COMPONENTS OF SPRINGER FIBERS ASSOCIATED TO CLOSED ORBITS FOR THE SYMMETRIC PAIRS $(Sp(2n), Sp(2p) \times Sp(2q))$ AND $(SO(2n), GL(n))$ I

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Introduction

The main result of this article is the description of components of Springer fibers associated to closed K-orbits in flag varieties. We consider pairs (G, K) of complex groups equal to

$$(Sp(2n), Sp(2p) \times Sp(2q)) \quad \text{and} \quad (SO(2n), GL(n)). \tag{1}$$

These pairs arise in the study of Harish-Chandra modules for the simple real Lie groups $G_{\mathbf{R}} = Sp(p, q)$ and $SO^*(2n)$; K is the complexification of a maximal compact subgroup. The articles [1] and [2] contain analogous results for pairs arising from the real groups $SU(p, q), Sp(2n, \mathbf{R})$ and $O(p, q)$. The main result expresses the structure of the components in terms of iterated orbits of a sequence of reductive subgroups and certain unipotent subgroups of K; in particular, the structure is more complicated than the cases considered in [1] and [2], where the unipotent subgroups do not occur.

The results of this paper have applications to the theory of Harish-Chandra modules. To describe this we introduce a small amount of notation. Let $\mathcal{Q}$ be a closed K-orbit in the flag variety $\mathfrak{B}$ of G. Let $\gamma_{\mathcal{Q}} : T^*_{\mathcal{Q}}\mathfrak{B} \to \mathfrak{g}$ be the restriction of the moment map μ of $T^*\mathfrak{B}$ to the conormal bundle $T^*_{\mathcal{Q}}\mathfrak{B}$ of $\mathcal{Q}$ in $\mathfrak{B}$. The image of $\gamma_{\mathcal{Q}}$ is the closure of a nilpotent K-orbit in $\mathfrak{g}$. Then $\gamma_{\mathcal{Q}}^{-1}(f) = \mu^{-1}(f) \cap T^*_{\mathcal{Q}}\mathfrak{B}$ is a single irreducible component of the Springer fiber $\mu^{-1}(f)$, which we refer to as a component associated to the K-orbit $\mathcal{Q}$. By the Beilinson-Bernstein theory of Harish-Chandra modules there is a discrete series representation X_π of $G_{\mathbf{R}}$ associated to the closed K-orbit $\mathcal{Q}$. It is a fact that the associated variety of X_π is $im(\gamma_{\mathcal{Q}}) = \overline{K \cdot f}$. Furthermore, the multiplicity of $\overline{K \cdot f}$ in the associated cycle of X_π is the dimension of a space of sections over $\gamma_{\mathcal{Q}}^{-1}(f)$. See [4]. In this article we give (i) an algorithm to compute the associated variety of any discrete series representation, i.e., an algorithm to compute $K \cdot f$ (and a convenient f) from $\mathcal{Q}$, and (ii) a description of $\gamma_{\mathcal{Q}}^{-1}(f)$. We remark that [9] contains an algorithm to compute associated varieties which is quite different from ours; the point of our algorithm is that it allows us to understand the structure of $\gamma_{\mathcal{Q}}^{-1}(f)$. In the sequel to the present article ([3]), (i) and (ii) are used to give an algorithm that computes the multiplicities in the associated cycle of any discrete series representation of $G_{\mathbf{R}} = Sp(p,q)$ or $SO^*(2n)$. The method in [3] is different from that of [1] and [2]. Rather that directly computing the dimension of the space of sections over $\gamma_{\mathcal{Q}}^{-1}(f)$ for each closed orbit $\mathcal{Q}$, as is done for the classical groups $SU(p,q), Sp(2n,\mathbf{R})$ and $O(p,q)$ in [1] and [2], we compute the space of sections for just one discrete series representation in each Harish-Chandra cell that contains a discrete series representation, then we argue that the multiplicity in the associated cycle of *any* representation such a Harish-Chandra can be calculated.

1. Preliminaries

This section gives some necessary information used throughout the remainder of the article.

