SMOOTH COMPONENTS OF SPRINGER FIBERS

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ABSTRACT. This article studies components of Springer fibers for $\mathfrak{gl}(n)$ that are associated to closed orbits of $GL(p) \times GL(q)$ on the flag variety of $GL(n)$, $n = p + q$. These components occur in any Springer fiber. In contrast to the case of arbitrary components, these components are smooth varieties. Using results of Barchini and Zierau we show these components are iterated bundles and are stable under the action of a maximal torus of $GL(n)$. We prove that if $\mathcal{L}$ is a line bundle on the flag variety associated to a dominant weight, then the higher cohomology groups of the restriction of $\mathcal{L}$ to these components vanish. We derive some consequences of localization theorems in equivariant cohomology and K-theory, applied to these components. In the appendix we identify the tableaux corresponding to these components, under the bijective correspondence between components of Springer fibers for $GL(n)$ and standard tableaux.

INTRODUCTION

Let G be a complex reductive algebraic group with flag variety $\mathfrak{B}$. The moment map $\mu : T^*\mathfrak{B} \to \mathfrak{g}^*$ is called the Springer resolution. If we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by using a nondegenerate G-invariant bilinear form, then the image of μ is the nilpotent cone in $\mathfrak{g}$, and the Springer fibers are the inverse images of nilpotent elements of $\mathfrak{g}$. A Springer fiber $\mu^{-1}(X)$ can be identified with its image under the projection $T^*\mathfrak{B} \to \mathfrak{B}$. This image is the set of Borel subalgebras of $\mathfrak{g}$ that contain X, or equivalently, the set $\mathfrak{B}^u$ of points in $\mathfrak{B}$ fixed by the unipotent element $u = \exp X$.

In the classic papers [27] and [28], Springer constructed a representation of the Weyl group W on the cohomology spaces of a Springer fiber. Subsequently, the Springer fibers and these Springer representations have played an important role in several areas in representation theory. Nevertheless, the geometry of the Springer fibers is still not well-understood. The fixed point scheme $\mathfrak{B}^s$ of a semisimple element $s \in G$ is smooth ([20]) and stable under the action of a maximal torus of G. In contrast, the Springer fibers $\mathfrak{B}^u$ are almost always singular. Indeed, they must be singular because the different irreducible components of $\mathfrak{B}^u$ often intersect. The individual components of $\mathfrak{B}^u$ can be singular as well. Moreover, the components are not in general stable under the action of a maximal torus of G. These facts all complicate the study of the Springer fibers.

In this paper, building on the work of Barchini and Zierau ([2]), we study certain components of Springer fibers in case $G = GL(n)$. As above, we view the Springer fiber as a subscheme of the flag variety $\mathfrak{B}$. Our main result is that these components are isomorphic to iterated fiber bundles constructed using subgroups of G. Thus, in contrast to the general case, these components are smooth and stable under the action of a maximal torus H of G. Using this description of the components, we calculate Betti numbers, we obtain a character formula related to associated cycles of discrete series representations, and we express the (equivariant) cohomology and K-theory classes defined by the components in terms of Schubert bases.

We now describe our results in more detail. Let (G, K) denote the pair of groups $(G, K) = (GL(n), GL(p) \times GL(q))$, $p + q = n$. The group K acts with finitely many orbits on $\mathfrak{B}$. Fix a closed K-orbit $\mathcal{Q}$. Note that $\mathcal{Q}$ is isomorphic to the flag variety for K. Let $\gamma_\mathcal{Q}$ denote the restriction of μ to the conormal bundle $T_\mathcal{Q}^* \mathfrak{B}$. The image $\gamma_\mathcal{Q}(T_\mathcal{Q}^* \mathfrak{B})$ is the closure of a single K-orbit $K \cdot f$; f is called generic. The inverse image $\gamma_\mathcal{Q}^{-1}(f)$ is a single component of the Springer fiber $\mu^{-1}(f)$. We say that such a component is associated to the closed K-orbit $\mathcal{Q}$. We view the component $\gamma_\mathcal{Q}^{-1}(f)$ as a subvariety of $\mathfrak{B}$. In fact, because the projection of $T_\mathcal{Q}^* \mathfrak{B}$ to $\mathfrak{B}$ is $\mathcal{Q}$, we see that $\gamma_\mathcal{Q}^{-1}(f) \subset \mathcal{Q} \subset \mathfrak{B}$.

In [2], the authors define a sequence of pairs $(G_0, K_0) = (G, K) \supset (G_1, K_1) \supset \cdots \supset (G_m, K_m)$, where (G_i, K_i) is a pair of the same type as (G, K). They define elements $f_i \in \mathfrak{g}_i = \operatorname{Lie} G_i$ so that $f = \sum f_i$ is generic. They also define parabolic subgroups $Q_i = L_i U_i \subset H K_i$, where H is the diagonal torus. The main results of [2] (see Proposition 1.8 below) imply that

$$\gamma_\mathcal{Q}^{-1}(f) = Q_m \cdots Q_1 Q_0 \cdot \mathfrak{b} = L_m \cdots L_1 L_0 \cdot \mathfrak{b} \subset G/B, \tag{1}$$

where B is a Borel subgroup of G, chosen so that the orbit $\mathcal{Q}$ is $K \cdot \mathfrak{b}$. Let $R_i = Q_i \cap Q_{i-1}$ and define

$$X_m = Q_m \underset{R_m}{\times} Q_{m-1} \underset{R_{m-1}}{\times} \cdots \underset{R_2}{\times} Q_1 \underset{R_1}{\times} Q_0 / R_0;$$

X_m is a bundle over Q_m/R_m with fibers isomorphic to X_{m-1}. Equation (1) implies that the map $F : X_m \to \gamma_\mathcal{Q}^{-1}(f)$ defined by $F([q_m, q_{m-1}, \ldots, q_0]) = q_m \ldots q_1 q_0 \cdot \mathfrak{b}$ is surjective. The main result of this paper (Theorem 2.10) is that F is an isomorphism of algebraic varieties. This theorem implies that the component $\gamma_\mathcal{Q}^{-1}(f)$ is a fiber bundle over Q_m/R_m, which is a generalized flag variety for a product of general linear groups, with fiber isomorphic to a component of the same type for a smaller pair (G', K') (Corollary 2.22).

The description of $\gamma_\mathcal{Q}^{-1}(f)$ as an iterated bundle has a number of applications. It implies that $\gamma_\mathcal{Q}^{-1}(f)$ is H-invariant, and makes it easy to calculate the Betti numbers of $\gamma_\mathcal{Q}^{-1}(f)$ (see Remark 2.26). Using our description of $\gamma_\mathcal{Q}^{-1}(f)$ we determine the H-fixed points (Proposition 4.1), and also the weights of H acting on the tangent spaces at these points (Corollary 4.4). This makes it possible to apply localization theorems in equivariant K-theory and Borel-Moore homology. Using these theorems we obtain a formula

for the character of H acting on $\sum_i (-1)^i H^i(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau))$, where $\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)$ is an invertible sheaf on $\gamma_{\mathcal{Q}}^{-1}(f)$ corresponding to the weight τ of H (Theorem 4.6). This formula is of interest because of a result of J.-T. Chang [5], which states that the dimension of $H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau))$ is the multiplicity of $\overline{K \cdot f}$ in the associated cycle of a discrete series representation of $U(p,q)$. If τ is sufficiently dominant, the higher cohomology groups vanish, so our formula gives the character of $H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau))$. As another application of our results and localization theorems, we obtain formulas expressing the classes defined by $\gamma_{\mathcal{Q}}^{-1}(f)$ in equivariant cohomology and K-theory in terms of Schubert bases (Theorems 4.8 and 4.9). These formulas imply corresponding non-equivariant formulas, answering (for these components) a question raised by T. A. Springer ([29]), and answered in some special cases by J. Güemes ([18]).

Typically one wants to understand the components in some Springer fiber $\mu^{-1}(f)$. In the approach taken here (and in [2]) the 'f' is a moving target. However, it is readily seen that often two closed orbits $\mathcal{Q}$ and $\mathcal{Q}'$ give generic elements f and f' in the *same* nilpotent G-orbit. When this occurs, by translating by some $g \in G$, the components $\gamma_{\mathcal{Q}}^{-1}(f)$ and $\gamma_{\mathcal{Q}'}^{-1}(f')$ may be identified with components in a single Springer fiber. The purpose of the appendix is to label the components studied in this article in terms of the usual parametrization of components. This will therefore identify the components of any Springer fiber that are of the form $\gamma_{\mathcal{Q}}^{-1}(f)$ (under the above identification) for some closed $\mathcal{Q}$ and generic f. Recall that the nilpotent orbits in $\mathfrak{gl}(n)$ are parametrized by partitions of n. If f is in the orbit corresponding to the partition λ, the components of the Springer fiber $\mu^{-1}(f)$ are parametrized by the set of standard tableaux on the shape of λ; see [26] and [30]. In the appendix, we describe the standard tableaux corresponding to the components we consider in this paper. In fact, the Springer fiber corresponding to *any* nilpotent orbit contains components of the type considered in this article.

Results about the smoothness of components of Springer fibers appear in the literature. An example of a nonsmooth component of a Springer fiber in $\mathfrak{sl}(6)$ is given in [32]. More recently L. Fresse ([10]) has determined exactly which components are smooth for Springer fibers of nilpotents in $\mathfrak{gl}(n)$ having tableau with exactly two columns. Springer fibers of nilpotent elements in $\mathfrak{gl}(n)$ have been studied by F. Fung ([12]) in the case where the Young diagram of f is either of hook shape or has two rows. He shows that the components are iterated bundles and he computes the Betti numbers of components in these cases. Combinatorial formulas for Betti numbers of Springer fibers are contained in the work of G. Lusztig (see for example [23]). A direct computation for $\mathfrak{gl}(n)$ is given in [9]. Note that our formulas are for Betti numbers of individual components. Some of our applications of our main result use a description of H-fixed points (Prop. 4.1); fixed points are determined in some special cases in [11]. The components of Springer fibers associated to closed K-orbits in $\mathfrak{B}$ have been studied for other classical groups by Barchini and Zierau, and they have obtained descriptions

that are similar to that in the case of $(G, K) = (GL(n), GL(p) \times GL(q))$, $p + q = n$. We believe that these descriptions imply that in the other classical cases these components are again isomorphic to iterated bundles. This will be pursued elsewhere.

Notation. We work over the field of complex numbers. We fix once and for all the pair of complex groups

$$(G, K) = (GL(n), GL(p) \times GL(q)), \ p + q = n.$$

Then K is the fixed point group of the involution Θ given by conjugation by the matrix $I_{p,q} := \operatorname{diag}(I_p, -I_q)$. The decomposition of $\mathfrak{g}$ into ± 1-eigenspaces is written as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

The diagonal Cartan subgroup is denoted by H. As is customary we denote the Lie algebras of G, K, H, etc. by $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$, etc. The root system $\Delta(\mathfrak{h}, \mathfrak{g})$ is $\{\epsilon_i - \epsilon_j : i \neq j\}$, where $\epsilon_k \in \mathfrak{h}^*$ is defined by $\epsilon_k(\operatorname{diag}(z_1, \ldots, z_n)) = z_k$. The system of roots of $\mathfrak{h}$ in $\mathfrak{k}$ is $\Delta_c = \{\epsilon_i - \epsilon_j : 1 \leq i, j \leq p \text{ or } p + 1 \leq i, j \leq n \, (\text{and } i \neq j)\}$. We will consider many positive systems for $\Delta(\mathfrak{h}, \mathfrak{g})$. However, we fix once and for all the following positive system in Δ_c:

$$\Delta_c^+ = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq p \text{ or } p + 1 \leq i < j \leq n\}.$$

For each root $\epsilon_i - \epsilon_j$, we let $X_{i,j}$ be the root vector equal to the matrix with a one in the (i, j) entry and zeros elsewhere. If L is a reductive subgroup of K containing the Cartan subgroup H, then the Weyl group is denoted by $W(L)$.

1. Preliminaries

A detailed description of the components of the Springer fibers associated to closed K-orbits in $\mathfrak{B}$ is given in [2]. Since this description plays a key role in the results of this article, we begin by carefully describing certain statements in [2]. Then we will give some consequences of these statements that will be needed later in the article.

1.1. **Components associated to closed K-orbits.** The closed K-orbits in $\mathfrak{B}$ are in one-to-one correspondence with positive systems $\Delta^+ \subset \Delta(\mathfrak{h}, \mathfrak{g})$ that contain Δ_c^+. Such a one-to-one correspondence is given by associating to Δ^+ the Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$, $\mathfrak{n}^- = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{(-\alpha)}$. Then $\mathcal{Q} = K \cdot \mathfrak{b}$ is the corresponding closed K-orbit in $\mathfrak{B}$.

Let us fix a positive system Δ^+ containing Δ_c^+, thus fixing a corresponding closed K-orbit $\mathcal{Q} = K \cdot \mathfrak{b}$. As described in [2, Section 2], the restriction of the moment map $\mu : T^*\mathfrak{B} \to \mathfrak{g}^*$ to the conormal bundle $T_{\mathcal{Q}}^*\mathfrak{B}$ may be identified with a map $\gamma_{\mathcal{Q}} : K \underset{B \cap K}{\times} (\mathfrak{n}^- \cap \mathfrak{p}) \to \mathcal{N}_\theta$, where $\mathcal{N}_\theta$ is the variety of nilpotent elements of $\mathfrak{g}$ contained in $\mathfrak{p}$. The map $\gamma_{\mathcal{Q}}$ is given by the formula

$$\gamma_{\mathcal{Q}}(k, Y) = k \cdot Y \ \ (:= \operatorname{Ad}(k)(Y)).$$

It is a well-known fact that the image of $\gamma_{\mathcal{Q}}$ is the closure of a single K-orbit in $\mathcal{N}_\theta$. An element f of $\mathfrak{n}^- \cap \mathfrak{p}$ is said to be *generic in* $\mathfrak{n}^- \cap \mathfrak{p}$ when $\gamma_{\mathcal{Q}}(T_{\mathcal{Q}}^*\mathfrak{B}) = \overline{K \cdot f}$.

Our first goal is to describe a particular generic element f in $\mathfrak{n}^- \cap \mathfrak{p}$. Note that each positive system Δ^+ containing Δ_c^+ is defined by a Δ-regular, Δ_c^+-dominant $\lambda \in \mathfrak{h}^*$ by $\Delta^+ = \{\alpha : \langle \alpha, \lambda \rangle > 0\}$. The notation $\lambda = (\lambda_1, \ldots, \lambda_n)$ is used for $\lambda = \sum \lambda_i \epsilon_i$. Note that λ is Δ_c^+-dominant means that the first p coordinates are decreasing, as are the last q coordinates. We build f from λ inductively; first f_0 is specified (as a sum of certain root vectors) and a subgroup G_1 (a lower rank general linear group) of G is determined. Then f_1 is chosen in $\mathfrak{g}_1$ and a subgroup G_2 of G_1 is given by the same procedure. One continues, obtaining $f_0, f_1, \ldots, f_{m-1}$; then $f = f_0 + \cdots + f_{m-1}$ is our generic element. The inductive procedure is easily described by forming an 'array' from λ. This array consists of two rows of numbered dots and is constructed as follows. If the greatest coordinate of λ is among the first p coordinates then place the first dot in the upper row, otherwise place it in the lower row. Working from left to right, place the next dot in the upper row if the next greatest coordinate of λ is among the first p coordinates and in the lower row otherwise. Continue in this manner. The jth dot (counting from the left) is in the upper row exactly when the jth greatest coordinate of λ is among the first p coordinates. Again working from left to right, number the dots in the upper row with $1, 2, \ldots, p$ and those in the lower row by $p+1, p+2, \ldots, p+q = n$. For example, if (G, K) is the pair $(GL(7), GL(4) \times GL(3))$ and $\lambda = (7, 6, 4, 3, 5, 2, 1)$, then the array is

$$
\begin{array}{cccccc}
\overset{1}{\bullet} & \overset{2}{\bullet} & & \overset{3}{\bullet} & \overset{4}{\bullet} & \\[2em]
& \underset{5}{\bullet} & & \underset{6}{\bullet} & \underset{7}{\bullet} &
\end{array}.
$$

We define a *block* in the array to be a set of dots in the array that is maximal with respect to the properties (i) all dots lie in the same row and (ii) the dots are consecutive. In the example the blocks are $\{1, 2\}, \{5\}, \{3, 4\}$ and $\{6, 7\}$.

