INDEFINITE HARMONIC THEORY AND HARMONIC SPINORS

L. BARCHINI AND R. ZIERAU

Abstract. We show how to formulate the indefinite harmonic theory of Rawnsley, Schmid and Wolf in the setting of harmonic spinors. A theorem on the existence of square integrable harmonic spinors on finite rank bundles over a semisimple symmetric space is proved.

1. Introduction

An important problem in representation theory is to find explicit realizations of irreducible unitary representations. In this article we discuss a method, known as indefinite harmonic theory, introduced by Rawnsley, Schmid and Wolf in [16], to associate irreducible unitary representations to elliptic coadjoint orbits of semisimple Lie groups. The significance of the method is that it is natural and it makes sense for an arbitrary elliptic orbit; however there are tremendous technical difficulties and it has not been carried out in general. We will point out some of these difficulties and some of the successes of the method. Then we will show how indefinite harmonic theory can be extended to the the setting of harmonic spinors.

The method given in [16] may very roughly be described as follows. One first observes that if $G/L = G \cdot \lambda$ is an elliptic coadjoint orbit, then G/L has a G-invariant complex structure. Under an integrality condition, λ exponentiates to a character of L and defines a holomorphic homogeneous line bundle $\mathcal{L}_\lambda \to G/L$. Geometric quantization (i.e., the orbit method) would suggest that one look for unitary representations in spaces of L_2 holomorphic sections of $\mathcal{L}_\lambda$. There are two immediate problems. First, the space of holomorphic sections is often zero, however irreducible representations often occur in higher degree Dolbeault cohomology. The second problem is that there is no good (G-invariant) notion of a square integrable differential form, since G/L typically carries an *indefinite* metric. An encouraging fact is that under a negativity condition on λ, the Dolbeault cohomology space vanishes except in one degree s, and $H^s(G/L, \mathcal{L}_\lambda)$ is an irreducible representation. This negativity condition will hold for some choice of complex structure on G/L, which we now assume has been made. The quantization procedure of [16] is the following.

(1) Consider the space of strongly harmonic forms of type $(0,s)$:

$$\mathcal{H}^{(0,s)}(G/L,\mathcal{L}_\lambda)=\{\omega:\overline{\partial}\omega=0 \text{ and } \overline{\partial}^*\omega=0\},$$

where $\overline{\partial}^*$ is the formal adjoint of $\overline{\partial}$ with respect the invariant metric on G/L. One needs to show that this harmonic space is nonzero.

(2) Define and auxiliary positive definite metric on G/L (which typically must be noninvariant). This metric may be used to define a notion of a square integrable differential form. It must be shown that the space $\mathcal{H}_2^{(0,s)}(G/L,\mathcal{L}_\lambda)$ of L_2 (strongly) harmonic forms is a nonzero G-invariant Hilbert space.

(3) The invariant hermitian form defined by integration of forms is well defined on $\mathcal{H}_2^{(0,s)}(G/L,\mathcal{L}_\lambda)$ (at least for a reasonable choice of auxiliary metric). Now the goal is to show that the image of $\mathcal{H}_2^{(0,s)}(G/L,\mathcal{L}_\lambda)$ in Dolbeault cohomology is infinitesimally equivalent to $H^s(G/L,\mathcal{L}_\lambda)$, and the invariant form is well-defined on this image in cohomology and is positive definite there.

In Section 2 we give some examples and discuss some of the what is known about when the procedure can be carried out. In the remainder of the paper we show how indefinite harmonic theory can be formulated for spinors on a reductive homogeneous space G/H. Section 3 reviews a construction of a formula for harmonic spinors ([12]). In Section 4 we show how to construct an auxiliary metric and we prove the following theorem.

Theorem. *If G/H is a semisimple symmetric space and E os a finite dimensional H-representation (with highest weight 'sufficiently regular'), then there is a nonzero space of L_2 harmonic spinors on the homogeneous vector bundle for E.*

It follows that, if E carries an H-invariant hermitian form, then this L_2- space of harmonic spinors carries a G-invariant hermitian form.

2. Comments on Indefinite harmonic theory

The strategy for constructing irreducible unitary representations that was briefly out lined in the introduction has had some success. We now discuss the construction in more detail and indicate the extent to which it is now known to produce irreducible unitary representations.

To begin, we need to better understand which representations *should* be attached to elliptic coadjoint orbits. Let G be a connected linear semisimple group and θ a Cartan involution of G. We let K be the fixed point group of θ, a maximal compact subgroup. Write the corresponding Cartan decomposition of the Lie algebra of G as

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}.$$

Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and extend it to a Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ of $\mathfrak{g}$ by choosing an appropriate $\mathfrak{a} \subset \mathfrak{s}$. Using the Killing form we consider $\mathfrak{t}^* \subset \mathfrak{h}^* \subset \mathfrak{g}^*$. Then an element $\lambda \in \mathfrak{t}^*$ is an elliptic element, and the orbit $G \cdot \lambda \subset \mathfrak{g}^*$ is an elliptic coadjoint orbit. We may identify this orbit with the homogeneous space G/L, where L is the centralizer in G of λ. On the other hand, λ defines a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbf{C}}$ as follows. Let $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$ be the roots of $\mathfrak{h}$ in $\mathfrak{g}$. Then the parabolic subalgebra associated to λ is

$$\mathfrak{q} = \mathfrak{l}_{\mathbf{C}} + \mathfrak{u}^-,$$

where $\mathfrak{l}_{\mathbf{C}}$ is spanned by $\mathfrak{h}_{\mathbf{C}}$ and all root spaces $\mathfrak{g}_{\mathbf{C}}^{(\alpha)}$ with $\langle \lambda, \alpha \rangle = 0$, and $\mathfrak{u}^-$ is spanned by all root spaces $\mathfrak{g}_{\mathbf{C}}^{(-\alpha)}$ with $\langle \lambda, \alpha \rangle > 0$. Let Q be the normalizer of $\mathfrak{q}$ in $G_{\mathbf{C}}$. One sees that $L = Q \cap G$, so G/L embeds into the (generalized) flag variety $G_{\mathbf{C}}/Q$ as an open subset. In particular, G/L has a G-invariant complex structure; the holomorphic tangent space at the identity coset is naturally identified with $\mathfrak{g}/\mathfrak{q} \simeq \mathfrak{u}$ (where $\mathfrak{u}$ is spanned by the the root spaces $\mathfrak{g}_{\mathbf{C}}^{(\alpha)}$ with $\langle \lambda, \alpha \rangle > 0$). Observe that each parabolic subalgebra conjugate to $\mathfrak{q}$ and containing $\mathfrak{l}_{\mathbf{C}}$ gives a complex structure on G/L; these are in fact all different. In the language of geometric quantization, these parabolics are the invariant (complex) polarizations (at λ). Typically, in geometric quantization, one chooses a particular polarization and this is what we will do here.