1.1. **Springer fibers.** The pairs (1) are symmetric pairs in the sense that K is the fixed point group of an involution Θ. Letting θ be the differential of Θ, we write the decomposition of $\mathfrak{g}$ into ± 1-eigenspaces as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The nilpotent cone in $\mathfrak{g}$ is denoted by $\mathcal{N}$ and we write $\mathcal{N}_\theta$ for $\mathcal{N} \cap \mathfrak{p}$. The cotangent bundle of $\mathfrak{B}$ may be identified with the homogeneous bundle $G \underset{B}{\times} \mathfrak{n}^-$, where $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ is some fixed base point (Borel subalgebra) in $\mathfrak{B}$ and $B = N_G(\mathfrak{b})$. Under this identification, and the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form, the moment map for the natural action of G on $T^*\mathfrak{B}$ is

$$\begin{aligned} &\mu : G \underset{B}{\times} \mathfrak{n}^- \to \mathcal{N} \\ &\mu(g,X) = g \cdot X \ (:= \mathrm{Ad}(g)X). \end{aligned}$$

Let $\mathcal{Q} \subset \mathfrak{B}$ be a closed K-orbit in $\mathfrak{B}$. Then the conormal bundle $T^*_{\mathcal{Q}}\mathfrak{B}$ of $\mathcal{Q}$ in $\mathfrak{B}$ may be identified with the homogeneous bundle

$$K \underset{K\cap B}{\times} (\mathfrak{n}^- \cap \mathfrak{p}).$$

We denote the restriction of μ to $T^*_{\mathcal{Q}}\mathfrak{B}$ by $\gamma_{\mathcal{Q}}$; it is given by the formula

$$\gamma_{\mathcal{Q}}(k,X) = k \cdot X.$$

The image of $\gamma_{\mathcal{Q}}$ is $K \cdot (\mathfrak{n}^- \cap \mathfrak{p})$, which contains a unique dense K-orbit. We say that $f \in \mathfrak{n}^- \cap \mathfrak{p}$ is *generic in* $\mathfrak{n}^- \cap \mathfrak{p}$ if the image of $\gamma_{\mathcal{Q}}$ is $\overline{K \cdot f}$. The fiber $\gamma_{\mathcal{Q}}^{-1}(f)$ may be identified with a subvariety of $\mathfrak{B}_K$ (the flag variety of K) via the natural map $T^*_{\mathcal{Q}}\mathfrak{B} \to \mathfrak{B}_K$. Under this identification

$$\gamma_{\mathcal{Q}}^{-1}(f) = N_K(f, \mathfrak{n}^- \cap \mathfrak{p})^{-1} \cdot \mathfrak{b} \tag{1.1}$$

where

$$N_K(f, \mathfrak{n}^- \cap \mathfrak{p}) = \{k \in K : k \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}\}.$$

This setup is carefully explained in [1, §2].

1.2. **Realizations of the pairs.** Each pair (G, K) we consider falls into one of the following types:

$$(Sp(2n), Sp(2p) \times Sp(2q)), n = p + q, \tag{C}$$
$$(SO(2n), GL(n)). \tag{D}$$

We refer to the two cases as types C and D. For the realizations, we use the matrices

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},\ S_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},\ \text{and}$$
$$K_{p,q} = \operatorname{diag}(I_p, -I_q, I_p, -I_q),\ \text{and}\ I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix},$$

where I_l is the identity $l \times l$ matrix.

Type C. In this case $G = \{g \in GL(2n) : gJ_ng^t = J_n\}$ and Θ is conjugation by $K_{p,q}$. It follows that $\theta = \operatorname{Ad}(K_{p,q})$ and $\mathfrak{k}$ equals

$$\left\{ \begin{pmatrix} z_1 & 0 & z_3 & 0 \\ 0 & z_2 & 0 & z_4 \\ z_5 & 0 & -z_1^t & 0 \\ 0 & z_6 & 0 & -z_2^t \end{pmatrix} : z_1 \in M_p(\mathbf{C}), z_2 \in M_q(\mathbf{C}) \text{ and } z_i^t = z_i,\ \text{for } i = 3, 4 \right\}$$

and $\mathfrak{k} \simeq \mathfrak{sp}(p) \times \mathfrak{sp}(q)$. The set of diagonal matrices in $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$ (and of $\mathfrak{g}$); we set

$$\mathfrak{h} = \{\operatorname{diag}(t_1, \dots, t_n, -t_1, \dots, -t_n) : t_i \in \mathbf{C}\}.$$

Define $\epsilon_i \in \mathfrak{h}^*$ by

$$\epsilon_i(\operatorname{diag}(t_1, \dots, t_n, -t_1, \dots, -t_n)) = t_i,\ 1 \le i \le n.$$

Then the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ is

$$\Delta(\mathfrak{h}, \mathfrak{g}) = \{\pm(\epsilon_i \pm \epsilon_j) : 1 \le i < j \le n\} \cup \{\pm 2\epsilon_i\}.$$

We fix once and for all a positive system of roots in $\mathfrak{k}$ by

$$\Delta_c^+ = \{\epsilon_i \pm \epsilon_j : 1 \le i < j \le p \text{ or } p + 1 \le i < j \le n\} \cup \{2\epsilon_i\}. \tag{1.2}$$