Suppose that our array has N blocks. Define $j_1, \ldots, j_N \in \{1, 2, \ldots, n\}$ so that j_i is the label of the dot farthest to the right in the ith block. Set

$$
f_0 = \sum_{i=1}^{N-1} X_{j_{i+1}, j_i},
$$

where X_{j_{i+1}, j_i} is the root vector for $\epsilon_{j_{i+1}} - \epsilon_{j_i}$ normalized as in the last paragraph of the introduction. Note that $f_0 \in \mathfrak{n}^- \cap \mathfrak{p}$. (Each X_{j_{i+1}, j_i} is noncompact since blocks alternate between lying in the upper and lower rows. Each root $\epsilon_{j_{i+1}} - \epsilon_{j_i}$ is negative since j_{i+1} is to the right of j_i, so $\lambda_{j_{i+1}} < \lambda_{j_i}$.) We call $j_1, \ldots, j_N$ the *first string* through the array. It is useful to express this by connecting the dots labelled by each pair j_i, j_{i+1}. In the example the first string is $2, 5, 4, 7$, which is depicted by
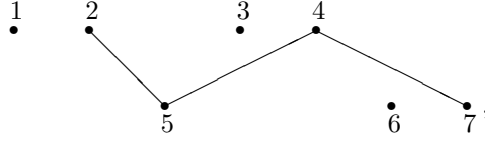
and

$$f_0 = X_{7,4} + X_{4,5} + X_{5,2}.$$

To complete the description of the generic element f in $\mathfrak{n}^- \cap \mathfrak{p}$ we introduce a subgroup G_1 of G. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbf{C}^n$ and set

$$V_0 = \mathrm{span}_{\mathbf{C}}\{e_{j_i} : i = 1, \ldots, N\}$$
$$W_0 = \mathrm{span}_{\mathbf{C}}\{e_j : j \notin \{j_1, \ldots, j_N\}\}.$$

Define

$$G_1 = \{g \in G : g(W_0) \subset W_0 \text{ and } g(e_{j_i}) = e_{j_i}, i = 1, \ldots, N\}.$$

Here are a few properties of G_1 that allow the inductive procedure to work; see [2].

(i) G_1 is isomorphic to $GL(n - N)$.

(ii) G_1 is Θ-stable and, setting $K_1 = (G_1)^{\Theta} = K \cap G_1$, the pair (G_1, K_1) is of the same type as (G, K).

(iii) $\mathfrak{h}_1 := \mathfrak{h} \cap \mathfrak{g}_1$ is a Cartan subalgebra of $\mathfrak{g}_1$ (and of $\mathfrak{k}_1$) and $\Delta^+_{1,c} := \{\alpha|_{\mathfrak{h}_1} : \alpha \in \Delta^+_c, \mathfrak{g}^{(\alpha)} \subset \mathfrak{g}_1, \alpha|_{\mathfrak{h}_1} \neq 0\}$ is a positive system of roots in $\mathfrak{k}_1$.

(iv) $\mathfrak{b}_1 := \mathfrak{b} \cap \mathfrak{g}_1$ is the Borel subalgebra of $\mathfrak{g}_1$ defined by $\lambda_1 := \lambda|_{\mathfrak{h}_1}$, which is $\Delta(\mathfrak{h}_1, \mathfrak{g}_1)$-regular and $\Delta^+_{1,c}$-dominant. $\mathcal{Q}_1 = K_1 \cdot \mathfrak{b}_1$ is a closed orbit in the flag variety for G_1.

(v) G_1 centralizes f_0.

Working in G_1 we choose $f_1 \in \mathfrak{n}_1 \cap \mathfrak{p}_1 (= \mathfrak{n} \cap \mathfrak{g}_1 \cap \mathfrak{p})$ in the same way that f_0 was chosen in $\mathfrak{g}$. This amounts to omitting the dots of the first string through the array to obtain a smaller array and forming a second string. This second string consists of the labels of the dots farthest to the right in the blocks of the smaller array. In the example the smaller array is
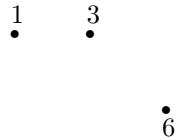
and $f_1 = X_{6,3}$. A crucial observation is that in passing to the smaller array it is possible (and likely) that several blocks 'collapse' to one block. (In the example, 1 and 3 are in different blocks of the array for $\mathfrak{g}$, but are in the same block of the array for $\mathfrak{g}_1$.)

Continue by defining a subgroup G_2 of G_1 just as G_1 was chosen in G. In this way there is a sequence of subgroups $G = G_0 \supset G_1 \supset \cdots \supset G_m$ and a sequence $f_i \in \mathfrak{n}^- \cap \mathfrak{p}$,

$i = 1, \ldots, m - 1$. The procedure ends when $f_m = 0$ (i.e., when there is at most one block in the array for $\mathfrak{g}_m$). It follows from [2, Thm. 3.2] that

$$f = f_0 + \cdots + f_{m-1} \tag{1.1}$$

is generic in $\mathfrak{n}^- \cap \mathfrak{p}$.

Remark 1.2. The construction given here differs slightly from that of [2] in that the strings in [2] pass through the dots farthest to the *left* in each block. One easily checks that there is a $k \in K$ so that our f and G_i are conjugate by k to the f and G_i of [2]. In fact, the element k may be chosen to represent an element of $W(K)$.

To describe $\gamma_{\mathcal{Q}}^{-1}(f)$ we need to define several subgroups of K and of $K_i = K \cap G_i$, $i = 1, \ldots, m$. Consider K first. Let Π be the set of simple roots in Δ^+ and set $S = \Pi \cap \Delta_c^+$. Define $\langle S \rangle = \operatorname{span}_{\mathbf{C}}\{S\} \cap \Delta_c$ and

$$\mathfrak{q}_K = \mathfrak{h} + \sum_{\alpha \in \langle S \rangle \cup \Delta_c^+} \mathfrak{g}^{(-\alpha)}.$$

Then $\mathfrak{q}_K$ is a parabolic subalgebra of $\mathfrak{k}$. Let Q_K denote the corresponding parabolic subgroup of K. Note that $\langle S \rangle$ consists of all roots $\epsilon_j - \epsilon_k$ for which j, k are labels of dots in the same block. Now define parabolic subgroups $Q_{i,K}, i = 1, \ldots, m$ of K_i in the same manner. It follows from Theorem 4.1 and Equation (4.3) of [2] that

$$\gamma_{\mathcal{Q}}^{-1}(f) = Q_{m,K} \cdots Q_{1,K} Q_K \cdot \mathfrak{b} \subset \mathfrak{B}. \tag{1.3}$$

1.2. **The action of the maximal torus H.** It is not the case that a maximal torus of K acts on each irreducible component of a Springer fiber. However, we will see that the maximal torus H does in fact act on each component associated to a closed K-orbit in $\mathfrak{B}$. In this subsection we establish this fact and give a variant of (1.3) for which the action of H is more apparent.

Lemma 1.4. *Let H be the diagonal Cartan subgroup. Then*

$$Q_{m,K} \cdots Q_{1,K} Q_K = (HQ_{m,K}) \cdots (HQ_{1,K})(HQ_K).$$

Proof. We show, by induction, that $Q_{i,K} \cdots Q_{1,K} Q_K = (HQ_{i,K}) \cdots (HQ_{1,K})(HQ_K)$ for each $i = 0, 1, \ldots, m$. The $i = 0$ case is $Q_K = HQ_K$, which clearly holds since Q_K is a parabolic subgroup of K. Assume that the $i - 1$ case holds. We may decompose H into the product of subgroups H' and H'' as follows. The subgroup H' consists of all h' so that

$$h'(e_j) = \begin{cases} e_j, & \text{if } j \text{ is in one of the first } i \text{ strings} \\ z_j e_j, & \text{otherwise} \end{cases}$$

and H'' consists of all h'' so that

$$h''(e_j) = \begin{cases} z_j e_j, & \text{if } j \text{ is in one of the first } i \text{ strings} \\ e_j, & \text{otherwise.} \end{cases}$$

(Here $z_j \in \mathbf{C}$.) Note that $H' \subset Q_{i,K}$ and H'' commutes with K_i (and so commutes with $Q_{i,K}$). Therefore

$$(HQ_{i,K})\cdots(HQ_{1,K})(HQ_K) = (H'Q_{i,K})(H''HQ_{i-1,K})\cdots(HQ_{1,K})(HQ_K)$$
$$= Q_{i,K}(HQ_{i-1,K})\cdots(HQ_{1,K})(HQ_K)$$
$$= Q_{i,K}\cdots Q_{1,K}Q_K, \text{ by the inductive hypothesis.}$$

$\square$

Corollary 1.5. *The maximal torus H acts on $\gamma_Q^{-1}(f)$.*

Definition 1.6. Let $\mathfrak{q}_i = \mathfrak{h} + \mathfrak{q}_{i,K}$ for $i = 1,\ldots,m$ and $\mathfrak{q}_0 = \mathfrak{q}_K$.

It is easy to see that $\mathfrak{q}_i$ is a parabolic subalgebra of the reductive Lie algebra $\mathfrak{h} + \mathfrak{k}_i$. Write $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i^-$ and write the corresponding parabolic subgroup Q_i as $L_iU_i^-$.

Remark 1.7. Note that L_i is slightly different than the group L_i in [2] in that we are including the full torus H in L_i. Also, the group Q_i appearing in [2] is some parabolic subgroup of G_i. We have no need to consider such a group and use Q_i to denote the group $HQ_{i,K}$ of [2].

The next proposition follows from Lemma 1.4, equation (1.3) and [2, Theorem 4.8].

Proposition 1.8. $\gamma_Q^{-1}(f) = Q_m\cdots Q_1Q_0\cdot\mathfrak{b} = L_m\cdots L_1L_0\cdot\mathfrak{b}$.

1.3. **The parabolics Q_i.** There are two useful descriptions of the parabolic subgroups Q_i, one in terms of roots and the other in terms of flags.

Suppose that there are N_i blocks in the array for $\mathfrak{g}_i$. List them (from left to right) as $B_1^i,\ldots,B_{N_i}^i$ (where the $i=0$ case is the case of the original array in $\mathfrak{g}$).

Let S_i be the set of simple roots in $\Delta(\mathfrak{h},\mathfrak{g}_i)\cap\Delta^+$ that are compact. Then $\langle S_i\rangle :=$ span$_\mathbf{C}\{S_i\}\cap\Delta(\mathfrak{h},\mathfrak{k}_i)$ is the set of roots $\epsilon_j - \epsilon_k$ so that j,k are both in a block B_l^i, for some l. Then

$$\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i^- \text{ with}$$
$$\mathfrak{l}_i = \mathfrak{h} + \sum_{\alpha\in\langle S_i\rangle}\mathfrak{g}^{(\alpha)}$$
$$\mathfrak{u}_i^- = \sum_{\alpha\in\Delta_{i,c}^+\setminus\langle S_i\rangle}\mathfrak{g}^{(-\alpha)}.$$

Alternatively, we may describe Q_i as the stabilizer of a flag. Let (F^i) be the flag $\{0\}\subset F_{N_i}^i\subset F_{N_i-1}^i\subset\cdots\subset F_2^i\subset F_1^i$, with

$$F_k^i = \text{span}_\mathbf{C}\{e_j : j\in B_k^i\cup\cdots B_{N_i}^i\}, k = 1,2,\ldots,N_i.$$

Then Q_i is the subgroup of HK_i stabilizing the flag (F^i).

Here are several immediate properties of $Q_i = L_iU_i^-$, $i = 0,1,\ldots,m$.

(1) L_i is isomorphic to $(Torus) \times \Pi_l GL(n_l^i)$ where n_l^i is the cardinality of B_l^i.

(2) Q_i stabilizes f_k, for $k < i$.

(3) $\mathfrak{u}_i^- \subset \mathfrak{u}_{i-1}^-$, $i = 1, 2, \ldots, m$.

The next lemma will be used in a crucial way in Section 2.

Lemma 1.9. *For $i = 1, \ldots, m$ the following facts hold.*

(i) $L_i \cap Q_{i-1}$ is a parabolic subgroup of L_i, and $Q_i \cap Q_{i-1}$ is a parabolic subgroup of Q_i.

(ii) The nilradical of $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$ is $\mathfrak{l}_i \cap \mathfrak{u}_{i-1}^-$, and a Levi factor of $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$ is $\mathfrak{l}_i \cap \mathfrak{l}_{i-1}$.

(iii) The nilradical of $\mathfrak{q}_i \cap \mathfrak{q}_{i-1}$ is $\mathfrak{u}_i^- + (\mathfrak{l}_i \cap \mathfrak{u}_{i-1}^-)$, and a Levi factor is $\mathfrak{l}_i \cap \mathfrak{l}_{i-1}$.

(iv) Letting $\mathfrak{v}_i = \mathfrak{u}_{i-1}^- \cap \mathfrak{l}_i$, $\mathfrak{v}_j \cap \mathfrak{v}_k = \{0\}$, for $j \neq k$.

Proof. Since $L_i \cap Q_{i-1} \supset L_i \cap (B \cap K_{i-1}) = L_i \cap B$, a Borel subgroup of L_i, we see that $L_i \cap Q_{i-1}$ is a parabolic subgroup of L_i. As the nilradical U_i^- of Q_i is contained in U_{i-1}^- by property (3) above, we see that $Q_i \cap Q_{i-1}$ contains a Borel subgroup of Q_i, so $Q_i \cap Q_{i-1}$ is a parabolic subgroup of Q_i.

The nilradical of $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$ is spanned by $\mathfrak{g}^{(-\alpha)}$ for $\alpha = \epsilon_a - \epsilon_b$, $a < b$, with a and b in the same block for $\mathfrak{g}_i$, but in different blocks for $\mathfrak{g}_{i-1}$. These are precisely the roots in $\mathfrak{u}_{i-1}^- \cap \mathfrak{l}_i$. A Levi factor of $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$ is spanned by $\mathfrak{h}$ along with the $\mathfrak{g}^{(\pm\alpha)}$ for $\alpha = \epsilon_a - \epsilon_b$, $a < b$, with a and b in the same block for $\mathfrak{g}_i$ and for $\mathfrak{g}_{i-1}$. This is $\mathfrak{l}_i \cap \mathfrak{l}_{i-1}$. This proves (ii); (iii) follows since $\mathfrak{q}_i \cap \mathfrak{q}_{i-1} = (\mathfrak{l}_i \cap \mathfrak{q}_{i-1}) \oplus \mathfrak{u}_i^-$ and $\mathfrak{l}_i \cap \mathfrak{q}_{i-1} = \mathfrak{l}_i \cap \mathfrak{l}_{i-1} + \mathfrak{l}_i \cap \mathfrak{u}_{i-1}^-$.

To verify (iv) we may assume that $j < k$. Suppose that $\mathfrak{v}_j \cap \mathfrak{v}_k$ is nonzero; since $\mathfrak{v}_j \cap \mathfrak{v}_k$ is $\mathfrak{h}$-stable it must contain some root space, say $\mathfrak{g}^{(-(\epsilon_a - \epsilon_b))} \subset \mathfrak{v}_j \cap \mathfrak{v}_k$. Then a, b are in the same block for $\mathfrak{g}_k$, but in different blocks for $\mathfrak{g}_{k-1}$. But since $j \leq k-1$, a and b must be in different blocks for both $\mathfrak{g}_{j-1}$ and $\mathfrak{g}_j$. But this contradicts $\mathfrak{g}^{(-(\epsilon_a - \epsilon_b))} \subset \mathfrak{v}_j$. $\square$

2. THE STRUCTURE OF $\gamma_\mathcal{Q}^{-1}(f)$

In this section we show that $\gamma_\mathcal{Q}^{-1}(f)$ is isomorphic, as an algebraic variety, to an iterated bundle. Note that we are giving $\gamma_\mathcal{Q}^{-1}(f)$ the reduced scheme structure induced by the closed embedding of $\gamma_\mathcal{Q}^{-1}(f)$ in $\mathfrak{B}$.

Definition 2.1. Let $R_i = Q_i \cap Q_{i-1}$, for $i = 1, 2, \ldots, m$, and $R_0 = Q_0 \cap B \ (= K \cap B)$.

For $k = 0, \ldots, m$, consider $Q_k \times Q_{k-1} \times \cdots \times Q_0$ with the mixing action of $R_k \times R_{k-1} \times \cdots \times R_0$ given by

$$(q_k, q_{k-1}, \ldots, q_1, q_0) \cdot (r_k, r_{k-1}, \ldots, r_1, r_0)$$
$$= (q_k r_k, r_k^{-1} q_{k-1} r_{k-1}, \ldots, r_2^{-1} q_1 r_1, r_1^{-1} q_0 r_0).$$

We denote the quotient by

$$X_k = Q_k \underset{R_k}{\times} Q_{k-1} \underset{R_{k-1}}{\times} \cdots \underset{R_2}{\times} Q_1 \underset{R_1}{\times} Q_0 / R_0. \tag{2.2}$$

The equivalence class of $(q_k, \ldots, q_1, q_0) \in Q_k \times \cdots \times Q_1 \times Q_0$ is denoted by $[q_k, \ldots, q_1, q_0]$. We will write $X = X_m$. The map

$$Q_m \times \cdots \times Q_1 \times Q_0 \to \gamma_{\mathcal{Q}}^{-1}(f) \tag{2.3}$$

defined by $(q_m, \ldots, q_1, q_0) \mapsto q_m \ldots q_1 q_0 \cdot \mathfrak{b}$ is a surjection from $Q_m \times \cdots \times Q_1 \times Q_0$ onto $\gamma_{\mathcal{Q}}^{-1}(f)$ (by Prop. 1.8) and clearly descends to a surjection

$$F : X \to \gamma_{\mathcal{Q}}^{-1}(f). \tag{2.4}$$

We define $F_k : X_k \to \gamma_{\mathcal{Q}}^{-1}(f)$ by the analogous formula (so $F = F_m$).