To attach a representation to $G \cdot \lambda$, we assume that λ lifts to a character χ_λ of L. There is then a holomorphic homogeneous line bundle associated to χ_λ. The natural thing is to attach cohomology representations $H^p(G/L, \mathcal{O}(\mathcal{L}_\lambda))$ to the orbit $G \cdot \lambda$. This is, however, a long story as the cohomology spaces are difficult to study directly. For example, it is not at all clear that they have a topology for which the natural action (by left translation) is a continuous representation. In fact it was such analytic difficulties that motivated Zuckerman ([25]) to define an algebraic analogue, which is now known as cohomological induction ([10]). Wong ([24]), generalizing and extending [18], confirmed that the cohomologically induced modules are in fact the proper algebraic analogues of the cohomology representations. Viewing the sheaf cohomology spaces as Dolbeault cohomology, he proved that (1) the $H^p(G/L, \mathcal{L}_\lambda)$

are continuous Fréchet representations (by showing that the image of $\overline{\partial}$ is closed in the C^∞ topology on forms), (2) the Harish-Chandra module of $H^p(G/L, \mathcal{L}_\lambda)$ (i.e., the $(\mathfrak{g}, K)$-module of K-finite vectors) is a cohomologically induced module, and (3) $H^p(G/L, \mathcal{L}_\lambda)$ is a maximal globalization of its Harish-Chandra module in the sense of [17].

Using these fundamental facts of Wong about the cohomology representations, along with results about the cohomologically induced representations, one may conclude that under the negativity condition $\langle \lambda + \rho, \beta \rangle < 0$, for β a root in $\mathfrak{u}$ (and ρ equal to half the sum of the positive roots for a positive system containing the roots $\Delta(\mathfrak{u})$ of $\mathfrak{u}$), $H^p(G/L, \mathcal{L}_\lambda) = \{0\}, p \neq s := \dim_{\mathbf{C}}(K/K \cap L)$, and $H^s(G/L, \mathcal{L}_\lambda)$ is irreducible ([22]) and unitarizable ([21]). It follows from Harish-Chandra ([7, Theorem 9]) that there is a unitary representation infinitesimally equivalent to $H^s(G/L, \mathcal{L}_\lambda)$. As $H^s(G/L, \mathcal{L}_\lambda)$ is a maximal globalization, this unitary representation embeds into $H^s(G/L, \mathcal{L}_\lambda)$ (as a proper subrepresentation, unless G/L is compact). Thus, the goal is to construct this unitary representation in a explicit way, perhaps as a subspace of $H^s(G/L, \mathcal{L}_\lambda)$. Let us give a couple of examples indicating how this might go.

Suppose that G is compact. Then (under a negativity condition on λ) the Borel-Weil Theorem tells us that $H^s(G/L, \mathcal{L}_\lambda)$ is an irreducible finite dimensional representation. The homogeneous space G/L has a G-invariant positive definite metric. This gives rise to an elliptic G-invariant Laplace-Beltrami operator $\square = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. The Hodge Theorem says that each cohomology class is represented by a unique harmonic form. Letting $A^{(0,s)}(G/L, \mathcal{L}_\lambda)$ be the space of smooth $\mathcal{L}_\lambda$-valued differential forms of type $(0, s)$, we define

$$\mathcal{H}^{(0,s)}(G/L, \mathcal{L}_\lambda) := \{\omega \in A^{(0,s)}(G/L, \mathcal{L}_\lambda) : \square\omega = 0\},$$

the space of Harmonic forms. Since each each harmonic form is L_2 (as G/L is compact), we conclude that $\mathcal{H}^{(0,s)}(G/L, \mathcal{L}_\lambda)$, with the L_2-inner product, is a unitary representation equivalent to $H^s(G/L, \mathcal{L}_\lambda)$.

If G is a simple group so that G/K is of hermitian type, then G/K has an invariant complex structure and is an elliptic orbit. In this case $s = 0$ and $H^0(G/K, \mathcal{L}_\lambda)$ is the maximal globalization of its Harish-Chandra module. In [8], Harish-Chandra proved that (under a negativity condition on λ) the space of L_2 sections is an irreducible unitary representation infinitesimally equivalent to $H^0(G/K, \mathcal{L}_\lambda)$. Note that G/K has an invariant positive metric and for an L_2 section η, $\square\eta = 0$ if and only if $\overline{\partial}\eta = 0$. Therefore, the space of L_2 harmonic sections is the L_2-harmonic space.

Another example is that of the regular elliptic orbits. Let us assume that G and K have the same complex rank. Therefore, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$; we let T be the corresponding cartan subgroup of G. Again, G/T has a G-invariant positive metric and there is a G-invariant elliptic Laplace-Beltrami operator $\square = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. Schmid proved ([19], [20]) that the L_2-harmonic space (under a negativity condition on λ) is a unitary representation infinitesimally equivalent to $H^s(G/T, \mathcal{L}_\lambda)$. He also proved that these unitary representations are in the discrete series of G, and all discrete series representations occur this way. If L is compact then G/L has a positive invariant metric, and the corresponding L_2-harmonic spaces are realizations of discrete series representations in much the same way as for G/T. We point out that in this case the invariant metric is positive definite, and serves as the auxiliary metric.

For regular semisimple orbits, Wolf has given geometric realizations of the corresponding representations ([23]). This includes a class of groups with relative discrete series.