Our construction in §2 uses a specific normalization of root vectors. We specify this normalization using the standard basis $\{E_{i,j}\}$ of $M_{2n}(\mathbf{C})$ by setting

$$\begin{aligned} &X_{i-j} = E_{i,j} - E_{n+j,n+i},\ \text{for } \epsilon_i - \epsilon_j, \\ &X_{i+j} = E_{i,n+j} + E_{j,n+i},\ \text{for } \epsilon_i + \epsilon_j, i \ne j, \\ &X_{2\cdot i} = E_{i,n+i},\ \text{for } 2\epsilon_i, \\ &X_{-(i+j)} = E_{n+i,j} + E_{n+j,i},\ \text{for } -(\epsilon_i + \epsilon_j), i \ne j, \\ &X_{-2\cdot i} = E_{n+i,i},\ \text{for } -2\epsilon_i. \end{aligned} \tag{1.3}$$

Each root vector determines a linear transformation of $\mathbf{C}^{2n}$. Taking $\{e_j\}$ to be the standard basis of $\mathbf{C}^{2n}$ these linear transformations are given by

$$\begin{aligned}
&X_{i-j}e_k = \delta_{k,j}e_i - \delta_{k,n+i}e_{n+j},\\
&X_{i+j}e_k = \delta_{k,n+j}e_i + \delta_{k,n+i}e_j, i \neq j,\\
&X_{2\cdot i}e_k = \delta_{k,n+i}e_i,\\
&X_{-(i+j)}e_k = \delta_{k,j}e_{n+i} + \delta_{k,i}e_{n+j}, i \neq j,\\
&X_{-2\cdot i}e_k = \delta_{k,i}e_{n+i}.
\end{aligned} \tag{1.4}$$

Let ω be the symplectic form on $\mathbf{C}^{2n}$ having matrix J_n with respect to the ordered basis $\{e_1, \dots, e_{2n}\}$.

There is an involution of $\{1, 2, \dots, 2n\}$ defined by

$$\tau(i) = \begin{cases} i+n, & \text{if } 1 \leq i \leq n \\ i-n, & \text{if } n+1 \leq i \leq 2n. \end{cases} \tag{1.5}$$

Note that $\omega(e_i, e_{\tau(i)}) = \pm 1$. We will use the following fact. If S_1 is a τ-stable subset of $\{1, 2, \dots, 2n\}$, then $W_1 := \operatorname{span}_{\mathbf{C}}\{e_i : i \in S_1\}$ is a subspace of $\mathbf{C}^{2n}$ on which ω is nondegenerate. Furthermore, the orthogonal complement (w.r.t. ω) of W_1 is $V_1 := \operatorname{span}_{\mathbf{C}}\{e_j : j \notin S_1\}$ and $\mathbf{C}^{2n} = W_1 \oplus V_1$.

Type D. In this case $G = \{g \in GL(2n) : gS_ng^t = S_n\}$, Θ is conjugation by $I_{n,n}$ and $\theta = \operatorname{Ad}(I_{n,n})$. Therefore,

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} : A \in \mathfrak{gl}(n) \right\} \simeq \mathfrak{gl}(n).$$

As for type C, we take $\mathfrak{h}$ to be the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{g}$. Then, with ϵ_i as for type C, the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ is

$$\Delta(\mathfrak{h}, \mathfrak{g}) = \{\pm(\epsilon_i \pm \epsilon_j) : 1 \leq i < j \leq n\}$$

and we fix a system of positive roots in $\mathfrak{k}$ by setting

$$\Delta_c^+ = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq n\}. \tag{1.6}$$

We specify normalized noncompact root vectors by taking

$$\begin{aligned}
&X_{i+j} = E_{i,n+j} - E_{j,n+i}, \text{ for } \epsilon_i + \epsilon_j, i < j,\\
&X_{-(i+j)} = E_{n+j,i} - E_{n+i,j}, \text{ for } -(\epsilon_i + \epsilon_j), i < j.
\end{aligned} \tag{1.7}$$

When viewed as linear transformations of $\mathbf{C}^{2n}$

$$\begin{aligned}
&X_{i+j}e_k = \delta_{k,n+j}e_i - \delta_{k,n+i}e_j,\\
&X_{-(i+j)}e_k = \delta_{k,i}e_{n+j} - \delta_{k,j}e_{n+i}.
\end{aligned} \tag{1.8}$$

Let $(\ ,\ )$ be the nondegenerate symmetric form on $\mathbf{C}^{2n}$ having matrix S_n with respect to the standard basis. Then the involution τ defined above has properties parallel to those stated for type C.

1.3. **Nilpotent orbits.** We recall the parametrization of K-orbits in $\mathcal{N}_\theta$ for the pairs (1) under consideration. This parametrization is given in terms of signed tableaux, and is well known; see for example [6, Ch. 9].