Proposition 2.5. *The space* $X = Q_m \underset{R_m}{\times} Q_{m-1} \underset{R_{m-1}}{\times} \cdots \underset{R_2}{\times} Q_1 \underset{R_1}{\times} Q_0/R_0$ *is a smooth projective variety and* $F : X \to \gamma_{\mathcal{Q}}^{-1}(f)$ *is a morphism of varieties.*

Proof. This type of argument is fairly standard, but we include it for completeness. We show by induction on k that X_k is a smooth projective variety, that each X_k has a natural Q_k action induced by the action of left multiplication of Q_k on $Q_k \times X_{k-1}$, and that $F_k : X_k \to \gamma_{\mathcal{Q}}^{-1}(f)$ is a morphism of varieties. If $k = 0$, then $X_0 = Q_0/R_0$ is a partial flag variety for Q_0, since R_0 contains a Borel subgroup of Q_0. In particular, X_0 is a smooth projective Q_0-variety. Moreover, since the map $Q_0 \to \gamma_{\mathcal{Q}}^{-1}(f)$ is constant on R_0-orbits, the universal mapping property of quotients ([3, II.6.3]) implies that the induced map $F_0 : X_0 \to \gamma_{\mathcal{Q}}^{-1}(f)$ is a morphism of varieties.

Assume that our assertions have been proved for X_{k-1}. Let R_m act by the mixing action on $Q_k \times X_{k-1}$. Now, $Q_k \to Q_k/R_k$ is a principal R_k-bundle, and X_{k-1} is projective. Moreover, some power of any line bundle on X_{k-1} is Q_{k-1}-equivariant, by [24, Cor. 1.6], so X_{k-1} has a Q_{k-1}-equivariant ample line bundle. This line bundle is R_k-equivariant, as $R_k \subset Q_{k-1}$. By [24, Prop. 7.1], this implies the existence of a principal bundle

$$Q_k \times X_{k-1} \to X_k := Q_k \underset{R_k}{\times} X_{k-1},$$

where X_k is quasi-projective. To see that X_k is projective, we need to show that X_k is complete. As in the proof of [24, Prop. 7.1], we have a fiber square

$$\begin{array}{ccc} Q_k \times X_{k-1} & \longrightarrow & Q_k \\ \downarrow & & \downarrow \\ X_k & \longrightarrow & Q_k/R_k. \end{array}$$

Since X_{k-1} is projective, the top map is proper. The vertical maps are flat and surjective (as they are principal bundle maps), hence faithfully flat. Therefore, since the top map is proper, by descent ([17, Section 8.4-5]), so is the bottom map. As Q_k/R_k is a partial flag variety (since R_k contains a Borel subgroup of Q_k), Q_k/R_k is complete. Therefore X_k is complete. Also, since X_{k-1} is smooth, the top morphism is smooth; so by descent, the bottom morphism is smooth. As Q_k/R_k is smooth, we see that X_k is smooth. Since $Q_k \times R_k$ acts (algebraically) on $Q_k \times X_{k-1}$, by [3, II.6.10], Q_k acts

algebraically on X_k. Thus, X_k is a smooth projective variety with a Q_k-action induced from the left multiplication action of Q_k on $Q_k \times X_{k-1}$, as desired. $\qquad\square$

Remark 2.6. In the preceding proof, we constructed X_k inductively as the quotient (in the sense of algebraic geometry, as in [3, II.6]) of $Q_k \times X_{k-1}$ by R_k. From this, one can show inductively that X_k is the quotient (in the same sense) of $Q_k \times Q_{k-1} \times \cdots \times Q_0$ by $R_k \times R_{k-1} \cdots \times R_0$. Indeed, we have a surjective map

$$Q_k \times Q_{k-1} \times \cdots \times Q_0 \to Q_k \times X_{k-1} \to X_k.$$

Since $Q_k \times \cdots \times Q_0$ is irreducible, this implies that X_k is also. The second map is a quotient by R_k; by induction, the first map is a quotient by $R_{k-1} \times \cdots \times R_0$. Therefore, the composition is a quotient by $R_k \times \cdots R_0$. (This follows, for example, because each map is open, being a quotient morphism. Therefore the composition is open; by [3, Lemma II.6.2], the composition is a quotient morphism. The composition is also an orbit map (that is, constant on $R_k \times \cdots \times R_0$-orbits). So by definition, X_k is the quotient of $Q_k \times Q_{k-1} \times \cdots \times Q_0$ by $R_k \times R_{k-1} \cdots \times R_0$.)

Remark 2.7. Consider the map

$$Q_m \times Q_{m-1} \times \cdots \times Q_0 \to K/R_m \times K/R_{m-1} \times \cdots \times K/R_0$$

defined by

$$(q_m, q_{m-1}, \ldots, q_0) \mapsto (q_m R_m, q_m q_{m-1} R_{m-1}, \ldots, q_m q_{m-1} \cdots q_0 R_0).$$

This map is constant on $R_m \times \cdots \times R_0$-orbits, so by the universal mapping property, it induces a map $\phi : X_m \to K/R_m \times K/R_{m-1} \times \cdots K/R_0$. If V is a representation of $R_m \times \cdots \times R_0$, there is an induced vector bundle on $\prod_i K/R_i$. Pulling back by ϕ yields a vector bundle on X whose sheaf of sections we denote by $\mathcal{O}_X(V)$. If V is a 1-dimensional representation corresponding to a character τ we will denote this sheaf simply by $\mathcal{O}_X(\tau)$. We will mostly be interested in this when V is simply a representation of R_0 (that is, for $i > 0$, R_i acts trivially on V).

Remark 2.8. The analogues of the preceding proposition and remarks hold for other varieties constructed as quotients by mixing actions. We will use this below.

The following lemma is based on a suggestion of Peter Trapa. The lemma applies when $\mathcal{Q}$ is a closed orbit, proving that the components $\gamma_{\mathcal{Q}}^{-1}(f)$ under consideration are smooth varieties.

Lemma 2.9. *Let $\mathcal{Q} \subset \mathfrak{B}$ be a K-orbit whose closure $\overline{\mathcal{Q}} \subset \mathfrak{B}$ is smooth. Let γ denote the restriction of the moment map to the conormal bundle $T_{\overline{\mathcal{Q}}}^* \mathfrak{B}$ of $\overline{\mathcal{Q}}$ in $\mathfrak{B}$, and let $f \in \mathcal{N}_\theta$ be such that $\gamma(T_{\overline{\mathcal{Q}}}^* \mathfrak{B}) = \overline{K \cdot f}$. Then $\gamma^{-1}(f)$ is smooth.*

Proof. Since $T_{\overline{\mathcal{Q}}}^* \mathfrak{B}$ is nonsingular, by [19, III, Cor. 10.7], there is a nonempty open subscheme $V \subset \overline{K \cdot f}$ such that the restriction of γ to $\gamma^{-1}(V)$ is smooth. For any

$k \in K$, the restriction of γ to $\gamma^{-1}(kV)$ is also smooth, so replacing V by $\cup_{k \in K} kV$ we may assume that V is K-invariant. Therefore $V \supset K \cdot f$. For any $v \in V$, $\gamma^{-1}(v)$ is smooth ([19, III, Theorem 10.2]). As $f \in V$, $\gamma^{-1}(f)$ is smooth.   □

Theorem 2.10. *The map $F : X \to \gamma_{\mathcal{Q}}^{-1}(f)$ is an isomorphism of varieties.*

Proof. By Zariski's Main Theorem ([3, Section AG.18.2]), a bijective morphism onto a smooth variety is an isomorphism, so the theorem will be proved once we see that F is bijective.

We prove that F is bijective by proving a little more. Consider X_k as in (2.2) and $F_k : X_k \to \gamma_{\mathcal{Q}}^{-1}(f)$ as above. We apply induction on k to prove that each F_k, $k = 0, 1, \ldots, m$, is a bijection onto its image $Q_k \cdots Q_1 Q_0 \cdot \mathfrak{b}$.

The $k = 0$ case is immediate since $X_0 = Q_0/R_0$ and $R_0 = Q_0 \cap B$. Suppose $k \geq 1$ and

$$q_k \ldots q_1 q_0 \cdot \mathfrak{b} = q_k' \ldots q_1' q_0' \cdot \mathfrak{b}.$$

Then for some $b \in B \cap K$,

$$q_k^{-1} q_k' = q_{k-1} \ldots q_0 b q_0'^{-1} \ldots q_{k-1}'^{-1}. \tag{2.11}$$

Claim : $q_k^{-1} q_k' \in R_k$.

Once the claim is proved, it will follow that $q_{k-1} \ldots q_1 q_0 \cdot \mathfrak{b} = r_k q_{k-1}' \ldots q_1' q_0' \cdot \mathfrak{b}$, for some $r_k \in R_k$. The inductive hypothesis is that F_{k-1} is a bijection, so

$$[q_{k-1}, \ldots, q_0] = [r_k q_{k-1}', \ldots, q_0'] \in X_{k-1}.$$

Therefore,

$$[q_k, q_{k-1}, \ldots, q_0] = [q_k' r_k^{-1}, r_k q_{k-1}', q_{k-2}', \ldots q_0'] = [q_k', q_{k-1}', \ldots, q_0']$$

in X_k.

To prove the claim it is enough to show that $q_k^{-1} q_k' \in Q_{k-1}$. For this we use the following lemma.

Lemma 2.12. *There is a sequence of parabolic subalgebras $\mathfrak{p}^{(k)}$, $k = 0, 1, \ldots, m$ of $\mathfrak{k}$ so that*
   (i) $\mathfrak{q} = \mathfrak{p}^{(0)} \subset \mathfrak{p}^{(1)} \subset \cdots \subset \mathfrak{p}^{(m)} = \mathfrak{k}$, and
   (ii) $\mathfrak{p}^{(k)} \cap (\mathfrak{h} + \mathfrak{k}_k) = \mathfrak{q}_k, k = 0, 1, \ldots m.$

Proof of lemma. For each $k = 0, 1, \ldots, m - 1$ consider sets $C_l^{(k)} \subset \{1, 2, \ldots, n\}$ with the following properties.

   (1) $\{1, 2, \ldots, n\}$ is the disjoint union of $C_1^{(k)}, \ldots C_{J_k}^{(k)}$.
   (2) Each $C_l^{(k)}$ consists of consecutive integers and lies either in $\{1, \ldots, p\}$ or in $\{p + 1, \ldots, n\}$.
   (3) Every $\mathfrak{g}_k$-block has labels that are contained in exactly one $C_l^{(k)}$ and each $C_l^{(k)}$ contains at most one $\mathfrak{g}_k$-block.

(4) Each $C_l^{(k)}$ is the union of $C_j^{(k-1)}$ for several j.

Set $C_1^{(m)} = \{1, \ldots, p\}$ and $C_2^{(m)} = \{p+1, \ldots, n\}$.

It follows from (3) that $J_0 = N$ (the number of blocks in the original array) and $C_1^{(0)}, \ldots, C_N^{(0)}$ are the original blocks.

We need to establish the existence of a family $C_l^{(k)}$ that satisfies (1)-(4). Intuitively, each $C_l^{(k)}$ is the union of all $C_j^{(k-1)}$ meeting a common $\mathfrak{g}_k$-block. This is not quite the case because (2) must be satisfied. The sets $C_l^{(k)}$ are not uniquely determined by (1)-(4), but we give one choice below.

Let N_k be the number of $\mathfrak{g}_k$-blocks. Let's list these blocks as $B_1^{(k)}, \ldots, B_{N_k}^{(k)}$ by first listing, from left to right, those blocks $B_1^{(k)}, \ldots, B_{p_k}^{(k)}$ occurring in the upper row. Then continue with the blocks $B_{p_k+1}^{(k)}, \ldots, B_{N_k}^{(k)}$ in the lower row. Define an increasing sequence of integers $a_0, a_1, \ldots, a_{N_k+1}$ by

$$
a_i = \begin{cases} 1, & i = 0 \\ \text{the index of the leftmost dot in } B_i^{(k)}, & i = 1, \ldots, p_k \\ p+1, & i = p_k + 1 \\ \text{the index of the leftmost dot in } B_{i-1}^{(k)}, & i = p_k + 2, \ldots, N_k + 1 \end{cases} \tag{2.13}
$$

Using the notation $[a, b) = \{r \in \mathbf{Z} : a \le r < b\}$ we define

$$
\begin{aligned}
C_i^{(k)} &= [a_i, a_{i+1}), \text{ for } i = 0, 1, \ldots, N_k \text{ and} \\
C_{N_k+1}^{(k)} &= [a_{N_k+1}, n],
\end{aligned} \tag{2.14}
$$

for $k = 0, 1, \ldots, m-1$. Note that these sets are not necessarily nonempty, even for a given value of k. For example, often the first dot in the first block is labelled by 1; in this case $a_1 = 1$, so $C_0^{(k)} = \emptyset$.

Since $\{1, \ldots, p\}$ is the disjoint union of $C_0^{(k)}, \ldots, C_{p_k}^{(k)}$ and $\{p+1, \ldots, n\}$ is the disjoint union of $C_{p_k+1}^{(k)}, \ldots, C_{N_k+1}^{(k)}$, properties (1) and (2) hold. Property (3) holds since each a_j is leftmost in a $\mathfrak{g}_k$-block.

It remains to show that (4) holds. For this it suffices to show that each $C_i^{(k-1)}$ is contained in some $C_j^{(k)}$. Let $a_0', a_1', \ldots$ be the sequence of integers defined in (2.13), but for $\mathfrak{g}_{k-1}$-blocks. We need to check that for any $i = 0, \ldots, N_{k-1}$,

$$
\begin{aligned}
[a_i', a_{i+1}') &\subset [a_j, a_{j+1}), \text{ for some } j \text{ and} \\
[a_{N_{k-1}+1}', n] &\subset [a_{N_k+1}, n].
\end{aligned}
$$

For this it suffices to show that $\{a_j\} \subset \{a_i'\}$. Suppose a_j is (leftmost) in a $\mathfrak{g}_k$-block $B_j^{(k)}$. Then a_j lies in some $\mathfrak{g}_{k-1}$-block $B_i^{(k-1)}$. Since several $\mathfrak{g}_{k-1}$-blocks collapse to one $\mathfrak{g}_k$-block and the kth string does not pass through the leftmost dot in $B_i^{(k-1)}$, a_j is

leftmost in $B_i^{(k-1)}$, so equal to a_i'. This proves the claim and completes the verification of (4).

The existence of sets $C_l^{(k)}$ satisfying the properties (1)-(4) is now established and we are ready to finish the proof of the lemma.

Define $S^{(k)} = \{\epsilon_{a+1} - \epsilon_a : a, a+1 \in C_l^{(k)} \text{ for some } l\}$. By Property (2), $\langle S^{(k)} \rangle :=$ $\mathrm{span}_{\mathbf{C}}\{S^{(k)}\} \cap \Delta = \{\epsilon_a - \epsilon_b : a, b \in C_l^{(k)}, \text{ for some } l\}$.

Definition 2.15. $\mathfrak{p}^{(k)} = (\mathfrak{h} + \sum_{\alpha \in \langle S^{(k)} \rangle} \mathfrak{g}^{(\alpha)}) + \sum_{\alpha \in \Delta_c^+ \setminus \langle S^{(k)} \rangle} \mathfrak{g}^{(-\alpha)}$.

The fact that $\mathfrak{p}^{(k)}$ is a parabolic subalgebra of $\mathfrak{k}$ follows from Property (2). Part (i) of the lemma follows from Property (4). The third property implies that for each l, $C_l^{(k)} \cap (\{1, 2, \cdots, n\} \setminus \{\text{the first } k \text{ strings}\})$ is a $\mathfrak{g}_k$-block (or is empty). Part (ii) of the lemma follows from this. $\square$

We are now in position to prove that $q_k^{-1} q_k' \in Q_{k-1}$. Let $P^{(k)} = N_K(\mathfrak{p}^{(k)})$, the parabolic subgroup with Lie algebra $\mathfrak{p}^{(k)}$. Since $Q_0, \ldots, Q_{k-1} \subset P^{(k-1)}$, the right hand side of (2.11) is in $P^{(k-1)}$. But the left hand side is in $HK_k \subset HK_{k-1}$. We conclude that $q_k^{-1} q_k' \in P^{(k-1)} \cap HK_{k-1} = Q_{k-1}$. $\square$

Remark 2.16. An alternate proof of Theorem 2.10 may be given as follows. The map F is finite since it has finite fibers and X is complete. There is a natural injection $H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{L}^n) \to H^0(X, F^*\mathcal{L}^n)$. If this injection is an isomorphism for some ample line bundle $\mathcal{L}$ and all sufficiently large n, then F is an isomorphism. The injection is an isomorphism because the G-module structure of $H^0(X, F^*\mathcal{L}^n)$ can be computed using the Borel-Weil Theorem, and it coincides with the G-module structure of $H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{L}^n)$, which is computed in [2]. This alternative proof implies that $\gamma_{\mathcal{Q}}^{-1}(f)$ is smooth without using Lemma 2.9.