We now return to an arbitrary elliptic orbit G/L. Assume that L is now noncompact. Then G/L typically does not have a G-invariant metric. (For example, if G is simple and $L \subsetneq G$ then G/L has no invariant positive metric.) If we wish to construct a unitary realization of $H^s(G/L, \mathcal{L}_\lambda)$ in analogy with the previous examples, the initial obstacle is that there is no natural G-invariant notion of an L_2 form. However, G/L has a G-invariant indefinite hermitian form. For example, the Killing form restricted to $\mathfrak{u} + \mathfrak{u}^-$ is nondegenerate and L-invariant, so may be used to define a G-invariant hermitian form $\langle\ ,\ \rangle$ on G/L. This form may be used to define $\overline{\partial}^*$ and a harmonic space

$$\mathcal{H}^{(0,s)}(G/L, \mathcal{L}_\lambda) := \{\omega \in A^{(0,s)}(G/L, \mathcal{L}_\lambda) : \overline{\partial}\omega = 0 \text{ and } \overline{\partial}^*\omega = 0\}.$$

The differential forms in this space are referred to as *strongly harmonic* forms. The invariant hermitian form may be used to define a G-invariant hermitian form

$$\langle \omega_1, \omega_2 \rangle_{\mathrm{inv}} := \int_{G/L} \langle \omega_1(g), \omega_2(g) \rangle \, dg, \tag{2.1}$$

provided the integral converges. The strategy of [16] is to define an auxiliary positive metric on G/L, which is necessarily noninvariant This metric should be $L \cap K$-invariant and should bound the invariant metric in an appropriate sense. A reasonable choice is

$$\langle X, Y \rangle_{\mathrm{pos}} := -\langle X, \theta(Y) \rangle.$$

This metric is used to define the notion of an L_2-form as follows. Use the Mostow decomposition ([13]):

$$\begin{aligned} G &= K\exp(\mathfrak{l}^{\perp}\cap\mathfrak{s})\exp(\mathfrak{l}\cap\mathfrak{s}),\\ g &= k(g)\exp(X(g))\exp(Y(g)). \end{aligned} \tag{2.2}$$

Then $\langle\omega_1(k(g)\exp(X(g)),\omega_2(k(g)\exp(X(g))\rangle_{\mathrm{pos}}$ is a well-defined function on G/L and it makes sense to ask when

$$\langle\omega_1,\omega_2\rangle_{\mathrm{pos}} = \int_{G/L}\langle\omega_1(k(g)\exp(X(g)),\omega_2(k(g)\exp(X(g))\rangle_{\mathrm{pos}}\,dg$$

is finite. Define

$$\mathcal{H}_2^{(0,s)}(G/L,\mathcal{L}_\lambda) := \{\omega\in\mathcal{H}^{(0,s)}(G/L,\mathcal{L}_\lambda) : \langle\omega,\omega\rangle<\infty\}.$$

With a little more care one may consider L_2 solutions to $\overline{\partial}\omega=0$ and $\overline{\partial}^*\omega=0$, and see that $\mathcal{H}_2^{(0,s)}(G/L,\mathcal{L}_\lambda)$ is a Hilbert space. One may also see that $\mathcal{H}_2^{(0,s)}(G/L,\mathcal{L}_\lambda)$ is invariant under left translation by G ([15]) and defines a continuous (but not unitary) representation of G. In addition, the invariant hermitian form (2.1) is well-defined on $\mathcal{H}_2^{(0,s)}(G/L,\mathcal{L}_\lambda)$. Formally, one has $\langle\omega,\overline{\partial}\eta\rangle_{\mathrm{inv}} = \langle\overline{\partial}^*\omega,\eta\rangle_{\mathrm{inv}}=0$, for ω strongly harmonic, so one expects that the nullspace of $\langle\ ,\ \rangle_{\mathrm{inv}}$ contains the exact forms. Therefore, if we write $q: A^{(0,s)}(G/L,\mathcal{L}_\lambda)\to H^s(G/L,\mathcal{L}_\lambda)$ for the natural quotient map, then one expects that $\langle\ ,\ \rangle_{\mathrm{inv}}$ is defined on $\overline{\mathcal{H}}_2^s := q(\mathcal{H}_2^{(0,s)}(G/L,\mathcal{L}_\lambda))$. Then two things must be shown. First, it needs to be shown that $\overline{\mathcal{H}}_2^s$ is infinitesimally equivalent to $H^s(G/L,\mathcal{L}_\lambda)$. It is not clear that either $\mathcal{H}^{(0,s)}(G/L,\mathcal{L}_\lambda)$ or $\mathcal{H}_2^{(0,s)}(G/L,\mathcal{L}_\lambda)$ is nonzero. Then one must show that the invariant form is positive definite on $\overline{\mathcal{H}}_2^s$.

The first success in this indefinite metric setting is that of Rawnsley, Schmid and Wolf ([16]). They consider the following situation. Suppose G is simple and G/K is a symmetric space of hermitian type and there are G-invariant complex structures on G/L and $G/K\cap L$ so that the natural double fibration

$$\begin{array}{ccc} & G/K\cap L & \\ \swarrow & & \searrow \\ G/K & & G/L \end{array}$$

is holomorphic. Writing the K-decomposition of $\mathfrak{s}$ as $\mathfrak{s}=\mathfrak{s}_+ + \mathfrak{s}_-$, the existence of such a holomorphic double fibration is equivalent to $\mathfrak{u}\cap\mathfrak{s}$ being contained in either $\mathfrak{s}_+$ or $\mathfrak{s}_-$. Under this condition the Harish-Chandra module of $H^s(G/L,\mathcal{L}_\lambda)$ is a (unitarizable) highest weight module. If, in addition, G/L is a semisimple symmetric space, then a unitary representation is constructed by the procedure outlined in the preceding paragraph. In other words, $\overline{\mathcal{H}}_2^s$ (with the invariant hermitian form) is an irreducible unitary representation infinitesimally equivalent to $H^s(G/L, \mathcal{L}_\lambda)$. The condition that G/L is semisimple symmetric is relaxed somewhat.