Consider $f \in \mathcal{N}_\theta$ and form a standard triple $\{e, h, f\}$ with $e \in \mathfrak{p}$ and $h \in \mathfrak{h}$. Denote by $SL(2)_f$ the corresponding complex subgroup of G. The standard representation of G on $\mathbf{C}^{2n}$ restricts to a representation of $SL(2)_f$. This restriction may be decomposed into a direct sum of irreducible subrepresentations. We choose these irreducible constituents to be stable under $K_{p,q}$ (resp., $I_{n,n}$) for type C (resp., type D). Furthermore, each such constituent is isotropic with respect to the symplectic form ω (resp., symmetric form $(\ ,\ )$). The constituents may be paired off as follows. For each constituent V there is a constituent V', which is equivalent to V, so that ω (resp., $(\ ,\ )$) is nondegenerate on $V \oplus V'$. If v_1 is a highest weight vector for V and v_1' a highest weight vector for V', then

$$v_i := f^{i-1}v_1 \text{ and } v_i' := f^{i-1}v_1', \text{ for } i = 1, 2, \ldots, d,$$

where $d = \dim(V) = \dim(V')$, form bases of V and V'. Note that each v_i and v_i' is an h-weight vector. Therefore, each v_i and v_i' is an eigenvector for $K_{p,q}$ (resp., $I_{n,n}$) for type C (resp., type D). The vectors v_i and v_i' may be normalized so that $\omega(v_i, v_j') = (-1)^{i-1}\delta_{i,d-j+1}$ (resp., $(v_i, v_j') = (-1)^{i-1}\delta_{i,d-j+1}$). The eigenvalues of v_i alternate in the sense that the eigenvalue of v_{i+1} is the negative of the eigenvalue of v_i; the same holds for the v_i'.

The signed tableau associated to an orbit $K \cdot f \subset \mathcal{N}_\theta$ has one row for each irreducible constituent in the decomposition of $\mathbf{C}^{2n}$ into $SL(2)_f$-representations. The number of boxes in each row is the dimension of the constituent. A plus or minus sign is placed in each box so as to alternate along each row. The k^{th} row begins with the sign of the eigenvalue of the lowest weight vector of the k^{th} constituent. Two signed tableau are considered to be the same when they differ only by a permutation of the rows. The following proposition gives the parametrization of $K\backslash\mathcal{N}_\theta$.

Proposition 1.9. *When (G, K) is of type C (resp., type D) the K-orbits on $\mathcal{N}_\theta$ are in one-to-one correspondence with signed tableau having $2n$ boxes that are filled with $\pm$ signs that alternate along each row and*

(a) *the number of rows of a given even (resp., odd) length starting with a $+$ sign coincides with the number starting with a $-$ sign, and*
(b) *the number of rows of a given odd (resp., even) length starting with a $+$ sign is even and the number starting with a $-$ sign is also even.*

The number of $+$ signs is p (resp., n) and the number of $-$ signs is q (resp., n).

In §4 we will use the following formulas for the dimensions of the centralizers of nilpotent elements. These formulas may be found in [6].

Suppose that the size of the i^{th} column of the tableau of f is c_i. Then

$$\dim(Z_G(f)) = \frac{1}{2}\big(\sum c_i^2 \pm \#(\text{odd rows})\big), \tag{1.10}$$

where '+' occurs for type C and '$-$' occurs for type D.

1.4. **A comment on centralizers.** For the special case of a nilpotent orbit $K \cdot f$ having tableau consisting of just two rows, we will need to specify certain elements of the centralizer explicitly.

In the case of two rows, the two rows are of the same length (by Prop 1.9). Let $v_1, \dots, v_n$ and $v_1', \dots, v_n'$ be bases of V and V' described in §1.3. Observe that if $Z \in \mathfrak{z}_{\mathfrak{k}}(f)$, then Z is determined by $Z(v_1)$ and $Z(v_1')$ (since $Z(v_l) = Z(f^{l-1}v_1) = f^{l-1}(Z(v_1))$, and similarly for $Z(v_l')$). The elements of the centralizer that we wish to write down have a slightly different form in each of four cases (types C or D, and n even or odd). In each case $Z|_V = 0$ and $Z(V') \subset V$.

Type C, n even. For $i = 1, 2, \dots, \frac{n}{2}$, define Z_i to be the element of $\mathfrak{z}_{\mathfrak{k}}(f)$ determined by $Z_i(v_1) = 0$ and $Z_i(v_1') = v_{2i}$. Therefore,

$$\begin{aligned} &Z_i(v_l) = 0 \ \text{ and} \\ &Z_i(v_l') = f^{l-1}v_{2i} \ \ (= v_{l+2i-1} \text{ or } 0), \end{aligned}$$

for $l = 1, \dots, n$.