Remark 2.17. If $L_i \subset L_0$, for all i, then $\gamma_{\mathcal{Q}}^{-1}(f) \cong Q_0/(Q_0 \cap B) \cong L_0/(L_0 \cap B)$, so $\gamma_{\mathcal{Q}}^{-1}(f)$ is a homogeneous variety. However, if the condition $L_i \subset L_0$ fails for some i (for example, whenever there is a block in the array having fewer dots than its two neighbors), then $\gamma_{\mathcal{Q}}^{-1}(f)$ is not in any obvious way a homogeneous variety.

2.1. **A vanishing theorem.** The next result is a vanishing theorem for the higher cohomology groups of invertible sheaves on $\gamma_{\mathcal{Q}}^{-1}(f)$ associated to dominant weights.

Let $\tau \in \mathfrak{h}^*$ correspond to a character χ_τ of H; extend χ_τ to a character of $K \cap B = R_0$. The closed K-orbit $\mathcal{Q}$ may be identified with $K/(K \cap B)$, so there is an invertible sheaf $\mathcal{O}_{\mathcal{Q}}(\tau)$ on $\mathcal{Q}$ (cf. Remark 2.7). As noted in the introduction, $\gamma_{\mathcal{Q}}^{-1}(f)$ can be viewed as a subvariety of $\mathcal{Q}$; let $\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)$ denote the pullback of the sheaf $\mathcal{O}_{\mathcal{Q}}(\tau)$ to $\gamma_{\mathcal{Q}}^{-1}(f)$.

Theorem 2.18. *If $\tau \in \mathfrak{h}^*$ is Δ_c^+-dominant and integral, then for all $i > 0$, we have* $H^i(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)) = 0$.

Proof. The proof is by induction on the number of strings m. If $m = 0$ then $\gamma_{\mathcal{Q}}^{-1}(f) = \mathcal{Q}$ and the result holds by the Borel-Weil-Bott theorem. Suppose the result holds in case there are $m - 1$ strings.

Define
$$X(1) = Q_m \underset{R_m}{\times} Q_{m-1} \underset{R_{m-1}}{\times} \cdots \underset{R_2}{\times} Q_1/R_1',$$
where $R_1' = Q_1 \cap B \subset R_1 = Q_1 \cap Q_0$. Note that $R_1' = R_1 \cap B$ is a Borel subgroup of R_1. By Theorem 2.10, $X(1) \simeq \gamma_{\mathcal{Q}_1}^{-1}(f')$, where $f' = f - f_0$ and $\mathcal{Q}_1$ is a closed K_1-orbit in the flag variety for G_1.

Suppose $\tau \in \mathfrak{h}^*$ is Δ_c^+-dominant and integral. We wish to show that $H^i(X, \mathcal{O}_X(\tau))$ vanishes when $i > 0$.

The map that forgets the last factor of X is a fibration
$$g : X \to M = Q_m \underset{R_m}{\times} Q_{m-1} \underset{R_{m-1}}{\times} \cdots \underset{R_2}{\times} Q_1/R_1.$$
This is a fiber bundle associated to the R_1-principal bundle
$$Q_m \underset{R_m}{\times} Q_{m-1} \underset{R_{m-1}}{\times} \cdots \underset{R_2}{\times} Q_1 \to M,$$
with fibers isomorphic to the R_1-variety Q_0/R_0.

Let $U_{-\tau}$ be the irreducible representation of L having lowest weight $-\tau$. We claim that
$$H^i(X, \mathcal{O}_X(\tau)) \cong H^i(M, \mathcal{O}_M(U_{-\tau}^*)). \tag{2.19}$$

The push-forward sheaf $R^i g_* \mathcal{O}_X(\tau)$ is the sheaf of sections on M associated to the R_1-module $H^i(Q_0/R_0, \mathcal{O}_{Q_0/R_0}(\tau))$. Since L_0 is a Levi factor of Q_0, and $R_0 = B \cap K$ is a Borel subgroup of Q_0, there is an isomorphism $L_0/(L_0 \cap B) \cong Q_0/R_0$. Therefore, $H^i(Q_0/R_0, \mathcal{O}_{Q_0/R_0}(\tau)) \cong H^i(L_0/(L_0 \cap B), \mathcal{O}_{L_0/(L_0 \cap B)}(\tau))$. By the Borel-Weil-Bott Theorem, this group is 0 for $i > 0$. For $i = 0$, the group is a Q_0-module whose restriction to L_0 is isomorphic to $U_{-\tau}^*$. We use the same notation $U_{-\tau}^*$ for the restriction of this module to $R_1 \subset Q_0$. Thus, we have shown that
$$g_* \mathcal{O}_X(\tau) = \mathcal{O}_M(U_{-\tau}^*),$$
and that for $i > 0$,
$$R^i g_* \mathcal{O}_X(\tau) = 0.$$
The Leray spectral sequence now implies that (2.19) holds for all i.

By Lemma 1.9, $L_1 \cap L_0$ is a Levi factor of R_1. Thus $U_{-\tau}^*$ has a filtration whose associated graded module is a representation on which the unipotent radical of R_1 acts trivially and on which $L_1 \cap L_0$ acts as $U_{-\tau}^*|_{L_0 \cap L_1} \simeq \oplus E_{-\tau_i}^*$. This induces a corresponding filtration on the sheaf $\mathcal{O}_M(U_{-\tau}^*)$. Induction and the long exact sequence in cohomology imply that our desired vanishing will follow if we can show that
$$H^i(M, \mathcal{O}_M(E_{-\tau_j}^*)) = 0, \text{ for } i > 0. \tag{2.20}$$

For this we consider the fibration $h : X(1) \to M$, a fiber bundle with fibers isomorphic to R_1/R_1'.

To show that (2.20) holds we prove

$$H^i(X(1), \mathcal{O}_{X(1)}(\tau_j)) \cong H^i(M, \mathcal{O}_M(E^*_{-\tau_j})). \tag{2.21}$$

In the course of the proof of [2, Theorem 6.6], it is proved that each τ_j is dominant with respect to the positive system $\Delta^+_{1,c}$. Therefore, assuming (2.21), the inductive hypothesis implies that (2.20) holds.

We now prove that (2.21) holds. Each τ_i defines a character of R_1', and the push-forward sheaf $R^i h_* \mathcal{O}_{X(1)}(\tau_i)$ is the sheaf of sections on M associated to the R_1-module $H^i(R_1/R_1', \mathcal{O}_{R_1/R_1'}(\tau_i))$. Since $L_1 \cap L_0$ is a Levi factor of R_1, and $R_1' = R_1 \cap B$ is a Borel subgroup of R_1, there is an isomorphism $(L_1 \cap L_0)/(L_1 \cap L_0 \cap B) \cong R_1/R_1'$. Therefore,

$$H^i(R_1/R_1', \mathcal{O}_{R_1/R_1'}(\tau_i)) \cong H^i((L_1 \cap L_0)/(L_1 \cap L_0 \cap B), \mathcal{O}_{(L_1 \cap L_0)/((L_1 \cap L_0 \cap B)}(\tau_i)).$$

By the Borel-Weil-Bott Theorem, this group is 0 for $i > 0$. For $i = 0$, the group is a R_1-module whose restriction to $L_1 \cap L_0$ is isomorphic to $E^*_{-\tau_i}$. Moreover, we claim that the unipotent radical of R_1 acts trivially on the module $H^0(R_1/R_1', \mathcal{O}_{R_1/R_1'}(\tau_i))$. Indeed, if we denote this unipotent radical by N, then since N is unipotent and normal in R_1, the space of N-fixed vectors in this module is nonzero and R_1-stable. Since the R_1-module $H^0(R_1/R_1', \mathcal{O}_{R_1/R_1'}(\tau_i))$ is irreducible (as its restriction to $L_1 \cap L_0$ is), the space of N-fixed vectors must be the entire module, proving the claim. Thus, we have shown that

$$h_* \mathcal{O}_{X(1)}(\tau_i) = \mathcal{O}_M(E^*_{-\tau_i}),$$

and that for $i > 0$,

$$R^i h_* \mathcal{O}_{X(1)}(\tau_i) = 0.$$

Again using the Leray spectral sequence we see that for all i and all τ_j,

$$H^i(X(1), \mathcal{O}_{X(1)}(\tau_j)) \cong H^i(M, \mathcal{O}_M(E^*_{-\tau_j})).$$

Therefore (2.21) holds and the proof of the theorem is complete.

$\square$

2.2. **Some topological consequences of Theorem 2.10.** A consequence of the theorem is that any component of a Springer fiber associated to a closed K-orbit is a fiber bundle over a generalized flag variety for a (smaller) general linear group having fiber that is a component of a Springer fiber associated to a closed orbit for a smaller pair (G', K'). To make this precise, let S' be the set of all labels of dots in the array that are contained in one of the m strings. Set $U = \text{span}_{\mathbf{C}}\{e_i : i \in S'\}$ and

$$G' = \{g \in G : g(U) \subset U \text{ and } g(e_j) = e_j, \text{ when } j \notin S'\}$$

Let $K' = K \cap G'$, $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$ and $\lambda' = \lambda|_{\mathfrak{h}'}$. If $\mathcal{Q}'$ is the corresponding closed K'-orbit in the flag variety for G', then Theorem 2.10 tells us that $X_{m-1} \simeq \gamma_{\mathcal{Q}'}^{-1}(f)$. (Note that the algorithm gives the same generic element f.) This proves the following corollary.

Corollary 2.22. *In the setting of the Theorem, there is a fibration*

$$X \to Q_m/R_m \tag{2.23}$$

having fiber X_{m-1}. Here X_{m-1} is a component of a Springer fiber for the smaller pair (G', K') associated to a closed K'-orbit in $\mathfrak{B}'$, and Q_m/R_m is a generalized flag variety for L_m.

We remark that often Q_m/R_m is just a point (so $X = X_{m-1}$). In this case one may take S' to be the labels of the dots in the first $m-1$ strings. Then there is a fibration

$$X \to Q_{m-1}/R_{m-1}$$

and the fiber is X_{m-2}, which is again a component of a Springer fiber for the smaller pair (G', K') associated to a closed K'-orbit in $\mathfrak{B}'$.

If Z is a topological space with finite-dimensional rational cohomology, we define the Poincaré polynomial of Z to be

$$P_t(Z) = \sum_i \dim H^i(Z, \mathbf{Q}) t^i.$$

As an application of Theorem 2.10, we obtain the Poincaré polynomial of $\gamma_{\mathcal{Q}}^{-1}(f)$.

Corollary 2.24. *The variety $\gamma_{\mathcal{Q}}^{-1}(f)$ is simply connected, and the cohomology ring $H^*(\gamma_{\mathcal{Q}}^{-1}(f); \mathbf{Z})$ is torsion-free and vanishes in odd dimensions. The Poincaré polynomial of $\gamma_{\mathcal{Q}}^{-1}(f)$ is*

$$P_t(\gamma_{\mathcal{Q}}^{-1}(f)) = \prod_{i=0}^{m} P_t(Q_i/R_i) = \prod_{i=0}^{m} P_t(L_i/(L_i \cap R_i)).$$

Proof. We use the notation of (2.2), so $X = X_m$. We prove by induction on k that the cohomology $H^*(X_k; \mathbf{Z})$ is torsion-free and vanishes in odd dimensions, and

$$P_t(X_k) = \prod_{i=0}^{k} P_t(Q_i/R_i).$$

For $k = 0$ the result holds since $X_0 = Q_0/R_0 = L_0/(L_0 \cap B)$ is a flag variety for the reductive group L_0. Suppose the proposition holds for X_{k-1}. As in Corollary 2.22, there is a fiber bundle $X_k \to Q_k/R_k$ with fiber X_{k-1}. Since Q_k/R_k and X_{k-1} are simply connected, the long exact sequence for homotopy implies that X_k is simply connected. Because $H^*(X_{k-1}; \mathbf{Z})$ is free and Q_k/R_k is simply connected, the Leray spectral sequence for the cohomology of this fiber bundle has E_2 term

$$E_2 = H^*(Q_k/R_k; \mathbf{Z}) \otimes H^*(X_{k-1}; \mathbf{Z}).$$

Since base and fiber have no odd-dimensional cohomology, the spectral sequence degenerates at E_2. This implies that as $\mathbf{Z}$-modules, $H^*(X_k; \mathbf{Z})$ and $H^*(Q_k/R_k; \mathbf{Z}) \otimes H^*(X_{k-1}; \mathbf{Z})$ are isomorphic, so

$$P_t(X_k) = P_t(X_{k-1})P_t(Q_k/R_k) = \prod_{i=0}^{k} P_t(Q_i/R_i),$$

proving the first equality of the proposition. The second equality holds because L_i is a Levi factor of Q_i. $\qquad\square$

Remark 2.25. As pointed out by the referee, the cohomology of every smooth component of a Springer fiber in type A is torsion-free and vanishes in odd dimensions. This is because for any nilpotent $f \in \mathfrak{g}$, there is a regular rank 1 torus $S \simeq \mathbf{C}^* \subset H$ such that $\mathrm{Ad}(S)f = \mathbf{C}^*f$. This implies that the Springer fiber $\mu^{-1}(f)$, and hence each irreducible component of this fiber, is S-stable. Since S acts with isolated fixed points on $\mathfrak{B}$, it acts with isolated fixed points on each component of $\mu^{-1}(f)$. Hence, if the component is smooth, the Bialynicki-Birula theorem gives a decomposition of the component into cells of even real dimension, and the assertion follows (cf. [7, §1.10]).

Remark 2.26. The polynomials $P_t(L_i/(L_i \cap R_i))$ can be easily computed from the array by the following procedure. If M is a reductive group, write p_M for the Poincaré polynomial of the flag variety for M. Since all odd-dimensional cohomology vanishes, it is convenient to write $u = t^2$. If $n \geq 2$, then

$$p_{GL(n)} = \frac{(1 - u^2)(1 - u^3) \cdots (1 - u^n)}{(1 - u)^{n-1}};$$

$p_{GL(1)} = 1$. If P is a parabolic subgroup of M, with Levi factor M', then $P_t(M/P) = \frac{p_M}{p_{M'}}$, as follows by considering the fibration $M/B_M \to M/P$ with fibers P/B_M (here $B_M \subset P$ is a Borel subgroup of M). By Lemma 1.9, $L_i \cap L_{i-1}$ is a Levi factor of $L_i \cap R_i$, so if $i > 0$,

$$P_t(L_i/(L_i \cap R_i)) = \frac{p_{L_i}}{p_{L_i \cap L_{i-1}}},$$

and $P_t(L_0/(L_0 \cap R_0) = p_{L_0}$. Recall that we denoted the blocks in the array for $\mathfrak{g}_i$ as $B_1^i, \ldots, B_{N_i}^i$, and let n_l^i equal the cardinality of B_l^i. By the discussion before Lemma 1.9,

$$p_{L_i} = \prod_l p_{GL(n_l^i)}.$$

We define subblocks of the blocks for $\mathfrak{g}_i$ (in case $i > 0$) as follows. Let a and b be in the same block for $\mathfrak{g}_i$. If they are also in the same block for $\mathfrak{g}_{i-1}$, we say they are in the same subblock. List the lengths of the subblocks as $m_1^i, m_2^i, \ldots$. By the proof of Lemma 1.9, the roots of $\mathfrak{l}_i \cap \mathfrak{l}_{i-1}$ are $\epsilon_a - \epsilon_b$, $a < b$, with a and b in the same subblock. Hence

$$p_{L_i \cap L_{i-1}} = \prod_l p_{GL(m_l^i)}.$$

From these facts one can readily calculate $P_t(\gamma_{\mathcal{Q}}^{-1}(f))$. For example, in the case of the array considered in Section 1.1, we find that $P_t(\gamma_{\mathcal{Q}}^{-1}(f)) = (1+u)^4 = (1+t^2)^4$.