For a general elliptic orbit, an approach to studying $\overline{\mathcal{H}}_2^s$ has been used with some success in [4]. The tool is an intertwining operator

$$S : C^\infty(G/P, \mathcal{W}) \to A^{(0,s)}(G/L, \mathcal{L}_\lambda),$$

where $C^\infty(G/P, \mathcal{W})$ is a principal series representation having a unique Langlands quotient (infinitesimally equivalent to $H^s(G/L, \mathcal{L}_\lambda)$). The image of S consists of strongly harmonic forms and $q(Im(S))$ is nonzero ([3], [2], [5]). In fact, $Im(S)$ contains all K-finite vectors in $H^s(G/L, \mathcal{L}_\lambda)$; each K-finite cohomology class is represented by a strongly harmonic form. In [4] it is shown that if G/L is a semisimple symmetric space, then for each K-finite $\varphi \in C^\infty(G/P, \mathcal{W})$, $S\varphi$ is square integrable. We may conclude that $\overline{\mathcal{H}}_2^s$ is infinitesimally equivalent to $H^s(G/L, \mathcal{L}_\lambda)$ and carries a G-invariant form $\langle\ ,\ \rangle_{\text{inv}}$. By [21, Thm. 1.3], this form must be positive definite or zero. A condition in [4] is given for the form to be nonzero. These results extend the the scope of indefinite harmonic theory in the construction of unitary representations.

3. Harmonic Spinors

Suppose G/H is a homogeneous space so that

(a) H is connected reductive subgroup of G,

(b) the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is nondegenerate, and (3.1)

(c) $\text{rank}(\mathfrak{g}_{\mathbf{C}}) = \text{rank}(\mathfrak{h}_C)$.

Let $\mathfrak{q}$ denote the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form. Then the Killing form is nondegenerate on $\mathfrak{q}$ and there is an orthogonal direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}.$$

The Clifford algebra of $\mathfrak{q}$ is denoted by $C\ell(\mathfrak{q})$ and $S_\mathfrak{q}$ denotes the corresponding spin representation of H.

Kostant ([11]) has defined the cubic Dirac operator in this setting. It is an element of $\{\mathcal{U}(\mathfrak{g}) \otimes C\ell(\mathfrak{q})\}^\mathfrak{h}$. This determines geometric Dirac operators on sections of homogeneous vector bundles on G/H. If E is a finite dimensional representation of H, then there is a homogeneous vector bundle $S_{\mathfrak{q}} \otimes \mathcal{E} \to G/H$. The geometric Dirac operator is a first order differential operator on sections:

$$D_{G/H,E} : C^\infty(G/H, S_{\mathfrak{q}} \otimes \mathcal{E}) \to C^\infty(G/H, S_{\mathfrak{q}} \otimes \mathcal{E}).$$

A formula may be found in [12, Section 2]. Note that G acts on the space of sections by left translation. It is easily seen that $D_{G/H,E}$ is a G-equivariant operator. We refer to the kernel of $D_{G/H,E}$ as the space of *harmonic spinors*.

An important example occurs for riemannian symmetric spaces. In this case $H = K$, a maximal compact subgroup of G. Note that the Killing form is therefore positive definite on $\mathfrak{q}$ and E has a K-invariant positive definite hermitian form. This gives a K-invariant inner product $\langle\ ,\ \rangle$ on $S_{\mathfrak{q}} \otimes E$. It follows that $D_{G/K,E}$ is an elliptic operator. An L_2-harmonic space $\mathcal{H}_2(G/K, E)$ may be defined as the space of harmonic spinors F so that

$$||F||_2^2 := \int_{G/K} \langle F(g), F(g)\rangle\, dg < \infty \tag{3.2}$$

This defines a Hilbert space and the inner product is G-invariant; $\mathcal{H}_2(G/K, E)$ is a unitary representation. It is shown in [14] and [1] that (under some conditions on E) this L_2-harmonic space is an irreducible representation and is in the discrete series of G, and every discrete series representation of G occurs this way.

We now return to the general setting of (3.1). It is often the case that a finite dimensional representation has an invariant hermitian form. We assume, for now, that this is the case. Typically, this form will have indefinite signature, unless H is compact. As $S_{\mathfrak{q}}$ has an H-invariant hermitian form, it follows that $S_{\mathfrak{q}} \otimes E$ has an H-invariant form. Denote this form by $\langle\ ,\ \rangle$ and define

$$\langle F_1, F_2\rangle_{\mathrm{inv}} := \int_{G/H} \langle F_1(g), F_2(g)\rangle\, dg.$$

The goal of indefinite harmonic theory for spinors is to identify a space of harmonic spinors on which $\langle\ ,\ \rangle_{\mathrm{inv}}$ is well-defined (i.e., the integral converges), then understand the invariant form. For example, one might find a subspace on which the form is positive definite, thus constructing a unitary representation.

Our main tool for understanding $\mathcal{H}_2(G/H, E)$ is an analogue of the intertwining map (3.3) constructed in [12]. This is an integral transform

$$\mathcal{P} : C^\infty(G/P, \mathcal{W}) \to \mathcal{H}(G/H, E), \tag{3.3}$$

where G/P is a real flag manifold and $\mathcal{W}$ is the homogenous vector bundle associated to an irreducible representation W of P. For our purposes here we do not need the full details of the construction of $\mathcal{P}$, but we will need several properties. We give a quick outline of the construction and refer to [12, §3] for more detail. Given G/H we may choose a Cartan involution θ of G that preserves H. As in §2, we write the Cartan decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}.$$

Let $\mathfrak{a}$ be maximal abelian in $\mathfrak{h} \cap \mathfrak{s}$. Let MA be the centralizer of $\mathfrak{a}$ in G and choose a Cartan subalgebra $\mathfrak{t}_M$ in $\mathfrak{m}_{\mathbf{C}}$. Then $\mathfrak{a}_{\mathbf{C}} + \mathfrak{t}_M$ is a Cartan subalgebra in both $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{h}_{\mathbf{C}}$. Various systems of positive roots are chosen as in [12, §3]. We denote by $\rho_{\mathfrak{g}}$ half the sum of the positive $\mathfrak{a}$-roots in $\mathfrak{g}$, and similarly for $\rho_{\mathfrak{h}}$. The positive roots of $\mathfrak{a}$ in $\mathfrak{g}$ determine a real parabolic subgroup $P = MAN$.

Lemma 3.4. ([12, Lemma 3.1]) The following hold.
(a) $P \cap H$ is a minimal parabolic subgroup of H, in particular $M \cap H$ is compact.
(b) M and $M \cap K$ have the complex ranks, so M has a nonempty discrete series.