Type C, n odd. For $i = 1, 2, \dots, \frac{n+1}{2}$, elements $Z_i \in \mathfrak{z}_{\mathfrak{k}}(f)$ are determined by $Z_i(v_1) = 0$ and $Z_i(v_1') = v_{2i-1}$. Therefore,

$$\begin{aligned} &Z_i(v_l) = 0 \ \text{ and} \\ &Z_i(v_l') = f^{l-1}v_{2i-1} \ \ (= v_{l+2i-2} \text{ or } 0), \end{aligned}$$

for $l = 1, \dots, n$.

Type D, n even. For $i = 1, 2, \dots, \frac{n}{2}$, define Z_i to be the element of $\mathfrak{z}_{\mathfrak{k}}(f)$ determined by $Z_i(v_1) = 0$ and $Z_i(v_1') = v_{2i-1}$. Therefore,

$$\begin{aligned} &Z_i(v_l) = 0 \ \text{ and} \\ &Z_i(v_l') = f^{l-1}v_{2i-1} \ \ (= v_{l+2i-2} \text{ or } 0), \end{aligned}$$

for $l = 1, \dots, n$.

Type D, n odd. For $i = 1, 2, \dots, \frac{n-1}{2}$, elements $Z_i \in \mathfrak{z}_{\mathfrak{k}}(f)$ are determined by $Z_i(v_1) = 0$ and $Z_i(v_1') = v_{2i}$. Therefore,

$$\begin{aligned} &Z_i(v_l) = 0 \ \text{ and} \\ &Z_i(v_l') = f^{l-1}v_{2i} \ \ (= v_{l+2i-1} \text{ or } 0), \end{aligned}$$

for $l = 1, \dots, n$.

Lemma 1.11. *In each case, the Z_i are independent elements of $\mathfrak{z}_{\mathfrak{k}}(f)$.*

Proof. We check that $Z_i \in \mathfrak{z}_{\mathfrak{k}}(f)$ in the first case; the proofs in the other cases are essentially the same. Let $\varepsilon = \pm 1$ be the eigenvalue of the $K_{p,q}$-eigenvector v_1'. First, each Z_i clearly commutes with f. To check that $Z \in \mathfrak{g}$ we show that the bilinear form ω is preserved.

$$\begin{aligned}
&\omega(Z_i(v_l'), v_j') + \omega(v_l', Z_i(v_j')) \\
&\quad = \omega(v_{l+2i-1}, v_j') + \omega(v_l', v_{j+2i-1}) \\
&\quad = (-1)^{l+2i-1}\delta_{l+2i-1,n-j+1} - (-1)^{j+2i-1}\delta_{l,n-(j+2i-1)+1} \\
&\quad = 0,
\end{aligned}$$

To check that $Z_i \in \mathfrak{k}$, we check that Z_i preserves eigenspaces of $K_{p,q}$, so commutes with $K_{p,q}$. Therefore Z_i is fixed by θ, so is in $\mathfrak{k}$. First, the $K_{p,q}$-eigenvalues of v_n and v_1' are equal. (Since n is even, the eigenvalue of v_n is $\omega(K_{p,q}v_n, v_1') = \omega(v_n, K_{p,q}v_1') = \varepsilon\omega(v_n, v_1') = \varepsilon$.) It follows that v_j has eigenvalue $(-1)^j\varepsilon$ and v_l' has eigenvalue $(-1)^{l-1}\varepsilon$. Now

$$K_{p,q}(z_i(v_l')) = K_{p,q}(v_{l+2i-1}) = (-1)^{l+2i-1}\varepsilon v_{l+2i-1}.$$

But $(-1)^{l+2i-1}\varepsilon = (-1)^{l-1}\varepsilon$ is the eigenvalue of v_l', so Z_i preserves eigenspaces.

Independence is easy to check. $\square$

1.5. **Closed K-orbits in $\mathfrak{B}$.** Let $\mathfrak{h}$ and Δ_c^+ be as in §1.2. Each regular λ defines a positive system of roots $\Delta^+ = \{\alpha : \langle\lambda, \alpha\rangle > 0\}$ and a Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-, \quad \mathfrak{n}^- = \sum_{\alpha\in\Delta^+} \mathfrak{g}^{(-\alpha)}.$$