Example 2.27. We illustrate these ideas with the following example. Consider the pair $(GL(14), GL(9) \times GL(4))$ and

$$\lambda = (13, 10, 9, 8, 7, 4, 3, 2, 1 \,|\, 12, 11, 6, 5).$$

The array with first string is

```
1            2  3  4  5           6   7   8   9
•            •  •  •  •           •   •   •   •
      •   •                 •  •
     10  11                12  13
```
.
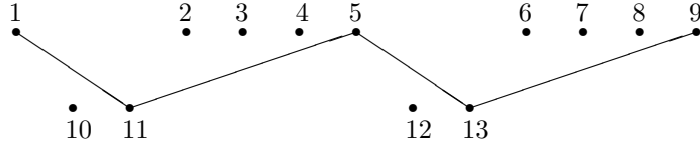
The arrays for (G_1, K_1) and (G_2, K_2) are

```
    2  3  4     6  7  8              2  3   6  7
    •  •  •     •  •  •              •  •   •  •
  •           •                              and
 10          12
```
.
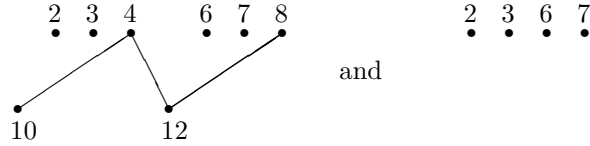
Then (except for a few factors of the torus, which play no role)

$$L = GL(1) \times GL(4) \times GL(4) \times GL(2) \times GL(2)$$
$$L_1 = GL(3) \times GL(3) \times GL(1), \quad L_1 \cap L = L_1$$
$$L_2 = GL(4), \quad L_2 \cap L_1 = GL(2) \times GL(2).$$

From this it is easy to see that the Poincaré polynomial is

$$(1+u)^3(1+u+u^2)^3(1+u+u^2+u^3)^3, \quad u = t^2.$$

The generalized flag variety Q_m/R_m ($m = 2$) is the Grassmannian $G_2(\mathbf{C}^4)$. Therefore, the fibration (2.23) is

$$X_1 \to G_2(\mathbf{C}^4),$$

and X_1 is the component of a Springer fiber for $(GL(9), GL(5) \times GL(4))$ with

$$\lambda' = (13, *, *, 8, 7, *, *, 2, 1 \,|\, 12, 11, 6, 5).$$

Here the coordinates in the $2, 3, 6$ and 7 places are omitted in passing to G'.

3. Representations on tangent spaces to iterated bundles at fixed points

The maximal torus H acts on $\gamma_{\mathcal{Q}}^{-1}(f)$ with finitely many fixed points, which we describe explicitly in the next section. If $x \in \gamma_{\mathcal{Q}}^{-1}(f)$ is an H-fixed point, then the tangent space $T_x\gamma_{\mathcal{Q}}^{-1}(f)$ is a representation of H. Using the structure of $\gamma_{\mathcal{Q}}^{-1}(f)$ as an

iterated bundle, we can describe the weights of this representation. Then we can apply localization theorems in equivariant cohomology and K-theory to the study of $\gamma_{\mathcal{Q}}^{-1}(f)$.

In this section we will describe the representations on tangent spaces to fixed points in case H is any algebraic group acting on any variety X constructed as an iterated bundle. Related calculations have been done, for example, in the case of Bott-Samelson-Demazure varieties (see e.g. [4]). However, for lack of a reference, we have decided to give the result for a general iterated bundle. In the next section, we will apply these results to the case where H is a maximal torus acting on $X = \gamma_{\mathcal{Q}}^{-1}(f)$.

3.1. **Generalities.** We begin with some notation. Given a left action of an algebraic group Q on a space X and an element $q \in Q$, let $L(q) : X \to X$ denote the map induced by the left action of q and let $C(q)$ denote the map $Q \to Q$ given by conjugation by q. If H is a subgroup of Q and $x \in X$ is H-fixed, then the map $H \to GL(T_x X)$ given by $h \mapsto L(h)_* = dL(h)$ defines a representation of H. Finally, if $\xi \in \mathfrak{q}$ and X is smooth, let $\xi^\#$ denote the induced vector field on X, whose value $\xi_x^\#$ at $x \in X$ is determined by the rule that if ϕ is a function on X, then

$$\xi_x^\# \, \phi = \frac{d}{dt}(\phi(\exp(t\xi)x)|_{t=0}.$$

Lemma 3.1. *Let $x \in X$ be fixed by $q \in Q$ and let $\xi \in \mathfrak{q}$. Then*

$$L(q)_*(\xi_x^\#) = (\mathrm{Ad}(q)\xi)_x^\#. \tag{3.2}$$

Proof. If we apply the left side of (3.2) to a function ϕ on X, we obtain

$$\frac{d}{dt}(\phi(q\exp(t\xi)x)|_{t=0}.$$

If we apply the right hand side of (3.2) to ϕ, we obtain

$$\frac{d}{dt}(\phi(\exp(t\mathrm{Ad}(q)\xi)x)|_{t=0} = \frac{d}{dt}(\phi(q\exp(t\xi)q^{-1}x)|_{t=0}.$$

Since x is fixed by q^{-1}, the two calculations agree. $\qquad\square$

3.2. **The case of a mixed space.** Throughout this subsection, Q will denote an algebraic group, and H and R will denote subgroups of Q. Suppose that M is a smooth algebraic variety with an R-action. We write $\widetilde{X} = Q \times M$ and we let X denote the "mixed space" $X = Q \underset{R}{\times} M$. Under mild hypotheses (cf. Remark 2.8) X is a smooth algebraic variety, and we assume this is the case. Let $\pi : \widetilde{X} \to X$ denote the quotient morphism. We write $[q, m]$ for $\pi(q, m)$ (for $q \in Q$, $m \in M$). The group $Q \times R$ acts on $\widetilde{X}$, so for $(q, r) \in Q \times R$ we have maps $L(q, r) := L(q) \times L(r)$ and $C(q) \times L(r)$ from $\widetilde{X}$ to $\widetilde{X}$. Also, Q acts on X, and we have $\pi \circ L(q, 1) = L(q) \circ \pi$. Note also that if $\xi \in \mathfrak{r}$ then there is an induced vector field $\xi^\#$ on M.

Observe that $L(q,1)_*$ maps $V = \mathfrak{q} \times T_m M$ isomorphically onto $T_{(q,m)}\widetilde{X}$. We define $\rho : V \to T_{[q,m]}X$ as the composition

$$\rho = \pi_* \circ L(q,1)_*. \tag{3.3}$$

Lemma 3.4. *The point* $[q,m] \in X$ *is* H-*fixed* $\Leftrightarrow q^{-1}Hq \subset R$, *and* m *is* $q^{-1}Hq$-*fixed.*

Proof. This is a straightforward computation. $\square$

Suppose now that $[q,m]$ is H-fixed. Let $V = \mathfrak{q} \oplus T_m M$, and define an H-module structure on V by the formula

$$h \cdot (\xi, v) = (\mathrm{Ad}(q^{-1}hq)\xi, L(q^{-1}hq)_* v),$$

for $\xi \in \mathfrak{q}$, $v \in T_m M$. We define an H-module structure on $\mathfrak{r}$ by the formula

$$h \cdot \xi = \mathrm{Ad}(q^{-1}hq)\xi.$$

Lemma 3.5. *The embedding* $\psi : \mathfrak{r} \to V$ *defined by* $\psi(\xi) = (\xi, -\xi_m^{\#})$ *is* H-*equivariant.*

Proof. Write $s = q^{-1}hq$. Using the preceding two lemmas, we have

$$\psi(h \cdot \xi) = (\mathrm{Ad}(s)\xi, -(\mathrm{Ad}(s)\xi)_m^{\#}) = (\mathrm{Ad}(s)\xi, -L(s)_*(\xi_m^{\#})) = h \cdot \psi(\xi),$$

as desired. $\square$

Let V_1 denote the H-submodule $\psi(\mathfrak{r})$ of V.

The main purpose of this subsection is to prove the following proposition.

Proposition 3.6. *With notation as above, assume that* $[q,m]$ *is an* H-*fixed point of* X. *The map* $\rho : V \to T_{[q,m]}X$ *is* H-*equivariant with kernel* V_1, *and hence induces an* H-*module isomorphism* $V/V_1 \to T_{[q,m]}X$.

Proof. To show that ρ is H-equivariant we must show that

$$L(h)_* \rho(\xi, v) = \rho(\mathrm{Ad}(s)\xi, L(s)_* v), \tag{3.7}$$

where $s = q^{-1}hq$. Now,

$$L(h)_* \circ \rho = L(h)_* \circ \pi_* \circ L(q,1)_* = (\pi \circ L(hq,1))_*.$$

Direct computation shows that

$$\pi \circ L(hq,1) = \pi \circ L(q,1) \circ (C(s) \times L(s)).$$

Therefore,

$$L(h)_* \circ \rho = \pi_* \circ L(q,1)_* \circ (C(s)_* \times L(s)_*) = \rho \circ (\mathrm{Ad}(s) \times L(s)_*),$$

which implies that ρ is H-equivariant.

Since V_1 and $\ker \rho$ have the same dimension, to show that they are equal it suffices to show that $V_1 \subset \ker \rho$. Observe that for all $r \in R$, the point $(qr, r^{-1}m)$ lies in the fiber $\pi^{-1}([q, m])$. This implies that the vector $v \in T_{(q,m)}\widetilde{X}$ defined by

$$v \cdot \phi = \frac{d}{dt}(\phi(q \exp(t\xi), \exp(-t\xi)m)|_{t=0}$$

satisfies $\pi_*(v) = 0$. But

$$v = L(q, 1)_*(\xi, -\xi_m^\#),$$

so we see that $\rho((\xi, -\xi_m^\#)) = 0$. Hence $V_1 \subset \ker \rho$, as desired. $\qquad\square$

Let H act on $\mathfrak{q}$ by the rule

$$h \cdot \xi = \mathrm{Ad}(q^{-1}hq)\xi.$$

If $[q, m]$ is H-fixed, then $q^{-1}Hq \subset R$. Thus, $\mathfrak{r}$ is an H-submodule of $\mathfrak{q}$, and hence $\mathfrak{q}/\mathfrak{r}$ is an H-module. Moreover, m is $q^{-1}Hq$-fixed, so the map $H \to GL(T_mM)$ given by $h \mapsto L(q^{-1}hq)_*$ defines an H-module structure on T_mM. By combining these H-module structures, we obtain an H-module structure on $\mathfrak{q}/\mathfrak{r} \oplus T_mM$. On the other hand, since $[q, m]$ is H-fixed, there is an H-module structure on $T_{[q,m]}X$.

Corollary 3.8. *Assume that the hypotheses of the preceding proposition hold, and assume in addition that H is reductive. Then $T_{[q,m]}X$ and $\mathfrak{q}/\mathfrak{r} \oplus T_mM$ are isomorphic H-modules.*

Proof. Let W be an irreducible H-module. If U is any H-module let $\mathrm{mult}_W(U)$ denote the multiplicity of W in U. By the preceding proposition, we have

$$\mathrm{mult}_W(T_{[q,m]}X) = \mathrm{mult}_W(V) - \mathrm{mult}_W(V_1).$$

Since $V = \mathfrak{q} \oplus T_mM$, and since by definition, $V_1 \cong \mathfrak{r}$ as H-modules, we have

$$\mathrm{mult}_W(T_{[q,m]}X) = \mathrm{mult}_W(\mathfrak{q}) + \mathrm{mult}_W(T_mM) - \mathrm{mult}_W(\mathfrak{r})$$
$$= \mathrm{mult}_W(\mathfrak{q}/\mathfrak{r}) + \mathrm{mult}_W(T_mM).$$

Since any irreducible H-module occurs with the same multiplicity in $T_{[q,m]}X$ and in $\mathfrak{q}/\mathfrak{r} \oplus T_mM$, we conclude that these H-modules are isomorphic. $\qquad\square$

3.3. **Iterated bundles.** We now apply the results of the preceding subsection to iterated bundles. Let $Q_0, \ldots, Q_n$ be subgroups of an algebraic group G and suppose that $H \subset Q_n$. Suppose that $R_0, \ldots, R_n$ are subgroups of G with $R_i \subset Q_{i-1} \cap Q_i$ for $i > 0$, and $R_0 \subset Q_0$. Let $\widetilde{X} = Q_n \times Q_{n-1} \times \cdots \times Q_0$, and let $X = Q_n \underset{R_n}{\times} Q_{n-1} \underset{R_{n-1}}{\times} \cdots \times Q_1 \underset{R_2}{\times} Q_0 / R_0 \underset{R_1}{}$.

Let $\pi : \widetilde{X} \to X$ denote the quotient morphism. If $\widetilde{x} = (q_n, \ldots, q_0) \in \widetilde{X}$, write $[q_n, \ldots, q_0]$ for $\pi(\widetilde{x})$. The following lemma is a straightforward computation.

Lemma 3.9. *The point $[q_n, \ldots, q_0] \in X$ is H-fixed $\Leftrightarrow$ for $i = 0, \cdots, n$, we have $C(q_i^{-1}q_{i+1}^{-1} \cdots q_n^{-1})(H) \subset R_i$.*

Assume that $[q_n, \ldots, q_0]$ is H-fixed. Let $V = \mathfrak{q}_n \oplus \cdots \oplus \mathfrak{q}_0$. Define an H-module structure on V by making $h \in H$ act on the $\mathfrak{q}_i$ summand by $\operatorname{Ad}(C(q_i^{-1} q_{i+1}^{-1} \cdots q_n^{-1})(h))$. Then $\mathfrak{r}_n \oplus \cdots \oplus \mathfrak{r}_0$ is an H-submodule of V. This gives an H-module structure on $V/(\mathfrak{r}_n \oplus \cdots \oplus \mathfrak{r}_0) \cong \mathfrak{q}_n/\mathfrak{r}_n \oplus \cdots \mathfrak{q}_0/\mathfrak{r}_0$.

As an immediate consequence of the definitions of the H-module structures, we have the following lemma.

Lemma 3.10. *Assume that $[q_n, \ldots, q_0] \in X$ is H-fixed. The embedding $\psi : \mathfrak{r}_n \oplus \cdots \oplus \mathfrak{r}_0 \to V$ which takes $(\xi_n, \ldots, \xi_0)$ to*

$$(\xi_n, -Ad(q_{n-1}^{-1})\xi_n + \xi_{n-1}, -Ad(q_{n-2}^{-1})\xi_{n-1} + \xi_{n-2}, \ldots, -Ad(q_0^{-1})\xi_1 + \xi_0)$$

is H-equivariant.

Let V_1 denote $\psi(\mathfrak{r}_n \oplus \cdots \oplus \mathfrak{r}_0)$; if $[q_n, \ldots q_0]$ is H-fixed, then V_1 is an H-submodule of V. Denote by $L(q_n, \ldots, q_0)$ the map of $\widetilde{X}$ to itself which sends $(a_n, \ldots, a_0)$ to $(q_n a_n, \ldots, q_0 a_0)$. Let

$$\rho = \pi_* \circ L(q_n, \ldots, q_0)_* : V \to T_{[q_n, \ldots q_0]} X. \tag{3.11}$$

Lemma 3.12. *Let $x = [q_n, \ldots, q_0] \in X$. Let $\xi \in \mathfrak{q}$, and let $\xi^\#$ denote the induced vector field on X. Then $\xi_x^\# = \rho(\operatorname{Ad}(q_n^{-1})\xi, 0, \ldots, 0)$.*

Proof. Let ϕ be a function on X. Then

$$\xi_x^\# \phi = \frac{d}{dt}\phi([\exp(t\xi)q_n, \ldots, q_0])|_{t=0}$$

$$= \frac{d}{dt}\phi([q_n \exp(t\operatorname{Ad}(q_n^{-1})\xi), \ldots, q_0])|_{t=0}.$$

Tracing through the definitions shows that this equals

$$\rho(\operatorname{Ad}(q_n^{-1})\xi, 0, \ldots, 0)\phi.$$

$\square$

The main purpose of this subsection is to prove the following proposition.