Let $\mu \in (\mathfrak{a}_{\mathbf{C}} + \mathfrak{t}_M)^*$ be the highest weight of E. Then the representation W of P is as follows. The action of N is trivial and A acts by the character e^{ν}, with $\nu = \mu|_{\mathfrak{a}} + \rho_{\mathfrak{h}} + \rho_{\mathfrak{g}}$.

As an M-representation, W is a discrete series representation. The precise parameters are not needed here, however, it is important that W be realized as an L_2-space of harmonic spinors. That this may be done is essentially the example given earlier in this section. (That construction in fact holds for the possibly disconnected reductive group M and the maximal compact subgroup $M \cap K$ replaced by the compact subgroup $M \cap H$.) Therefore, we take W to be an L_2-harmonic space

$$W := \mathcal{H}_2(M/M \cap H, E) \subset C^{\infty}(M/M \cap H, S_{\mathfrak{q}\cap\mathfrak{m}} \otimes U). \tag{3.5}$$

Here $S_{\mathfrak{m}\cap\mathfrak{q}}$ is the spin representation of $M \cap H$ for $M/M \cap H$ and U is a finite dimensional representation of $M \cap H$. The representation U is determined by a highest weight and a character (due to the disconnectedness of M) as specified in [12, §3].

One easily sees that the representation $E_{\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})}$ of highest weight $\mu + \rho(\mathfrak{q}) - 2\rho(\mathfrak{m} \cap \mathfrak{k} \cap \mathfrak{q})$ occurs in $S_{\mathfrak{q}} \otimes E$. The $\mathfrak{h} \cap \mathfrak{n}$-invariants

$$V_0 := (E_{\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})})^{\mathfrak{h}\cap\mathfrak{n}}$$

plays an important role. One can see that $V_0 \subset S_{\mathfrak{m}\cap\mathfrak{q}} \otimes U$ and we may define the projection

$$\pi_0 : S_{\mathfrak{m}\cap\mathfrak{q}} \otimes U \to V_0.$$

The intertwining map (3.3) has the formula

$$\mathcal{P}\varphi(g) = \int_{H\cap K} \ell \cdot \pi_o(\varphi(g\ell)(e))\, d\ell.$$

Note that $\varphi(g\ell) \in W \subset C^\infty(M/M\cap H, S_{\mathfrak{q}\cap\mathfrak{m}} \cap U)$, so when evaluated at the identity, gives an element of $S_{\mathfrak{q}\cap\mathfrak{m}} \cap U$. Under a condition that μ is sufficiently regular ([12, Eqn. (4.3)]), the following holds ([12, Thm. 4.6]).

Theorem 3.6. *$\mathcal{P}$ is a nonzero G-intertwining map into $\mathcal{H}(G/H, E)$.*

4. The L_2-theory

To set up indefinite harmonic theory for spinors, we need to define an L_2-space of harmonic spinors. We begin by considering hermitian forms on $S_{\mathfrak{q}} \otimes E$. Here E is the finite dimensional representation of H having highest weight μ.

Lemma 4.1. *Any finite dimensional representation of H has a positive definite hermitian form $\langle\ ,\ \rangle_{\text{pos}}$ with the property that*

$$\langle h \cdot v, w\rangle_{\text{pos}} = \langle v, \theta(h^{-1}) \cdot w\rangle_{\text{pos}},$$

for $h \in H$.

Proof. There is a positive definite form invariant under the compact real form $\mathfrak{h}_u = \mathfrak{h} \cap \mathfrak{k} + i(\mathfrak{h} \cap \mathfrak{s})$. Write $Z = X + Y \in \mathfrak{h} \cap \mathfrak{k} + \mathfrak{h} \cap \mathfrak{s}$. Then

$$\begin{aligned}
\langle Zv, w\rangle_{\text{pos}} &= \langle (X - i(iY))v, w\rangle_{\text{pos}} \\
&= \langle Xv, w\rangle_{\text{pos}} - i\langle iYv, w\rangle_{\text{pos}} \\
&= -\langle v, Xw\rangle_{\text{pos}} - i(-i\langle v, iYw\rangle_{\text{pos}}) \\
&= -\langle v, Xw\rangle_{\text{pos}} + \langle v, -i(iY)w\rangle_{\text{pos}} \\
&= -\langle v, (X - Y)w\rangle_{\text{pos}} \\
&= -\langle v, \theta(Z)w\rangle_{\text{pos}}.
\end{aligned}$$

□

We now assume that E has a nondegenerate H-invariant hermitian form. A necessary and sufficient condition for the existence of such a form is contained in [6, Prop. 2.3]. We note that one situation where such a form exists is when $\text{rank}(H) =$

rank$(H \cap K)$. It follows that $S_{\mathfrak{q}} \otimes E$ has a nondegenerate H-invariant hermitian form, which we denote by $\langle\ ,\ \rangle$. Now fix a positive definite form on $S_{\mathfrak{q}} \otimes E$ as in Lemma 4.1 and let $||\cdot||_{\text{pos}}$ denote the corresponding norm.

Lemma 4.2. *If E has an H-invariant hermitian form, then there is a constant C so that*

$$|\langle v, w\rangle| \leq C||v||_{\text{pos}}||w||_{\text{pos}},$$

for all $v \in S_{\mathfrak{q}} \otimes E$.

Proof. $S_{\mathfrak{q}} \otimes E$ may be decomposed as $V_+ \oplus V_-$, an orthogonal (with respect to $\langle\ ,\ \rangle$) direct sum with $\langle\ ,\ \rangle$ positive definite on V_+ and negative definite on V_-. Writing $v = v_+ + v_-$ and $w = w_+ + w_-$,

$$\langle v, w\rangle'_{\text{pos}} = \langle v_+, w_+\rangle - \langle v_-, w_-\rangle$$

is a positive definite hermitian form on $S_{\mathfrak{q}} \otimes E$. Since all norms on a finite dimensional vector space are equivalent, there is a constant C so that

$$\begin{aligned}
|\langle v, w\rangle| &\leq |\langle v_+, w_+\rangle| + |\langle v_-, w_-\rangle| \\
&= |\langle v_+, w_+\rangle'_{\text{pos}}| + |\langle v_-, w_-\rangle'_{\text{pos}}| \\
&\leq ||v_+||'_{\text{pos}}||w_+||'_{\text{pos}} + ||v_-||'_{\text{pos}}||w_-||'_{\text{pos}} \\
&\leq 2||v||'_{\text{pos}}||w||'_{\text{pos}} \\
&\leq C||v||_{\text{pos}}||w||_{\text{pos}}.
\end{aligned}$$

□

To define a Hilbert space of harmonic spinors we need to integrate over G/H. However $||F(g)||_{\text{pos}}$ is not a function on G/H. We use the Mostow decomposition (2.2). It is easy to check that

$$||F(k(g)\exp(X(g))||^2_{\text{pos}}$$

is a function on G/H. (Note that $||\cdot||_{\text{pos}}$ is $K \cap H$-invariant by Lemma 4.1.)