Therefore, each such λ determines a K-orbit $\mathcal{Q} = K \cdot \mathfrak{b} \subset \mathfrak{B}$. Such K-orbits $\mathcal{Q}$ are closed since $\mathcal{Q} \simeq K/K \cap B$ and $K \cap B$ is a Borel subgroup of K, so $\mathcal{Q} \simeq \mathfrak{B}_K$. The closed K-orbits in $\mathfrak{B}$ are in one-to-one correspondence with the W_K-orbits of Weyl chambers for $\mathfrak{g}$. For the two types of pairs considered here, these are in one-to-one correspondence with W-conjugates λ of $(n, n-1, \ldots, 2, 1)$ that are Δ_c^+-dominant. Thus for

$$\begin{aligned}
&\text{type C} : \lambda = (a_1, \ldots, a_p, b_1, \ldots, b_q), \\
&\qquad \text{with } a_1 > \cdots > a_p > 0,\ b_1 > \cdots > b_q > 0 \text{ and} \\
&\text{type D} : \lambda = (a_1, \ldots, a_p, b_1, \ldots, b_q), \\
&\qquad \text{with } a_1 > \cdots > a_p > 0 > b_1 > \cdots > b_q, \text{ for some } p, q.
\end{aligned} \tag{1.12}$$

In order to construct generic elements $f \in \mathfrak{n}^- \cap \mathfrak{p}$ we associate to a λ (as above) an array of numbered dots. This array consist of two horizontal rows of dots as follows. Begin with the coordinate of λ having the greatest absolute value. Note that in type C the absolute value plays no role and the greatest coordinate of λ is either a_1 or b_1. In type D the coordinate of greatest absolute value is either a_1 or b_q. Begin the array by placing a dot in the upper row if this coordinate is among the a_i's and in the lower row otherwise. Working from left to right, place the second dot in the upper (resp., lower) row if the coordinate having greatest absolute value is among the a_i's (resp., b_i's). Continue until n dots have been placed. Label each dot with the index of the corresponding coordinate of λ and give this label the sign of the corresponding coordinate. Note that in type C the dots are labeled $1, 2, \ldots, p$ along the upper row (left to right) and $p+1, p+2, \ldots, n$ along the lower row, and in type D the upper row is labeled the same way and the lower row is labeled with $-n, -(n-1), \ldots, -(p+1)$.

Here are two examples. In type C take $\lambda = (9, 8, 5, 2, 1 \,|\, 7, 6, 4, 3)$. Then $p = 5$ and $q = 4$ and the corresponding array is

$$\begin{array}{ccccccccc}
\overset{1}{\bullet} & \overset{2}{\bullet} & & & \overset{3}{\bullet} & & & \overset{4}{\bullet} & \overset{5}{\bullet} \\
\\
& & \underset{6}{\bullet} & \underset{7}{\bullet} & & \underset{8}{\bullet} & \underset{9}{\bullet} & &
\end{array}.$$

Note that the simple roots are

$$\begin{aligned}
&\epsilon_1 - \epsilon_2, \epsilon_4 - \epsilon_5, \epsilon_6 - \epsilon_7, \epsilon_8 - \epsilon_9, 2\epsilon_5 \text{ (compact)} \\
&\epsilon_2 - \epsilon_6, \epsilon_7 - \epsilon_3, \epsilon_3 - \epsilon_8, \epsilon_9 - \epsilon_4 \text{ (noncompact).}
\end{aligned}$$

The positive roots are $\epsilon_i - \epsilon_j$, with i the label of a dot to the left of the dot labeled by j, and all $\epsilon_i + \epsilon_j$. The compact (resp, noncompact) roots have i, j in the same (resp., different) rows.

For type D take $\lambda = (10, 9, 8, 5, 2, 1 \,|\, -3, -4, -6, -7)$. Then the array is

$$\begin{array}{cccccccccc}
\overset{1}{\bullet} & \overset{2}{\bullet} & \overset{3}{\bullet} & & & \overset{4}{\bullet} & & & \overset{5}{\bullet} & \overset{6}{\bullet} \\
\\
& & & \underset{-10}{\bullet} & \underset{-9}{\bullet} & & \underset{-8}{\bullet} & \underset{-7}{\bullet} & &
\end{array}.$$

Note that the simple roots are

$$\begin{aligned}
&\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_5 - \epsilon_6, \epsilon_9 - \epsilon_{10}, \epsilon_7 - \epsilon_8 \text{ (compact)} \\
&\epsilon_3 + \epsilon_{10}, -(\epsilon_9 + \epsilon_4), \epsilon_4 + \epsilon_8, -(\epsilon_7 + \epsilon_5), \epsilon_5 + \epsilon_6 \text{ (noncompact).}
\end{aligned}$$

2. Generic elements

An algorithm is given to associate to each closed K-orbit in $\mathfrak{B}$ a nilpotent element $f \in \mathcal{N}_\theta$ so that $im(\gamma_{\mathfrak{Q}}) = \overline{K \cdot f}$. Specifically, given λ as in (1.12) (and therefore a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$) we explicitly determine a generic element f in $\mathfrak{n}^- \cap \mathfrak{p}$. The proof that the nilpotent element we construct is in fact generic is postponed until §4.4. We also associate to f some subgroups which play a crucial role in our description of $\gamma_{\scriptscriptstyle Q}^{-1}(f)$.