Proposition 3.13. *With notation as above, assume that $x = [q_n, \ldots, q_0]$ is an H-fixed point of X. The map $\rho : V \to T_{[q_n, \ldots, q_0]} X$ is H-equivariant with kernel V_1, and hence induces an H-module isomorphism $V/V_1 \to T_{[q_n, \ldots, q_0]} X$.*

Proof. The proof is by induction on n. The case $n = 0$ is handled by Proposition 3.6, taking M to be a point. Suppose that the proposition is true for $n - 1$. Let $M = Q_{n-1} \underset{R_{n-1}}{\times} \cdots \underset{R_2}{\times} Q_1 \underset{R_1}{\times} Q_0/R_0$. and let $m = [q_{n-1}, \ldots, q_0] \in M$. Then we can identify X with $Q_n \underset{R_n}{\times} M$ and the point x with $[q_n, m]$. Let

$$\rho_n : \mathfrak{q}_n \oplus T_m M \to T_x X$$

be the map defined in (3.3) (with Q_n in place of Q, and R_n in place of R), and let

$$\rho_{n-1} : \mathfrak{q}_{n-1} \oplus \cdots \mathfrak{q}_0 \to T_m M$$

be the analog of the map ρ (with M in place of X and m in place of x). We have

$$\mathfrak{q}_n \oplus \mathfrak{q}_{n-1} \oplus \cdots \mathfrak{q}_0 \xrightarrow{1 \times \rho_{n-1}} \mathfrak{q}_n \oplus T_m M \xrightarrow{\rho_n} T_x X,$$

and

$$\rho = \rho_n \circ (1 \times \rho_{n-1}).$$

By hypothesis, $x = [q_n, m]$ is H-fixed. By Lemma 3.4, this implies that m is $H' = C(q_n^{-1})(H)$-fixed. By Proposition 3.6, the map ρ_n is H-equivariant, and our inductive hypothesis implies that ρ_{n-1} is H'-equivariant. Combining these, we can show that ρ is H-equivariant as follows. To simplify the notation, write $r_i = C(q_i^{-1} q_{i+1}^{-1} \cdots q_n^{-1})(h) \in R_i$. Then

$$L(h)_* \rho(\xi_n, \ldots, \xi_0) = L(h)_* \rho_n(\xi_n, \rho_{n-1}(\xi_{n-1}, \ldots, \xi_0))$$
$$= \rho_n(\mathrm{Ad}(r_n)\xi_n, L(r_n)_* \rho_{n-1}(\xi_{n-1}, \ldots, \xi_0))$$

by H-equivariance of ρ_n. Using H'-equivariance of ρ_{n-1}, this equals

$$\rho_n(\mathrm{Ad}(r_n)\xi_n, \rho_{n-1}(\mathrm{Ad}(r_{n-1})\xi_{n-1}, \ldots, \mathrm{Ad}(r_0)\xi_0))$$

which (as $\rho = \rho_n \circ (1 \times \rho_{n-1})$) is

$$\rho(\mathrm{Ad}(r_n)\xi_n, \mathrm{Ad}(r_{n-1})\xi_{n-1}, \ldots, \mathrm{Ad}(r_0)\xi_0) = \rho(h \cdot (\xi_n, \ldots, \xi_0)).$$

Hence ρ is H-equivariant, as desired.

It remains to check that $\ker\rho = V_1$. We will make use of the following observation: if $f : V \to W$ is a linear map of vector spaces, and if $A \subset V$, $B \subset W$ are subspaces such that $f(A) = B$, then $f^{-1}(B) = A + \ker f$. As $\rho = \rho_n \circ (1 \times \rho_{n-1})$, we see that

$$\ker\rho = (1 \times \rho_{n-1})^{-1}(\ker\rho_n).$$

By Proposition 3.6,

$$\ker\rho_n = \{(\xi, -\xi_m^{\#}) \mid \xi \in \mathfrak{r}_n\}.$$

By Lemma 3.12,

$$\rho_{n-1}(\mathrm{Ad}(q_{n-1}^{-1})\xi, 0, \ldots, 0) = \xi_m^{\#}.$$

Therefore, if we let

$$A = \{(\xi, -\mathrm{Ad}(q_{n-1}^{-1})\xi, 0, \ldots, 0) \mid \xi \in \mathfrak{r}_n\} \subset \mathfrak{q}_n \oplus \cdots \oplus \mathfrak{q}_0,$$

we see that $(1 \times \rho_{n-1})(A) = \ker\rho_n$. Therefore, by the observation above,

$$(1 \times \rho_{n-1})^{-1}(\ker\rho_n) = A + \ker(1 \times \rho_{n-1}) = A + (0 \oplus \ker\rho_{n-1}).$$

By our inductive hypothesis,

$$\ker\rho_{n-1} = \{(\xi_{n-1}, -Ad(q_{n-2}^{-1})\xi_{n-1} + \xi_{n-2}, \ldots, -Ad(q_0^{-1})\xi_1 + \xi_0) \mid \xi_i \in \mathfrak{r}_i\}.$$

Hence, from the definition of V_1, we see that

$$A + (0 \oplus \ker \rho_{n-1}) = V_1,$$

so $\ker \rho = V_1$, as desired.   □

Recall that $\mathfrak{q}_n/\mathfrak{r}_n \oplus \cdots \mathfrak{q}_0/\mathfrak{r}_0$ has an H-module structure defined after Lemma 3.9.

Corollary 3.14. *Assume that the hypotheses of the preceding proposition hold, and in addition assume that H is reductive. Then $T_{[q_n,\cdots,q_0]}X$ and $\mathfrak{q}_n/\mathfrak{r}_n \oplus \cdots \mathfrak{q}_0/\mathfrak{r}_0$ are isomorphic H-modules.*

Proof. This is proved by an argument similar to the proof of Corollary 3.8.   □

It is convenient to give an alternative formulation of the corollary in case H is a torus and each $q_i^{-1} \ldots q_n^{-1}$ is in the normalizer $N_G(H)$ of H. If V is a representation of the torus H, let $\Phi(V)$ denote the multiset of weights of H acting on V (this is the set of weights, where each weight is counted with multiplicity). We adopt the convention that if s occurs a times in the multiset A, and b times in the multiset B, then s occurs $a + b$ times in the multiset $A \cup B$.

The group $N_G(H)$ acts on weights by the rule that if λ is a weight of H (so $e^\lambda : H \to \mathbf{C}^*$ is a homomorphism), and $w \in N_G(H)$, then $w\lambda$ is the weight satisfying

$$e^{w\lambda}(h) = e^\lambda(w^{-1}hw).$$

With these definitions, we can reformulate the preceding corollary as follows.

Corollary 3.15. *Assume the hypotheses of Proposition 3.13 hold. Assume in addition that H is a torus and that each $q_i^{-1} \ldots q_n^{-1}$ is in the normalizer of H. Then*

$$\Phi(T_{[q_n,\cdots,q_0]}X) = q_n \cdot \Phi(\mathfrak{q}_n/\mathfrak{r}_n) \cup q_n q_{n-1} \cdot \Phi(\mathfrak{q}_{n-1}/\mathfrak{r}_{n-1}) \cup \cdots \cup q_n q_{n-1} \cdots q_0 \cdot \Phi(\mathfrak{q}_0/\mathfrak{r}_0).$$

Proof. This follows from Corollary 3.14 and the definition of the H-action on $\mathfrak{q}_n/\mathfrak{r}_n \oplus \cdots \oplus \mathfrak{q}_0/\mathfrak{r}_0$ given after Lemma 3.9.   □

Remark 3.16. The results of this section can be generalized to remove the assumption that the Q_i are subgroups of a group G. Instead of $R_i \subset Q_{i-1} \cap Q_i$, what we need are inclusions $R_i \subset Q_i$ and homomorphisms $R_i \to Q_{i-1}$.

3.4. **Induced vector bundles.** Keep the notation of the previous subsection. If V is a representation of R_0, then

$$\mathbf{V} = Q_n \underset{R_n}{\times} Q_{n-1} \underset{R_{n-1}}{\times} \cdots \underset{R_1}{\times} Q_0 \underset{R_0}{\times} V \to X$$

is a Q_n-equivariant vector bundle. If $x = [q_n, \ldots, q_0]$ is fixed by H, then the fiber $\mathbf{V}_x$ is a representation of H, which is described as follows.

Proposition 3.17. *With notation as above, if $x = [q_n, \ldots, q_0]$ is fixed by H, then as an H-module, $\mathbf{V}_x$ is isomorphic to V, with H-action given by*

$$h \cdot v = C(q_0^{-1} \cdots q_n^{-1})(h)v,$$

for $h \in H$, $v \in V$. (Note that the right side of this equation makes sense, since $C(q_0^{-1} \cdots q_n^{-1})(h) \in R_0$ and V is a representation of R_0.)

Proof. We have an isomorphism $f : V \to \mathbf{V}_x$ given by $v \mapsto [q_n, \ldots, q_0, v]$. An element $h \in H$ acts on $\mathbf{V}_x$ by

$$\begin{aligned}
h \cdot f(v) &= [hq_n, q_{n-1}, \ldots, q_0, v] \\
&= [q_n C(q_n^{-1})(h), q_{n-1}, \ldots, q_0, v] \\
&= [q_n, C(q_n^{-1})(h)q_{n-1}, \ldots, q_0, v] \\
&\quad \vdots \\
&= [q_n, \ldots, q_0, C(q_0^{-1} \cdots q_n^{-1})(h)v] \\
&= f(C(q_0^{-1} \cdots q_n^{-1})(h)v).
\end{aligned}$$

The proposition follows. $\square$

Corollary 3.18. *Keep the assumptions of the previous proposition. Assume in addition that H is a torus and that each $q_i^{-1} \ldots q_n^{-1}$ is in the normalizer of H, so in particular $H \subset R_0$. Suppose that V is a 1-dimensional representation of R_0 and that the weight of H on V (induced by the inclusion $H \subset R_0$) is λ. Then the weight of H acting on $\mathbf{V}_x$ is $q_n \cdots q_0 \cdot \lambda$.*

Proof. This follows immediately from the proposition and the definition of the action of the normalizer of H on weights of H given in the preceding subsection. $\square$

4. Localization and $\gamma_{\mathcal{Q}}^{-1}(f)$

We return to the situation of Section 2. Using the computations of Section 3, we apply localization theorems in equivariant cohomology and K-theory to $\gamma_{\mathcal{Q}}^{-1}(f)$. The first application gives a character formula for cohomology of $\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)$. This is related to associated cycles of discrete series representations. The second application is to express the homology and K-theory classes determined by $\gamma_{\mathcal{Q}}^{-1}(f) \subset \mathfrak{B}$ in terms of Schubert bases. This answers (for the components $\gamma_{\mathcal{Q}}^{-1}(f)$) a question of Springer.

The first step is to determine the fixed points of the action of H. Recall that the fixed point set of H on $\mathfrak{B}$ is $W \cdot \mathfrak{b}$, where W is the Weyl group of G. Since $\gamma_{\mathcal{Q}}^{-1}(f)$ is a subset of $\mathfrak{B}$, the fixed points on $\gamma_{\mathcal{Q}}^{-1}(f)$ are a subset of $W \cdot \mathfrak{b}$. In fact, $\gamma_{\mathcal{Q}}^{-1}(f)$ is contained in the flag variety for K, so the fixed points are a subset of $W(K) \cdot \mathfrak{b}$.

Proposition 4.1. *The fixed point set of the action of H on $\gamma_{\mathcal{Q}}^{-1}(f)$ is*

$$\{w_m \ldots w_1 w_0 \cdot \mathfrak{b} : w_i \in W(L_i)\}. \tag{4.2}$$

Proof. We prove the following statement by induction on k: the fixed point set of H on $Q_k \cdots Q_1 Q_0 \cdot \mathfrak{b}$ is

$$\{w_k \ldots w_1 w_0 \cdot \mathfrak{b} : w_i \in W(L_i)\}, \tag{4.3}$$

for $k = 0, 1, \ldots, m$. For $k = 0$ this is clear since $Q_0 \cdot \mathfrak{b} = L_0 \cdot \mathfrak{b}$ is the flag variety for L_0. Assume that (4.3) holds for $k-1$, and let $w \cdot \mathfrak{b} = q_k \cdots q_1 q_0 \cdot \mathfrak{b}$ be a fixed point in $Q_k \cdots Q_1 Q_0 \cdot \mathfrak{b}$. Using the fact (proved in Section 2) that $F_k : X_k \to Q_k \cdots Q_1 Q_0 \cdot \mathfrak{b}$ is an H-equivariant bijection, we have

$$h[q_k, \ldots, q_1, q_0] = [q_k, \ldots, q_1, q_0] \in X_k, \text{ for all } h \in H.$$

Therefore, $h q_k \equiv q_k$ modulo R_k, for all $h \in H$. In other words, $q_k \cdot \mathfrak{r}_k$ is a fixed point of the action of H on the (generalized) flag variety $Q_k/R_k = L_k/R_k \cap L_k$. We conclude that $q_k = w_k r_k$, for some representative w_k of $W(L_k)$ and $r_k \in R_k$. Therefore,

$$w_k^{-1} w \cdot \mathfrak{b} = r_k q_{k-1} \ldots q_1 q_0 \cdot \mathfrak{b}$$

is H-fixed. By the inductive hypothesis the righthand side is $w_{k-1} \ldots w_1 w_0 \cdot \mathfrak{b}$, $w_i \in W(L_i)$. The proposition now follows. $\qquad\square$

Recall that if V is an H-module then $\Phi(V)$ denotes the set of weights of V, counted with multiplicity.

Corollary 4.4. *The weights of H on the tangent space to the H-fixed point $w_m \ldots w_1 w_0 \cdot \mathfrak{b}$ of $\gamma_{\mathcal{Q}}^{-1}(f)$ (where $w_i \in W(L_i)$) are*

$$w_n \cdot \Phi(\mathfrak{q}_n/\mathfrak{r}_n) \cup w_n w_{n-1} \cdot \Phi(\mathfrak{q}_{n-1}/\mathfrak{r}_{n-1}) \cup \cdots \cup w_n w_{n-1} \cdots w_0 \cdot \Phi(\mathfrak{q}_0/\mathfrak{r}_0).$$

All the weights have multiplicity 1.

Proof. The description of the weights is an immediate consequence of Corollary 3.15. Each weight has multiplicity 1 because if $w \in W$, then $T_{w\mathfrak{b}}\gamma_{\mathcal{Q}}^{-1}(f) \subset T_{w\mathfrak{b}}\mathfrak{B}$, and H acts with multiplicity 1 on $T_{w\mathfrak{b}}\mathfrak{B}$. $\qquad\square$

Recall that the representation ring $R(H)$ is the free abelian group spanned by e^μ, as μ runs over all weights of H. The multiplication in $R(H)$ is defined by the rule $e^\lambda e^\mu = e^{\lambda+\mu}$. Let $\mu_1, \ldots, \mu_n$ be weights of H. Let $\mathbf{C}_{\mu_i}$ denote the 1-dimensional representation of H corresponding to μ_i, and let $V = \oplus_i \mathbf{C}_{\mu_i}$. Corresponding to V there is the element $\sum_i e^{\mu_i}$ of $R(H)$; by abuse of notation we will also write V for this element. Define $\lambda_{-1}(V) = \prod_i (1 - e^{\mu_i}) \in R(H)$. We will make use of the following general fact, which is a consequence of the localization theorem in equivariant K-theory (see [6, Remark 5.11.8]).

Proposition 4.5. *Let H be a torus acting on a smooth complete algebraic variety M, and assume that the set M^H of H-fixed points on M is finite. Let L be an H-equivariant line bundle on M, and let $\mathcal{L}$ denote the corresponding invertible sheaf on M. For $m \in M^H$, let $\mu(m)$ denote the weight of H on L_m. Then in $R(H)$, we have*

$$\sum_i (-1)^i H^i(M, \mathcal{L}) = \sum_{m \in M^H} \frac{e^{\mu(m)}}{\lambda_{-1}(T_m^* M)}.$$

In this proposition, the individual terms in the sum on the right hand side are in the quotient field of $R(H)$; they need not be in $R(H)$, but their sum is. Combining this proposition with our description of the weights of H on tangent spaces (Corollary 4.4), and Corollary 3.18, we obtain the following character formula (with notation as in Section 2).

Theorem 4.6. *Let $\tau \in \mathfrak{h}^*$ be an integral weight. Let A_i denote the set of weights of H on $\mathfrak{q}_i/\mathfrak{r}_i$. In $R(H)$, we have*

$$\sum_i (-1)^i H^i(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)) = \sum \frac{e^{w_m w_{m-1} \cdots w_0 \tau}}{\prod_i \prod_{\mu \in A_i}(1 - e^{-w_m w_{m-1} \cdots w_i \mu})}.$$

Here the sum is taken over all products $w_m w_{m-1} \cdots w_0$, where $w_i \in W(L_i)$.

Note that in this theorem the sum is not over $m+1$-tuples $(w_m, \cdots, w_0)$ where $w_i \in W(L_i)$, but rather over the distinct products $w_m w_{m-1} \cdots w_0$. Also, if τ is dominant with respect to the positive system Δ_c^+, then by the cohomology vanishing theorem (Theorem 2.18), Theorem 4.6 gives a formula for the character of the H-representation on $H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau))$.

Now we turn to the question of expressing homology classes of $\gamma_{\mathcal{Q}}^{-1}(f)$ in terms of Schubert bases. Because $\gamma_{\mathcal{Q}}^{-1}(f)$ is an H-invariant subvariety of $\mathfrak{B}$, it defines classes in the homology (or cohomology) and K-theory (ordinary or H-equivariant) of $\mathfrak{B}$. The homology and K-theory of $\mathfrak{B}$ have Schubert bases, that is, bases defined in terms of Schubert varieties. Using localization theorems in equivariant cohomology and K-theory, known results about Schubert classes, and Corollary 4.4, we can express the equivariant classes determined by $\gamma_{\mathcal{Q}}^{-1}(f)$ in terms of the Schubert bases. The expressions in ordinary homology or K-theory are obtained by specializing the corresponding H-equivariant expressions.