Definition 4.3. Let $\mathcal{H}_2(G/H, E)$ be the space of harmonic spinors F so that

$$||F||^2_{\text{pos}} := \int_{G/H} ||F(k(g)\exp(X(g))||^2_{\text{pos}}\, dg$$

is finite.

It follows from Lemma 4.2 that

$$\langle F_1, F_2\rangle_{\text{inv}} := \int_{G/H} \langle F_1(g), F_2(g)\rangle \, dg$$

is finite for all $F_1, F_2 \in \mathcal{H}_2(G/H, E)$, so defines a G-invariant hermitian form on $\mathcal{H}_2(G/H, E)$.

Our goal is to show that $\mathcal{H}_2(G/H, E)$ is nonzero when G/H is a semisimple symmetric space. This will be accomplished by using the formula of (3.3). Therefore we will not only show that $\mathcal{H}_2(G/H, E) \neq \{0\}$, but we will have an integral formula for L_2 harmonic spinors.

We begin by deriving an estimate for $||P\varphi(g)||_{\text{pos}}$ in general then we prove convergence when G/H is semisimple symmetric.

Some standard decompositions and integration formulas will be used. In particular we consider the Iwasawa decompositions with respect to $P = MAN$ and the opposite parabolic $\overline{P} = MA\overline{N}$. We write

$$\begin{aligned} g &= \kappa(g)m(g)n(g)\mathrm{e}^{H(g)} \in K\exp(\mathfrak{m} \cap \mathfrak{s})NA \\ g &= \overline{\kappa}(g)\overline{m}(g)\overline{n}(g)\mathrm{e}^{\overline{H}(g)} \in K\exp(\mathfrak{m} \cap \mathfrak{s})\overline{N}A. \end{aligned}$$

A formula relating these two decompositions is

$$H(g) = \overline{H}(g) + H(\overline{m}(g)\overline{n}(g)). \tag{4.4}$$

For $h \in H, m(h) = e$, as $P \cap H$ is a minimal parabolic subgroup of H. Therefore, we have

$$h = \kappa(h)n(h)\mathrm{e}^{H(h)}.$$

We will use the following integration formulas:

$$\int_{K\cap H} F(k)\, dk = \int_{\overline{N}\cap H} F(\kappa(\overline{n}_H))\mathrm{e}^{-2\rho_{\mathfrak{h}}(H(\overline{n}_H))}\, d\overline{n}_H \tag{4.5}$$

and, for right A-invariant functions F,

$$\begin{aligned} \int_{H/A} F(h)\, dh &= \int_{K\cap H}\int_{\overline{N}\cap H} F(k\overline{n}_H)\, d\overline{n}_H dk \quad \text{and} \\ \int_{G/A} F(g)\, dg &= \int_K \int_M \int_{\overline{N}} F(km\overline{n})\, d\overline{n} dm dk. \end{aligned} \tag{4.6}$$

Lemma 4.7. *For $\varphi \in C^\infty(G/P, \mathcal{W})$ and $g \in G$,*

$$||\mathcal{P}\varphi(g)||^2_{pos} = \int_{K\cap H}\int_{\overline{N}\cap H} \langle \pi_0(\varphi(g\ell)(e)), \pi_0(\varphi(g\ell\overline{n}_H)(e))\rangle_{\text{pos}}\, d\overline{n}_H d\ell. \tag{4.8}$$

Proof. Since $\varphi \in C^\infty(G/P, \mathcal{W})$,

$$\begin{aligned}\pi_0(\varphi(g\ell\overline{n}_H)(e)) &= \pi_0(\exp(-H(\overline{n}_H)) \cdot \varphi(g\ell\kappa(\overline{n}_H))(e)) \\ &= \mathrm{e}^{-2\rho_{\mathfrak{h}}(H(\overline{n}_H))} \exp(-H(\overline{n}_H)) \cdot \pi_0(\varphi(g\ell\kappa(\overline{n}_H))(e))),\end{aligned}$$

by [12, Lem. 4.4]. Therefore, since the image of π_0 consists of $N \cap H$-invariants,

$$\overline{n}_H \cdot \pi_0(\varphi(g\ell\overline{n}_H)(e)) = \mathrm{e}^{-2\rho_{\mathfrak{h}}(H(\overline{n}_H))}\kappa(\overline{n}_H) \cdot \pi_0(\varphi(g\ell\kappa(\overline{n}_H))(e)). \tag{4.9}$$

Now

$$\begin{aligned}&||\mathcal{P}\varphi(g)||^2_{\mathrm{pos}} \\ &\quad = \int_{K\cap H}\int_{K\cap H} \langle \ell \cdot \pi_0(\varphi(g\ell)(e)), \ell' \cdot \pi_0(\varphi(g\ell')(e))\rangle_{\mathrm{pos}}\, d\ell' d\ell\end{aligned}$$

(by the definition of $\mathcal{P}$)

$$= \int_{K\cap H}\int_{K\cap H} \langle \pi_0(\varphi(g\ell)(e)), \ell^{-1}\ell' \cdot \pi_0(\varphi(g\ell')(e))\rangle_{\mathrm{pos}}\, d\ell' d\ell$$

(by the $K \cap H$-invariance of $\langle\ ,\ \rangle_{\mathrm{pos}}$)

$$\begin{aligned}&= \int_{K\cap H}\int_{K\cap H} \langle \pi_0(\varphi(g\ell)(e)), \ell' \cdot \pi_0(\varphi(g\ell\ell')(e))\rangle_{\mathrm{pos}}\, d\ell' d\ell \\ &= \int_{K\cap H}\int_{\overline{N}\cap H} \langle \pi_0(\varphi(g\ell)(e)), \kappa(\overline{n}_H) \cdot \pi_0(\varphi(g\ell\kappa(\overline{n}_H))(e))\rangle_{\mathrm{pos}} \mathrm{e}^{-2\rho_{\mathfrak{h}}(H(\overline{n}_H))}\, d\overline{n}_H d\ell\end{aligned}$$