2.1. **The algorithm.** Let us assume that $\lambda \in \mathfrak{h}^*$ is regular and Δ_c^+-dominant. Form the array of numbered dots as in §1.5. In the array, call any maximal set of consecutive dots in a row a *block*. We construct a *string* through the array as follows. The first part of the string consists of the dots farthest to the right in each block. Let $a_1, a_2, \dots, a_{\ell+1}$ be the labels of these dots, listed from left to right. The second part of the string is formed as follows. In type C choose the dot farthest to the right that is in the row opposite to that of the dot labeled by $a_{\ell+1}$ and is not among the dots $a_1, a_2, \dots, a_\ell$. In type D choose the dot farthest to the right that is in the same row as the dot labeled by $a_{\ell+1}$ and is not among the dots labeled by $a_1, \dots, a_{\ell+1}$. For both types continue by choosing the dot farthest to the right that is (i) left of the most recently chosen dot, (ii) is in the row opposite to that of the most recently chosen dot, and (iii) is not among those already chosen. Continue in this manner until no dot satisfies (i)-(iii). Let $b_1, b_2, \dots, b_{N-\ell-1}$ be the labels of these dots, listed from left to right. The string is then the collection of the N dots and may be pictured in the array.

This string determines a nilpotent element in $\mathfrak{n}^- \cap \mathfrak{p}$:

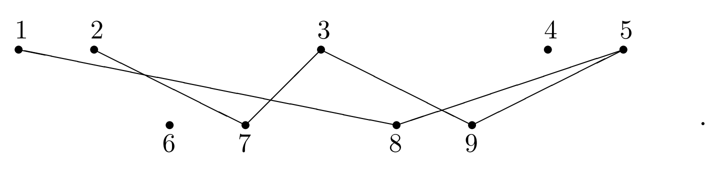
$$f_0 = \Big(\sum_{i=1}^{\ell} X_{a_{i+1}-a_i}\Big) + X_{-a_{\ell+1}-b_{N-\ell-1}} + \Big(\sum_{i=1}^{N-\ell-2} X_{b_{i+1}-b_i}\Big).$$

Now continue by deleting the dots in the first string, thus getting a smaller array, and forming a string in this array as specified above. Then f_1 is defined in the same manner as f_0 is defined. Repeat this procedure until no more strings can be formed (that is, until all dots are in the same row or there are no dots left). Then set

$$f = f_0 + f_1 + \cdots f_{m-1},$$

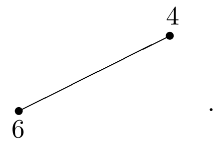
where m is the number of strings that can be formed.

For the earlier type C example we have



Then $\ell = 4$, $N = 7$, and $a_1 = 2, a_2 = 7, a_3 = 3, a_4 = 9, a_5 = 5, b_1 = 1$ and $b_2 = 8$.

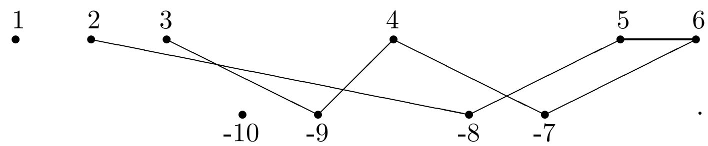
The smaller array, with the second string drawn in, is



Therefore,

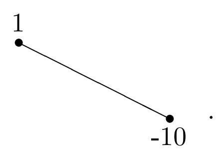
$$f = f_0 + f_1 = (X_{7-2} + X_{3-7} + X_{9-3} + X_{5-9} + X_{-5-8} + X_{8-1}) + (X_{4-6}).$$

In the type D example of §1.5 we have



Then $\ell = 4$, $N = 8$, and $a_1 = 3, a_2 = -9, a_3 = 4, a_4 = -7, a_5 = 6, b_1 = 2$, $b_2 = -8$ and $b_3 = 5$.

Removing the string gives the following smaller array (with the second string drawn in):



Therefore, $f = f_0 + f_1$ is

$$(X_{-(3+9)} + X_{9+4} + X_{-(4+7)} + X_{7+6} + X_{-(6+5)} + X_{5+8} + X_{-(8+2)}) + (X_{-(1+10)}).$$

Theorem 2.1. *The nilpotent element f constructed by the algorithm above is generic in $\mathfrak{n}^- \cap \mathfrak{p}$.*

This statement will be proved in §4.4, where it will follow easily from (4.18).