We begin by recalling some known facts about equivariant (co)homology and Schubert classes. The facts we need can be found in [15] or [22]. Given any space Z with H-action, one can define the equivariant cohomology groups $H_H^i(Z)$ and the equivariant Borel-Moore homology groups $H_i^H(Z)$. Then $H_*^H(Z) = \oplus_i H_i^H(Z)$ is a module for the ring $H_H^*(Z) = \oplus_i H_H^i(Z)$. We have $H_H^*(\text{point}) = S(\hat{H})$, the symmetric algebra on the group of characters of H. Note that $S(\hat{H})$ is the polynomial ring $\mathbf{Z}[\lambda_1, \ldots, \lambda_n]$, where $\lambda_1, \ldots, \lambda_n$ is a basis for the free abelian group $\hat{H}$. By pulling back from the map from

Z to a point, there is a map $S(\hat{H}) \to H^*_H(Z)$, and thus $H^H_*(Z)$ is an $S(\hat{H})$-module. If Z is a complete variety, there is a pairing

$$(\, , \,) : H^*_H(Z) \otimes_{S(\hat{H})} H^H_*(Z) \to S(\hat{H}).$$

Now consider $Z = \mathfrak{B}$. By definition, the Schubert class X_w is the closure of $B \cdot w\mathfrak{b}$ in $\mathfrak{B}$. Each X_w defines a class $[X_w]_H$ in the equivariant Borel-Moore homology $H^H_{\ell(w)}(\mathfrak{B})$. The space $H^H_*(\mathfrak{B})$ is a free $S(\hat{H})$-module with basis given by $[X_w]_H$. Moreover, there is a basis $\{x_w\}$ of $H^*_H(\mathfrak{B})$ with the property that

$$(x_u, [X_v]_H) = \delta_{u,v},$$

for all $u, v \in W$.

Given a class $\eta \in H^*_H(\mathfrak{B})$, and $u \in W$, write $\eta(u)$ for the pullback of η to the equivariant cohomology group $H^*_H(u\mathfrak{b})$. Because $u\mathfrak{b}$ is a point, this group is identified with $S(\hat{H})$. The statement of Theorem 4.8 will involve the polynomials $x_w(u)$. These polynomials are known. Indeed, [22, Lemma 11.1.9] contains an explicit formula for elements $R(w, u) \in S(\hat{H})$. In Sections 11.1-11.3 of that book, especially Prop. 11.3.10, the connection of these elements with equivariant cohomology is explained; the result, in our notation, is that,

$$x_w(u) = (-1)^{\ell(w)} R(w, u)$$

(the sign $(-1)^{\ell(w)}$ is necessary because we have taken the roots in $\mathfrak{b}$ to be negative, the opposite of Kumar's convention).

Because $\gamma_{\mathcal{Q}}^{-1}(f)$ is H-invariant, it defines a class $[\gamma_{\mathcal{Q}}^{-1}(f)]_H \in H^H_{2d}(\mathfrak{B})$, where d is the complex dimension of $\gamma_{\mathcal{Q}}^{-1}(f)$. Thus, we can write

$$[\gamma_{\mathcal{Q}}^{-1}(f)]_H = \sum_w A_w [X_w]_H.$$

Observe that each A_w is a polynomial in $\lambda_1, \ldots, \lambda_n$. Let a_w be obtained from A_w by setting all the λ_i equal to 0. Then in the ordinary homology $H_*(X)$ we have the equation

$$[\gamma_{\mathcal{Q}}^{-1}(f)] = \sum_w a_w [X_w];$$

Springer's original question was essentially to calculate the coefficients a_w. The A_w can be calculated by pairing with the dual basis:

$$A_w = (x_w, [\gamma_{\mathcal{Q}}^{-1}(f)]_H).$$

See [15] or [22] for proofs and references for the preceding facts.

If H acts on a smooth variety M and u is an H-fixed point, let $P_u(M) \in S(\hat{H})$ denote the product of the weights (with multiplicity) of H on $T_u M$. (As in Corollary 4.4, if M is an H-invariant subvariety of $\mathfrak{B}$, then all weights of $T_u M$ must occur with multiplicity 1.) The following proposition gives a formula for the pairing.

Proposition 4.7. *Let M be a smooth closed H-invariant subvariety of $\mathfrak{B}$, and let $\eta \in H_H^*(\mathfrak{B})$. Then*

$$(\eta, [M]_H) = \sum_{u \in W} \frac{\eta(u)}{P_u(M)},$$

where we have written $P_u(M)$ for $P_{u\mathfrak{b}}(M)$. (The individual terms of the sum on the right hand side are in the quotient field $Q(\hat{H})$ of $S(\hat{H})$, but the sum is in $S(\hat{H})$).

Proof. This is an immediate consequence of the "integration formula" of [8, Cor. 1] (the formula given there is an algebraic version of [1, Equation (3.8)]). In that paper, the calculation is done in equivariant Chow groups, but the same formula holds in equivariant Borel-Moore homology (in fact, for $\mathfrak{B}$ the two theories coincide because $\mathfrak{B}$ is paved by affines). $\qquad\square$

As an immediate consequence, we obtain the following theorem.

Theorem 4.8. *The class $[\gamma_Q^{-1}(f)]_H$ in $H_*^H(\mathfrak{B})$ is given by $\sum_w A_w[X_w]_H$, where*

$$A_w = \sum_u \frac{x_w(u)}{P_u(\gamma_Q^{-1}(f))}.$$

Here the sum is over all $u \in W$ which can be written as $u = w_m w_{m-1} \cdots w_0$, where $w_i \in W(L_i)$. Also, if A_i denotes the set of weights of H on $\mathfrak{q}_i/\mathfrak{r}_i$, then fixing an expression of u as a product $w_m w_{m-1} \cdots w_0$, we have

$$P_u(\gamma_Q^{-1}(f)) = \prod_i \prod_{\mu \in A_i} w_m w_{m-1} \cdots w_i \mu.$$

We can perform the analogous calculations in equivariant K-theory using almost identical arguments. We begin by recalling some facts about the equivariant K-theory of the flag variety; see [16] for more details and references. If Z is a scheme with an H-action, $K_H(Z)$ denotes the Grothendieck group of H-equivariant coherent sheaves on Z. This is a module for the representation ring $R(H)$, which we recall is the free abelian group spanned by e^μ, for $\mu \in \hat{H}$. If Z is smooth, then every equivariant coherent sheaf on Z admits a finite equivariant resolution by locally free sheaves, so $K_H(Z)$ can be identified with the Grothendieck group of H-equivariant vector bundles on Z. In this case, $K_H(Z)$ has a ring structure induced from the tensor product of vector bundles. If Z is complete, there is a pairing

$$(\, , \,) : K_H(Z) \otimes_{R(H)} K_H(Z) \to R(Z)$$

defined by

$$(v_1, v_2) = \sum_i (-1)^i H^i(Z, v_1 \cdot v_2).$$

The group $K_H(\mathfrak{B})$ is a free $R(H)$-module with basis $\{[\mathcal{O}_{X_w}]\}_{w \in W}$, that is, a basis given by the classes of structure sheaves of Schubert varieties. Just as in cohomology,

there is a dual basis $\{\xi^w\}_{w \in W}$, characterized by the property that

$$(\xi^u, [\mathcal{O}_{X_w}]) = \delta_{u,w}.$$

Write $\xi^u(w)$ for the pullback of ξ^u to the equivariant K-theory group $K_H(w\mathfrak{b}) \cong R(H)$. Observe that the $\xi^u(w)$ are known; [14] and [33] give explicit formulas for these classes (Willems uses a slightly different basis, but see [16] for a description of the relations between various bases).

If $w\mathfrak{b}$ is an H-fixed point of $\gamma_{\mathcal{Q}}^{-1}(f)$, we can write $w = w_m w_{m-1} \cdots w_0$, where $w_i \in W(L_i)$. Define

$$Q_w(\gamma_{\mathcal{Q}}^{-1}(f)) = \lambda_{-1}(T^*_{w\mathfrak{b}} \gamma_{\mathcal{Q}}^{-1}(f)) = \prod_i \prod_{\mu \in A_i} (1 - e^{-w_m w_{m-1} \cdots w_i \mu}),$$

where A_i is as in Theorem 4.8.

The expansion of the class $[\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}]$ in terms of the classes $[\mathcal{O}_{X_w}]$ is given by the following theorem.

Theorem 4.9. *The class $[\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}]$ in $K_H(\mathfrak{B})$ is given by $\sum_w B_w[\mathcal{O}_{X_w}]$, where*

$$B_w = \sum_u \frac{\xi^w(u)}{Q_u(\gamma_{\mathcal{Q}}^{-1}(f))}.$$

The sum is over all $u \in W$ that can be written as $u = w_m w_{m-1} \cdots w_0$, where $w_i \in W(L_i)$. The corresponding expansion in the ordinary K-theory $K(\mathfrak{B})$ is given by setting all the e^λ in each B_u equal to 1.

Proof. The proof of this theorem is almost the same as the proof of Theorem 4.8, with K-theory in place of cohomology. We omit the details. □

Remark 4.10. There is a similar expansion for $[\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}]$ in terms of ξ^w, obtained by replacing $\xi^w(u)$ by $[\mathcal{O}_{X_w}](u)$ in the above formula. An explicit expression for the elements $[\mathcal{O}_{X_w}](u)$ is given in [14] (see [21] for another proof).

APPENDIX A.
THE STANDARD TABLEAUX FOR COMPONENTS ASSOCIATED TO CLOSED ORBITS

A parametrization of the components of a Springer fiber for $GL(n)$ is given in [30]. This is done in a natural way by associating to each component of $\mu^{-1}(f)$ a standard tableau. Since this parametrization has become somewhat standard, in this appendix we identify the standard tableaux of the components studied in this article (and in [2]). Using this identification we determine which of the components we consider are Richardson (see Remark A.6).

We mention that the results of this appendix may be found in the literature by piecing together a number of results about irreducible Harish-Chandra modules and their annihilators. For $SU(p,q)$ the set of irreducible Harish Chandra-modules of a

given regular integral infinitesimal character is in one-to-one correspondence with K-orbits in $\mathfrak{B}$. On the other hand the primitive ideals in the enveloping algebra are parametrized in terms of standard tableau. In [13] an algorithm is given to associate to the K-orbit $\mathcal{Q}$ corresponding to a Harish-Chandra module X the standard tableau corresponding to $Ann(X)$. The algorithm in fact also constructs a signed tableau having the same shape as the standard tableau. As shown in [31], the signed tableau is the associated variety of X and the standard tableau is the component of the Springer fiber of a generic element for $T_{\mathcal{Q}}^{*}\mathfrak{B}$. Therefore, the statements of this appendix follow from [13] and [31]. We include the appendix as an alternative that is elementary and purely in terms of the geometric description of $\gamma_{\mathcal{Q}}^{-1}(f)$.

We begin by recalling the parametrization in [30]. Suppose that f is any nilpotent element of $\mathfrak{g} = \mathfrak{gl}(n)$. The Young diagram associated to f consists of rows of boxes; the lengths of these rows are the sizes of the Jordan blocks of f (which are equal to the lengths of the strings, in the language of Section 1). By a standard tableau we mean a numbering of a Young diagram (having n boxes) by the numbers in $\{1, 2, \ldots, n\}$ in such a way that the numbers increase from left to right along any row and down any column. The irreducible components of $\mu^{-1}(f)$ are parametrized by the standard tableaux having the same shape as the Young diagram of f. We will need to describe this parametrization carefully.

Suppose $\mathfrak{b} \in \mu^{-1}(f)$ (i.e., $\mathfrak{b}$ is a Borel subalgebra containing f). Let (E_i) be the flag $\{0\} \subset E_1 \subset \cdots \subset E_n = \mathbf{C}^n$ that corresponds to $\mathfrak{b}$ (that is, $\mathfrak{b}$ is the stabilizer of (E_i)). Then we may associate to $(\mathfrak{b}, f)$ a standard tableau $ST(\mathfrak{b}, f)$ as follows. The tableau of $f|_{E_i}$ is obtained from the tableau of $f|_{E_{i-1}}$ by attaching a new box onto the end of a row of the tableau of $f|_{E_{i-1}}$, or perhaps by starting a new row with one box, and putting the number i in the new box. The standard tableau of $f|_{E_i}$ is obtained from that of $f|_{E_{i-1}}$ by inserting the number i in the new block. The starting point for this procedure is a single block containing the number 1, corresponding to $f|_{E_1}$. The statement of [30] is that for each standard tableau ST having the same shape as the Young diagram of f there exists a unique irreducible component C of $\mu^{-1}(f)$ so that for $\mathfrak{b}$ in a dense open subset of C, $ST(\mathfrak{b}, f) = ST$. This gives a one-to-one correspondence between irreducible components and standard tableaux.

An example is to take f as in the example of Section 1. Let $\mathfrak{b}$ be as in that example. Then $\mathfrak{b}$ is the stabilizer of the flag (E_i) with E_i the span of the standard basis vectors e_j with j among the labels of the i dots farthest to the right in the array. Then the associated standard tableau is

1	3	5	6
2	4		
7			

For the generic elements f considered in this paper, there is a simple algorithm for computing $ST(\mathfrak{b}, f)$ in terms of the array. The algorithm is as follows. Relabel the dots in the array with $1, 2, \ldots, n$ in *decreasing* order from left to right. Each row in the tableau of f corresponds to a string or a dot not passed through by a string. Now fill in the boxes of each row with the new labels of the dots in the corresponding string. The rows with one box get the new labels of the dots in the array that do not lie in a string (in increasing order).

In the previous example, with all strings sketched in, we get
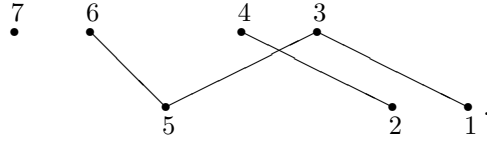
One now easily reads off the standard tableau obtained earlier.

The next proposition implies that in the preceding example, the component $\gamma_{\mathcal{Q}}^{-1}(f)$ corresponds to the standard tableau given above. To prove this one would need to find an open dense set of Borel subalgebras $\mathfrak{b}'$ in $\gamma_{\mathcal{Q}}^{-1}(f)$ for which $ST(\mathfrak{b}', f)$ is this same standard tableau. This is how the proof of the proposition proceeds.

Proposition A.1. *Given a closed K-orbit $\mathfrak{B}$ with base point $\mathfrak{b}$ as in Section 1.1, let $f \in \mathfrak{n}^- \cap \mathfrak{p}$ be the generic element constructed in Section 1. Then the standard tableau associated to $\gamma_{\mathcal{Q}}^{-1}(f)$ is $ST(\mathfrak{b}, f)$.*

To prove the proposition it suffices to prove Proposition A.5 below. We begin with two lemmas.

Lemma A.2. *Suppose $\mathfrak{b}, \mathfrak{b}' \in \mu^{-1}(f)$ and $\mathfrak{b}$ and $\mathfrak{b}'$ correspond to flags (E_i) and (E_i'). Then $ST(\mathfrak{b}, f) = ST(\mathfrak{b}', f)$ if and only if $\dim(\ker(f^k|_{E_i})) = \dim(\ker(f^k|_{E_i'}))$ for each $i = 1, 2, \ldots, n$ and each $k \in \mathbf{Z}_{>0}$.*

Proof. It follows from the above discussion that $ST(\mathfrak{b}, f)$ is determined by the Jordan forms of $f|_{E_i}, i = 1, 2, \ldots, n$. Therefore it is determined by $\dim(\ker(f^k|_{E_i}))$ for each $i = 1, 2, \ldots, n$ and each $k \in \mathbf{Z}_{>0}$. $\qquad\square$

Define $W_i = \mathrm{span}_{\mathbf{C}}\{e_j : j$ is a label of a dot in the ith string$\}$, for $i = 1, \ldots, m$. List the standard basis vectors not in any W_i as $e_{j_1}, \ldots, e_{j_{r-m}}$ (some r) with $j_1 \geq j_2 \geq \cdots$. Now set $W_{m+i} = \mathbf{C}e_{j_i}$. It follows that $\mathbf{C}^n = W_1 \oplus \cdots \oplus W_r$ and each W_i is f-stable.

Consider a closed K-orbit $\mathcal{Q} = K \cdot \mathfrak{b}$ and the generic f in $\mathfrak{n}^- \cap \mathfrak{p}$ as in Section 1.1. Let $\mathfrak{v}_1^-, \ldots, \mathfrak{v}_m^-$ be the nilradicals of the parabolic subalgebras $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$ of $\mathfrak{l}_i$, as discussed in Lemma 1.9. Let $\mathfrak{v}_0^- = \mathfrak{l}_0 \cap \mathfrak{n}^-$. Let $\mathfrak{v}_0, \mathfrak{v}_1, \ldots, \mathfrak{v}_m$ be the nilradicals of the opposite parabolic subalgebras. Then $\mathcal{V} := \exp(\mathfrak{v}_m) \cdots \exp(\mathfrak{v}_1) \exp(\mathfrak{v}_0) \cdot \mathfrak{b}$ is dense and open in $\gamma_{\mathcal{Q}}^{-1}(f)$.