(by formula (4.5))

$$= \int_{K\cap H}\int_{\overline{N}\cap H} \langle \pi_0(\varphi(g\ell)(e)), \overline{n}_H \cdot \pi_0(\varphi(g\ell\overline{n}_H))(e))\rangle_{\mathrm{pos}}\, d\overline{n}_H d\ell$$

(by formula (4.9))

$$= \int_{K\cap H}\int_{\overline{N}\cap H} \langle \theta(\overline{n}_H^{-1}) \cdot \pi_0(\varphi(g\ell)(e)), \pi_0(\varphi(g\ell\overline{n}_H))(e))\rangle_{\mathrm{pos}}\, d\overline{n}_H d\ell$$

(by Lemma 4.1)

$$= \int_{K\cap H}\int_{\overline{N}\cap H} \langle \pi_0(\varphi(g\ell)(e)), \pi_0(\varphi(g\ell\overline{n}_H))(e))\rangle_{\mathrm{pos}}\, d\overline{n}_H d\ell$$

(since V_0 is fixed by $N \cap H$ and $\theta(\overline{n}_H) \in N \cap H$). $\square$

We now assume that φ is K-finite. Therefore, there are a finite number of $\varphi_i \in C^\infty(G/P, \mathcal{W})$ so that $\mathrm{span}_{\mathbf{C}}\{\varphi_1, \ldots, \varphi_q\}$ is K-stable and contains φ. It follows that for each $k \in K$,

$$k^{-1} \cdot \varphi = \sum_{i=1}^{q} C_i(k)\varphi_i,$$

where each C_i is a continuous function on K. It follows that

$$\varphi(k) = \sum_{i=1}^{q} C_i(k)\varphi_i(e).$$

We may also choose an orthonormal basis $\{v_l\}$ of V_0 (with respect to $\langle\ ,\ \rangle_{\mathrm{pos}}$) so that for any $u \in S_{\mathfrak{m}\cap\mathfrak{q}} \otimes U$

$$\pi_0(u) = \sum_l \langle u, v_l\rangle v_l.$$

It follows that

$$\begin{aligned}
&\langle \pi_0(\varphi(g\ell)(e)), \pi_0(\varphi(g\ell\overline{n}_H)(e))\rangle_{\mathrm{pos}} \\
&\quad = \sum_l \langle \varphi(g\ell)(e), v_l\rangle_{\mathrm{pos}} \langle \varphi(g\ell\overline{n}_H)(e), v_l\rangle_{\mathrm{pos}} \\
&\quad = \sum_l \mathrm{e}^{-\nu(H(g\ell)+H(g\ell\overline{n}_H)} \langle \varphi(\kappa(g\ell))(m(g\ell)), v_l\rangle_{\mathrm{pos}} \langle \varphi(\kappa(g\ell\overline{n}_H))(m(g\ell\overline{n}_H)), v_l\rangle_{\mathrm{pos}} \\
&\quad = \sum_{i,j,l} \mathrm{e}^{-\nu(H(g\ell)+H(g\ell\overline{n}_H))} C_i(\kappa(g\ell)) C_j(\kappa(g\ell\overline{n}_H)) \\
&\qquad\qquad \langle \varphi_i(e)(m(g\ell)), v_l\rangle_{\mathrm{pos}} \langle \varphi_i(e)(m(g\ell\overline{n}_H)), v_l\rangle_{\mathrm{pos}}.
\end{aligned}$$

Lemma 4.10. *When φ is K-finite, the integrand in (4.8) is bounded by a constant multiple of*

$$\sum_{i,j,l} \mathrm{e}^{-\nu(H(g\ell)+H(g\ell\overline{n}_H))} ||\varphi_i(e)(m(g\ell))||_{\mathrm{pos}}\ ||\varphi_i(e)(m(g\ell\overline{n}_H))||_{\mathrm{pos}}.$$

Proof. This is a consequence of the preceding equalities, the fact that the C_i are continuous (hence bounded) and

$$|\langle \varphi(e)(m), v_l\rangle_{\mathrm{pos}}| \leq ||\varphi(e)(m)||_{\mathrm{pos}} ||v_l||_{\mathrm{pos}} = ||\varphi_i(e)(m)||_{\mathrm{pos}}.$$

□

At this point we assume that G/H is a semisimple symmetric space. Suppose σ is the involution having H as the fixed point group. This assumption gives us some additional identities involving the Iwasawa decompositions.

Lemma 4.11. *If $\sigma\theta(g) = g$, then $\overline{H}(g) = -H(g)$ and $\overline{m}(g) = m(g)$.*

Proof. Recall that $\mathfrak{a} \subset \mathfrak{h} \cap \mathfrak{s}$, so $\sigma\theta$ acts by -1 on $\mathfrak{a}$. Therefore, $\sigma\theta(N) = \overline{N}$ and $\sigma\theta$ preservers M. In fact, since $\mathfrak{m} \cap \mathfrak{s} = \mathfrak{m} \cap \mathfrak{q} \cap \mathfrak{s}$ (by Lemma 3.4(a)), $\sigma\theta$ acts by the identity on $\mathfrak{m} \cap \mathfrak{s}$. Therefore,

$$\begin{aligned} g = \sigma\theta(g) &= \sigma\theta(\kappa(g))\sigma\theta(m(g))\sigma\theta(n(g))\mathrm{e}^{\sigma\theta(H(g))} \\ &= \sigma(\kappa(g))m(g)\sigma\theta(n(g))\mathrm{e}^{-H(g)} \\ &\in KM\overline{N}A. \end{aligned}$$

