentation $W^{\tau_1,\mu+\rho(\mathfrak{h})}$ of $Ker\big(D_{G^+/H}(\mathcal{E}_{\tau,\mu})\big)$.*

Since $\mu + \rho(\mathfrak{g}) - \langle Q\rangle$ is $\Delta^+(\mathfrak{h})$-dominant for all $Q \subset \Delta^+(\mathfrak{q})$ and is dominant regular when $Q = \Delta^+(\mathfrak{k}\cap\mathfrak{q})$ (compare with (3.4)), $E_{\tau',\mu-2\rho(\mathfrak{k}\cap\mathfrak{q})+\rho(\mathfrak{q})}$ is a constituent of multiplicity one in $S_\mathfrak{q} \otimes E_{\tau,\mu}$. (Note: $\mu - \rho(\mathfrak{k} \cap \mathfrak{q}) + \rho(\mathfrak{s}) = \mu - 2\rho(\mathfrak{k} \cap \mathfrak{q}) + \rho(\mathfrak{q})$.) We may define

$$\pi_0 : S_\mathfrak{q} \otimes E_{\tau,\mu} \to E_{\tau',\mu-2\rho(\mathfrak{k}\cap\mathfrak{q})+\rho(\mathfrak{q})}$$

to be the corresponding projection.

Corollary A.24. *For any constituent $W^{\tau_1,\mu+\rho(\mathfrak{h})}$ as in the theorem*

$$f \mapsto \pi_0(f(e))$$
$$W^{\tau_1,\mu+\rho(\mathfrak{h})} \to E_{\tau',\mu-\rho(\mathfrak{k}\cap\mathfrak{q})+\rho(\mathfrak{s})}$$

is a nonzero homomorphism.

Proof. Recall from (A.4) and (A.21) there are projections

$$\pi_1 : S_{\mathfrak{s}} \otimes F_{\tau,\lambda} \longrightarrow F_{\tau',\lambda+\rho(\mathfrak{s})},\ \text{for } \lambda = \mu - \rho(\mathfrak{k}\cap\mathfrak{q})$$
$$\pi_2 : S_{\mathfrak{k}\cap\mathfrak{q}} \otimes E_{\tau,\mu} \to E_{\tau',\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}.$$

Consider also

$$\pi' : S_{\mathfrak{s}} \otimes E_{\tau',\mu-\rho(\mathfrak{k}\cap\mathfrak{q})} \to E_{\tau',\mu-2\rho(\mathfrak{k}\cap\mathfrak{q})+\rho(\mathfrak{q})}.$$

Note that $\pi_0 = \pi' \circ (1 \otimes \pi_2)$ since each map is nonzero and $\dim\big(\mathrm{Hom}_H(S_{\mathfrak{q}} \otimes E_{\tau,\mu}, E_{\tau',\mu-2\rho(\mathfrak{k}\cap\mathfrak{q})+\rho(\mathfrak{q})})\big) = 1$.

Let us use the following (temporary) notation:

$W_0 \simeq F_{\tau_1,\mu-2\rho(\mathfrak{k}\cap\mathfrak{q})+\rho(\mathfrak{q})}$: $G^+ \cap K$-type in $W^{\tau_1,\mu+\rho(\mathfrak{h})}$ see Proposition A.6,

w^+ : highest weight vector in W_0,

v^+ : highest weight vector in $F_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}$,

u^+ : highest weight vector in $E_{\tau',\mu-\rho(\mathfrak{k}\cap\mathfrak{q})}$,

s^+ : highest weight vector in $S_{\mathfrak{s}}$.

Consider

$$\begin{array}{ccccccc} W_0 & \xrightarrow{\imath_1} & S_{\mathfrak{s}} \otimes F_{\tau_1,\mu-\rho(\mathfrak{k}\cap\mathfrak{q})} & \xrightarrow{\imath_2} & S_{\mathfrak{s}} \otimes E_{\tau',\mu-\rho(\mathfrak{k}\cap\mathfrak{q})} & \xrightarrow{\pi'} & E_{\tau',\mu-2\rho(\mathfrak{k}\cap\mathfrak{q})+\rho(\mathfrak{q})} \\ f & \longrightarrow & F_f(e) & \longrightarrow & (1\otimes\pi_2)(F_f(e)(e)) & \to & \pi'(1\otimes\pi_2)(F_f(e)(e)) = \pi_0(f(e)). \end{array} \tag{A.25}$$

We claim that this is nonzero. For this we follow w^+ through each homomorphism. Note that the $\mu - 2\rho(\mathfrak{k}\cap\mathfrak{q}) + \rho(\mathfrak{q})$-weight space occurs in each representation in (A.25). By Proposition A.6 the image of $\imath_1$ is $F_{\tau_1',\mu-2\rho(\mathfrak{k}\cap\mathfrak{q})+\rho(\mathfrak{q})}$, so contains $\imath_1(w^+) = s^+ \otimes v^+$. By Corollary A.22, $\imath_2$ is nonzero, therefore $\imath_2(s^+ \otimes v^+) = s^+ \otimes u^+$. Also, $\pi'(s^+ \otimes u^+)$ is a highest weight vector in $E_{\tau',\mu-2\rho(\mathfrak{k}\cap\mathfrak{q})+\rho(\mathfrak{q})}$. Therefore, $\pi' \cdot \imath_2 \cdot \imath_1(w^+) \neq 0$. □

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