Lemma A.3. *Suppose $X_l \in \mathfrak{v}_l$ and j is arbitrary. Then $\exp(X_l)e_j = e_j + y$, where y is a linear combination of standard basis vectors e_a with $a < j$ and a in the same $\mathfrak{g}_l$-block as j.*

Proof. Consider $X_l \in \mathfrak{v}_l$. Then X_l is a linear combination of root vectors for $\epsilon_a - \epsilon_b$ with a, b in the same $\mathfrak{g}_l$-block and $a < b$. Then $X_l(e_j)$ is 0 if j is not in a $\mathfrak{g}_l$-block and is a linear combination of e_a with a in the same $\mathfrak{g}_l$-block as j and $a < j$. The lemma follows. □

Lemma A.4. *For any $v \in \exp(\mathfrak{v}_m) \cdots \exp(\mathfrak{v}_1)\exp(\mathfrak{v}_0)$, $\ker(f^k) = \ker(f^k \circ v)$.*

Proof. Since v is invertible, $\ker(f^k)$ and $\ker(f^k \circ v)$ have the same dimension. So it is enough to show that $\ker(f^k) \subset \ker(f^k \circ v)$. The following observation will be used. If a, b are in the same $\mathfrak{g}_l$-block (for some l) and $a < b$, then $f^k(e_b) = 0$ implies $f^k(e_a) = 0$. To see this it suffices to assume that $l = 0$. Then a lies in a later string than the string of b (since strings pass through the rightmost dot in a block). The statement follows from the construction of f.

Combining this observation with the preceding lemma, along with the fact that the kernel of f^k is spanned by standard basis vectors, we see that for all l,

$$\exp(X_l)\ker(f^k) \subset \ker(f^k).$$

Induction easily gives

$$\exp(X_l) \cdots \exp(X_0)\ker(f^k) \subset \ker(f^k), \text{ all } l.$$

In particular,

$$v(\ker(f^k)) \subset \ker(f^k).$$

The inclusion follows. □

Proposition A.5. *The standard tableaux $ST(\mathfrak{b}', f)$ coincide for all Borel subalgebras $\mathfrak{b}'$ in the dense open set $\mathcal{V}$.*

Proof. Let $\mathfrak{b}' = v \cdot \mathfrak{b} \in \mathcal{V}$. Then the flag defining $\mathfrak{b}'$ is $(E_i') = (v(E_i))$. We have

$$\ker(f^k|_{E_i'}) = v(\ker((f^k \circ v)|_{E_i})) = v(\ker(f^k|_{E_i})),$$

where the second equality is by the preceding lemma. Since v is invertible, we see that $\dim(\ker(f^k|_{E_i})) = \dim(\ker(f^k|_{E_i'}))$. Now $ST(\mathfrak{b}', f) = ST(\mathfrak{b}, f)$ follows from Lemma A.2. □

Remark A.6. By Prop A.1 the standard tableau of any $\gamma_{\mathcal{Q}}^{-1}(f)$ is easily found. As observed in Remark 2.17, when $L_i \subset L$, for all $i = 1, 2, \ldots, m$, $\gamma_{\mathcal{Q}}^{-1}(f)$ is homogeneous for L (and for Q). In this case, it is in fact homogeneous for a parabolic of G as well. A component of a Springer fiber is called *Richardson* if it is homogeneous for a parabolic subgroup of G. A general criterion for a component C of a Springer fiber for $GL(n)$ to be Richardson is given in [25] in terms of the standard tableau of C. This criterion,

along with Prop. A.1, allows us to say exactly which of the components associated to closed orbits $\mathcal{Q}$ are Richardson. It is not difficult to show that the following are equivalent.

(1) $\gamma_{\mathcal{Q}}^{-1}(f)$ is Richardson.
(2) The sizes of the blocks in the array (after reordering if necessary) give the partition dual to the partition of f.
(3) No 'collapse' of blocks occurs in the construction of f.
(4) $L_i \subset L$, for all $i = 1, 1, \ldots, m$.
(5) $\gamma_{\mathcal{Q}}^{-1}(f) = L \cdot \mathfrak{b} \ (= Q \cdot \mathfrak{b})$.

It follows that many components of the form $\gamma_{\mathcal{Q}}^{-1}(f)$ are not Richardson. The simplest such example is for $(GL(5), GL(4) \times GL(1))$ and $\lambda = (5, 4, 2, 1 \,|\, 3)$.

As briefly discussed in the introduction, given a Young diagram T there is a closed K-orbit $\mathcal{Q}$ in $\mathfrak{B}$ and a generic f having Young diagram T. In fact there may be several such $\mathcal{Q}$ and f (with perhaps different K, i.e., different p, q). Let $\mathcal{O}_{\mathrm{T}}$ be the nilpotent orbit corresponding to T. Fix a closed K-orbit $\mathcal{Q}$ with corresponding generic f, and a closed K'-orbit $\mathcal{Q}'$ with corresponding generic f'. Suppose that f and f' are both contained in $\mathcal{O}_{\mathrm{T}}$. Then there exists $g \in G$ so that $f = g \cdot f'$. It follows that the Springer fiber $\mu^{-1}(f)$ is the g-translate of $\mu^{-1}(f')$. Therefore the components of $\mu^{-1}(f)$ are the g-translates of the components of $\mu^{-1}(f')$. As the parametrization is given in terms of the linear algebra, it is clear that the standard tableau of a component is the same as that of its g-translate. We may conclude that $\gamma_{\mathcal{Q}}^{-1}(f)$ and $\gamma_{\mathcal{Q}'}^{-1}(f')$ may be viewed as components in a single Springer fiber, and the standard tableaux tell us which components.

The following example illustrates the above discussion. We see how a number of components of a single Springer fiber are of the form $\gamma_{\mathcal{Q}}^{-1}(f)$, and therefore have the structure described in this article.

Consider the Young diagram

$$\mathrm{T} = \ \begin{array}{c}\text{(Young diagram)}\end{array} \ .$$

Let $\mathcal{O}_{\mathrm{T}}$ be the corresponding nilpotent orbit in $\mathfrak{g}$. We write down all $\gamma_{\mathcal{Q}}^{-1}(f)$ that occur as components in a Springer fiber for an element of $\mathcal{O}_{\mathrm{T}}$. There are several pairs (G, K) to consider.

When $(p, q) = (1, 7)$ or $(7, 1)$, any array has just one dot in one of the two rows. Therefore no string has length 4, and a generic f cannot lie in $\mathcal{O}_{\mathrm{T}}$. A similar argument shows the same holds when $(p, q) = (2, 6)$ or $(6, 2)$.

Now consider $(p, q) = (3, 5)$. There are four closed orbits $\mathcal{Q}$ with generic elements $f \in \mathcal{O}_T$. We list below these orbits $\mathcal{Q}$ (by giving the corresponding $\lambda \in \mathfrak{h}^*$), the standard tableau for $\gamma_{\mathcal{Q}}^{-1}(f)$, and the signed tableau corresponding to $K \cdot f$. (The orbits of K on $\mathcal{N}_\theta$ are parametrized by signed tableaux. See [2] for a discussion of these signed tableaux, and how the tableau corresponding to a generic f is obtained from the array.)

$$\lambda = (8, 4, 3 | 7, 6, 5, 2, 1)$$

1	3	5	8
2	4	6	
7			

-	+	-	+
-	+	-	
-			

$$\lambda = (8, 5, 4 | 7, 6, 3, 2, 1)$$

1	4	6	8
2	5	7	
3			

-	+	-	+
-	+	-	
-			

$$\lambda = (5, 4, 1 | 8, 7, 6, 3, 2)$$

1	2	4	6
3	5	7	
8			

+	-	+	-
-	+	-	
-			

$$\lambda = (6, 5, 1 | 8, 7, 4, 3, 2)$$

1	2	5	7
3	6	8	
4			

+	-	+	-
-	+	-	
-			

Now consider $(p, q) = (4, 4)$; there are again four cases.

$$\lambda = (8, 5, 4, 3 | 7, 6, 2, 1)$$

1	3	6	8
2	4	7	
5			

-	+	-	+
-	+	-	
+			

$$\lambda = (8, 7, 3, 2 | 6, 5, 4, 1)$$

1	2	4	7
3	5	8	
6			

-	+	-	+
+	-	+	
-			

$$\lambda = (7, 6, 2, 1 | 8, 5, 4, 3)$$

1	3	6	8
2	4	7	
5			

+	-	+	-
+	-	+	
-			

$$\lambda = (6, 5, 4, 1 | 8, 7, 3, 2)$$

1	2	4	7
3	5	8	
6			

+	-	+	-
-	+	-	
+			

(Note that the last two come from the first two by interchanging the rows of the array.)

For $(p, q) = (5, 3)$ there are four cases, each obtained from the $(3, 5)$ case by interchanging the rows of the array. We get the following.

$\lambda = (7,6,5,2,1|8,4,3)$

1	3	5	8
2	4	6	
7			

+	-	+	-
+	-	+	
+			

$\lambda = (7,6,3,2,1|8,5,4)$

1	4	6	8
2	5	7	
3			

+	-	+	-
+	-	+	
+			

$\lambda = (8,7,6,3,2|5,4,1)$

1	2	4	6
3	5	7	
8			

-	+	-	+
+	-	+	
+			

$\lambda = (8,7,4,3,2|6,5,1)$

1	2	5	7
3	6	8	
4			

-	+	-	+
+	-	+	
+			

In summary, there are 12 closed orbits $\mathcal{Q}$ with generic element in $\mathcal{O}_{\mathrm{T}}$. The 12 orbits pair off (under the symmetry of interchanging the rows of the arrays) into pairs giving the same components, so there are six components of the type $\gamma_{\mathcal{Q}}^{-1}(f)$. A further observation is that for $(p,q) = (3,5)$ (or $(5,3)$), of the four orbits $\mathcal{Q}$, two orbits $K \cdot f \subset \mathcal{N}_\theta$ occur and all four components are different. This is not the case for $(p,q) = (4,4)$.

In the example above, where the Springer fiber corresponds to the Jordan form $(4,3,1)$, all the components of the form $\gamma_{\mathcal{Q}}^{-1}(f)$ turn out to be homogeneous (in fact, Richardson), but for other Jordan forms, this need not be the case (cf. Remark A.6). However, the situation described in the example, where a number of components of a single Springer fiber are of the form $\gamma_{\mathcal{Q}}^{-1}(f)$, occurs for Springer fibers corresponding to other Jordan forms, as follows. Let us fix $G = GL(n)$ and consider all pairs $(G, K_{p,q})$ with $K_{p,q} = GL(p) \times GL(q)$, $p + q = n$. Fix a Young diagram T and corresponding nilpotent orbit $\mathcal{O}_{\mathrm{T}}$ in $\mathfrak{g}$. The key observation is the following.

Lemma A.7. *Suppose $\mathcal{Q} = K_{p,q} \cdot \mathfrak{b}$ is a closed $K_{p,q}$-orbit and $\mathcal{Q}' = K_{p',q'} \cdot \mathfrak{b}'$ is a closed $K_{p',q'}$-orbit, and f and f' are the corresponding generic elements. Then $ST(\mathfrak{b}, f) = ST(\mathfrak{b}', f')$ if and only if either $\mathcal{Q} = \mathcal{Q}'$, or $(p,q) = (q', p')$ and the array for $\mathcal{Q}$ is obtained from the array for $\mathcal{Q}'$ by switching the two rows.*

Proof. This is clear from our description of $ST(\mathfrak{b}, f)$, since there are two ways to reconstruct an array from a standard tableau (one with the last dot in the upper row and one with it in the lower row). $\square$

The lemma implies the following proposition.

Proposition A.8. *Let (G, K) be one of our pairs $(GL(n), GL(p) \times GL(q))$. For each Young diagram T, the components of the Springer fiber for $\mathcal{O}_{\mathrm{T}}$ of the form $\gamma_{\mathcal{Q}}^{-1}(f)$,*

with $\mathcal{Q}$ a closed K-orbit and f generic, are distinct when $p \neq q$. When $p = q$ there are an even number of such components, and each occurs for exactly two $\mathcal{Q}$.

References

[1] M. F Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), no. 1, 1–28.

[2] L. Barchini and R. Zierau, *Certain components of Springer fibers and associated cycles for discrete series representations of $SU(p,q)$*, Representation Theory **12** (2008), 403–434, with appendix by P. Trapa.

[3] A. Borel, *Linear Algebraic Groups*, 2nd ed., Graduate Texts in Math, vol. 126, Springer, New York, 1991.

[4] M. Brion, *Equivariant cohomology and equivariant intersection theory*, Representation Theories and Algebraic Geometry (Montreal, PQ, 1997) (Dordrecht), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., 1998, pp. 1–37.

[5] J.-T. Chang, *Characteristic cycles of discrete series for $\mathbf{R}$-rank one groups*, Trans. A.M.S. **341** (1994), no. 2, 603–622.

[6] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhauser Boston, Boston, MA, 1997.

[7] D. De Concini and G. Lusztig and C. Procesi, *Homology of the zero-set of a nilpotent vector field on a flag manifold*, J. Amer. Math. Soc. **1** (1988), no. 1, 15–34.

[8] D. Edidin and W. Graham, *Localization in equivariant intersection theory and the Bott residue formula*, Amer. J. Math. **120** (1998), no. 3, 619–636.

[9] L. Fresse, *Betti numbers of Springer fibers in type A*, J. Algebra **322** (2009), no. 7, 2566–2579.

[10] ———, *Singular components of the Springer fiber in the two-column case*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 6, 2429-2444.

[11] ———, *A unified approach on Springer fibers in the hook, two-rows and two-columns cases*, Transform. Groups **15** (2010), no. 2, 285–331.

[12] F. Fung, *On the topology of components of some Springer fibers and their relation to Kazhdan-Lusztig theory*, Adv. in Math. **178** (2003), 244–276.

[13] D. Garfinkle, *The annihilators of irreducible Harish-Chandra modules for $\mathrm{SU}(p,q)$ and other type A_{n-1} groups*, Amer. J. Math. **115** (1993), no. 2, 305–369.

[14] W. Graham, *Equivariant K-theory and Schubert varieties*, preprint.

[15] ———, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), no. 3, 599–614.

[16] W. Graham and S. Kumar, *On positivity in T-equivariant K-theory of flag varieties*, Int. Math. Res. Not. IMRN **2008** (2008).

[17] A. Grothendieck, *Séminaire de géométrie algébrique. Revêtements étales et groupe fondemental*, Lecture Notes in Math., vol. 224, Springer-Verlag, Heidelberg, 1971.

[18] J. J. Güemes, *On the homology classes for the components of some fibres of Springer's resolution*, Orbites unipotentes et reprsentations, III, Astérisque, no. 173-174, 1989, pp. 257–269.

[19] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math., vol. 52, Springer Verlag, 1977.

[20] B. Iversen, *A fixed point formula for action of tori on algebraic varieties*, Inventiones Math. **16** (1972), 229–236.

[21] A. Knutson, *Schubert patches degenerate to subword complexes*, Transformation Groups **13** (2008), 715–726.

[22] S. Kumar, *Kac-Moody Groups, Their Flag Varieties and Representation Theory*, Progress in Mathematics, vol. 204, Birkhäuser, Boston, 2002.

[23] G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. **42** (1981), no. 2, 169–178.

[24] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, pringer-Verlag, 1994.

[25] N. G. J. Pagnon and N. Ressayre, *Adjacency of Young tableaux and the Springer fibers*, Selecta Math. **12** (2006), 517–540.

[26] N. Spaltenstein, *Classes Unipotentes et Sous-groupes de Borel*, Lecture Notes in Mathematics, vol. 946, Springer-Verlag, 1982.

[27] T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207.

[28] ———, *A construction of representations of Weyl groups*, Invent. Math. **44** (1978), 279–293.

[29] ———, Contribution to *Open problems in algebraic groups*, Katata (1983), Taniguchi Foundation.

[30] R. Steinberg, *An occurrence of the Robinson-Schensted correspondence*, Journal of Algebra **113** (1988), 523–528.

[31] P. Trapa, *Generalized Robinson-Schensted algorithms for real Lie groups*, Int. Math. Res. Not. (1999), no. 15, 803–834.

[32] J. Vargas, *Fixed points under the action of unipotent elements of $SL(n)$ in the flag variety*, Bol. Soc. Mat. Mexicana **24** (1979), no. 1, 1–14.

[33] M. Willems, *Cohomologie et K-théorie équivariantes des variétés de Bott-Samelson et des variétiés de drapeaux*, Bull. Soc. Math. France **132** (2004), no. 4, 569–589.

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