The statement of the lemma now follows. □

Several useful identities follow from the lemma. Suppose $\ell \in K \cap H$. Since $X(g) \in \mathfrak{q} \cap \mathfrak{s}$, $\sigma\theta(\exp(X(g))\ell) = \exp(X(g))\ell$. Therefore,

$$\begin{aligned} H(\exp(X(g))\ell) &= -\overline{H}(\exp(X(g))\ell) \\ m(\exp(X(g))\ell) &= \overline{m}(\exp(X(g))\ell) = \overline{m}(\exp(X(g)\ell\overline{n}_H), \end{aligned}$$

for $\overline{n}_H \in \overline{N} \cap H$. Therefore, we have

$$\begin{aligned} &H(\exp(X(g))\ell) + H(\exp(X(g))\ell\overline{n}_H) \\ &\quad = H\exp(X(g))\ell) + \overline{H}(\exp(X(g))\ell\overline{n}_H) + H(\overline{m}(\exp(X(g))\ell\overline{n}_H)\overline{n}(\exp(X(g))\ell\overline{n}_H)) \\ &\quad = H\exp(X(g))\ell) + \overline{H}(\exp(X(g))\ell) + H(\overline{m}(\exp(X(g))\ell\overline{n}_H)\overline{n}(\exp(X(g))\ell\overline{n}_H)) \\ &\quad = H(\overline{m}(\exp(X(g))\ell\overline{n}_H)\overline{n}(\exp(X(g))\ell\overline{n}_H)). \end{aligned}$$

To prove that $||\mathcal{P}\varphi(g)||_{\mathrm{pos}}$ is finite, it suffices (by Lemma 4.10), to show that

$$\begin{aligned} &\int_{G/H}\int_{K\cap H}\int_{\overline{N}\cap H} \mathrm{e}^{-\nu(H(\overline{m}(\exp(X(g))\ell\overline{n}_H)\overline{n}(\exp(X(g))\ell\overline{n}_H)))} \\ &\qquad ||\varphi_i(e)(\overline{m}(\exp(X(g))\ell\overline{n}_H))||_{\mathrm{pos}}||\varphi_j(e)(m(\exp(X(g))\ell\overline{n}_H))||_{\mathrm{pos}}d\overline{n}_H d\ell dg \end{aligned} \tag{4.12}$$

is finite. This expression equals

$$\begin{aligned} &\int_{G/H}\int_{H/A} \mathrm{e}^{-\nu(H(\overline{m}(\exp(X(g))h)\overline{n}(\exp(X(g))h)))} \\ &\qquad\qquad ||\varphi_i(e)(\overline{m}(\exp(X(g))h))||_{\mathrm{pos}}||\varphi_j(e)(m(\exp(X(g))h))||_{\mathrm{pos}}\, dhdg \\ &\quad = \int_{G/A} \mathrm{e}^{-\nu(H(\overline{m}(g)\overline{n}(g)))}||\varphi_i(e)(\overline{m}(g))||_{\mathrm{pos}}||\varphi_j(e)(m(g))||_{\mathrm{pos}}dg \end{aligned}$$

(by the change of variables $h \to \exp(Y(g))h$)

$$= \int_K\int_M\int_{\overline{N}} \mathrm{e}^{-\nu(H(m_0\overline{n}))}||\varphi_i(e)(m_0)||_{\mathrm{pos}}||\varphi_j(e)(m(m_0\overline{n}))||_{\mathrm{pos}}d\overline{n}dm_0dk.$$

One easily checks that for $m_0 \in M$

$$\begin{aligned} m(m_0\overline{n}) &= m(m_0\overline{n}m_0^{-1})m_0 \text{ and} \\ H(m_0\overline{n}) &= H(m_0\overline{n}m_0^{-1}). \end{aligned}$$

Applying the change of variables $\overline{n} \to m_0^{-1}\overline{n}m_0$ we see that Equation (4.12) equals

$$\int_M \int_{\overline{N}} \mathrm{e}^{-\nu(H(\overline{n}))} ||\varphi_i(e)(m_0)||_{\mathrm{pos}} ||\varphi_j(e)(m(\overline{n})m_0)||_{\mathrm{pos}}\, d\overline{n} dm_0. \tag{4.13}$$

Since $W = \mathcal{H}_2(M/M \cap H, U)$ has inner product given by integrating over $M/M \cap H$ (or M),we have

$$\begin{aligned} \int_M &||\varphi_i(e)(m_0)||_{\mathrm{pos}} ||\varphi_j(e)(m(\overline{n})m_0)||_{\mathrm{pos}} dm_0 \\ &\leq ||\varphi_i(e)||_2 ||m(\overline{n})^{-1} \cdot \varphi_j(e)||_2, \\ &= ||\varphi_i(e)||_2 ||\varphi_j(e)||_2 \end{aligned}$$

by the unitarity of the M-representation W. We now conclude that (4.13) is bounded by

$$||\varphi_i(e)||_2 ||\varphi_j(e)||_2 \int_{\overline{N}} \mathrm{e}^{-\nu(H(\overline{n}))} d\overline{n},$$

which is finite when $\nu - \rho_{\mathfrak{g}}$ is regular dominant for the $\mathfrak{a}$-roots in $\mathfrak{n}$ (by, for example, [9, Ch. VII.7]). However this is the case since $\nu = \mu|_{\mathfrak{a}} + \rho_{\mathfrak{h}} + \rho_{\mathfrak{g}}$, with $\mu|_{\mathfrak{a}} + \rho_{\mathfrak{h}}$ dominant regular.

The following theorem is now proved (under the condition that μ is sufficiently regular ([12, Eqn. (4.3)])).

Theorem 4.14. *If G/H is a semisimple symmetric space, then $\mathcal{H}_2(G/H, E) \neq \{0\}$. If, in addition, E has an invariant hermitian form, then $\mathcal{H}_2(G/H, E)$ carries a G-invariant hermitian form.*

The intertwining map $\mathcal{P}$ gives an explicit integral formula for L_2 harmonic spinors; $\mathcal{P}$ may be considered an analogue of the Poisson transform.

Remark 4.15. The proof of square integrability given above uses some estimates in common with the proof of square integrability for $(0, s)$-forms given in [4]. However, the argument here is more direct. In [4] certain matrix coefficients are bounded by Harish-Chandra's spherical functions. Here, we do not need such bounds.

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Mathematics Department, Oklahoma State University, Stillwater, Oklahoma 74078

E-mail address: leticia@math.okstate.edu

Mathematics Department, Oklahoma State University, Stillwater, Oklahoma 74078

E-mail address: zierau@math.okstate.edu