2.2. **Subgroups.** A number of subgroups play an important role in our description of $\gamma_{_Q}^{-1}(f)$. We now define a sequence of subgroups of G which will allow us to formulate and prove our results inductively. Set

$$G_0 = G, K_0 = K \text{ and } V_0 = \mathbf{C}^{2n}.$$

Define $G_{2j}, j = 1, 2, \dots, m$, inductively as follows. Let S_{2j} be the labels of the dots in the j^{th} string. Therefore, in the notation of the previous subsection, we have

$$S_2 = \{a_1, \dots, a_{\ell+1}, b_1, \dots, b_{N-\ell-1}\}.$$

Then

$$W_{2j} := \operatorname{span}_{\mathbf{C}}\{e_i : i \in S_{2j} \cup \tau(S_{2j})\} \text{ and}$$
$$V_{2j} := \operatorname{span}_{\mathbf{C}}\{e_i : i \notin S_{2j} \cup \tau(S_{2j})\} \cap V_{2(j-1)}$$

are nondegenerate subspaces of $V_0 = \mathbf{C}^{2n}$. Also, $V_{2(j-1)} = V_{2j} \oplus W_{2j}$. The subgroups G_{2j} are defined by

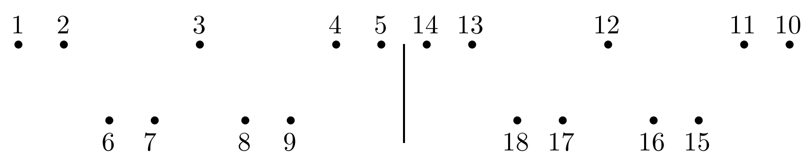
$$G_{2j} := \{g \in G_{2(j-1)} : g(V_{2j}) \subset V_{2j} \text{ and } g|_{W_{2j}} = Id\}.$$

Then we define $K_{2j} := K \cap G_{2j}$ and $\mathfrak{h}_{2j} := \mathfrak{h} \cap \mathfrak{g}_{2j}$ (a Cartan subalgebra of both $\mathfrak{k}_{2j}$ and $\mathfrak{g}_{2j}$). The following statements are easily verified.

(1) (G_{2j}, K_{2j}) is a pair of the same type as (G, K).
(2) $\lambda_{2j} := \lambda|_{\mathfrak{h}_{2j}}$ determines the Borel subalgebra $\mathfrak{b}_{2j} := \mathfrak{b} \cap \mathfrak{g}_{2j}$, which we write as $\mathfrak{b}_{2j} = \mathfrak{h}_{2j} + \mathfrak{n}_{2j}^-$.
(3) For any $j = 0, 1, \ldots, m-1$, $f_j + f_{j+1} + \cdots + f_{m-1}$ is the generic element of $\mathfrak{n}_{2j}^- \cap \mathfrak{p}$ constructed by the algorithm applied to the pair (G_{2j}, K_{2j}) and λ_{2j}.
(4) G_{2j} centralizes $f_0 + \cdots + f_{j-1}$; in particular $G_{2m} \subset Z_G(f)$.

2.3. **The doubled array.** It is useful to consider a 'doubled array'. Types C and D will be described separately.

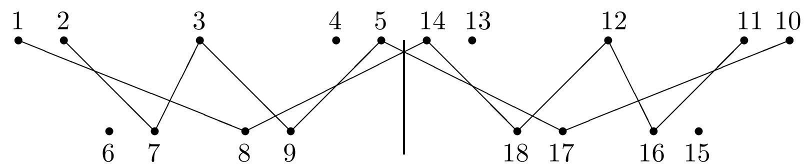
Type C. The doubled array is formed by reflecting the array about a vertical line just to the right of the array. This array is numbered by keeping the numbering on the left-hand side and labeling the top row, right of center, with $n+p, n+p-1, \ldots, n+1$, and the dots in the lower row, right of center, with $2n, 2n-1, \ldots, n+p+1$. Note that if the label of a dot is k, then the label of its reflection is $\tau(k)$. Here is the doubled array of the earlier example in type C.



The string through the array is represented by two sets of labels of dots in the doubled array as follows. Recall that we have $a_1, \ldots, a_{\ell+1}$ and $b_1, \ldots, b_{N-\ell-1}$, which define f_0. Extend to $a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ by setting

$$a_i = \tau(b_{N-i+1}), \quad \text{for } i = \ell+2, \ldots, N$$
$$b_i = \tau(a_{N-i+1}), \quad \text{for } i = N-\ell, \ldots, N.$$

These two sets of dots are displayed on the doubled array by connecting the consecutive dots labeled by the a_i and also connecting those labeled be the b_i. In the example this is



Note that in this example the action of f_0, up to a multiple of ± 1, on the standard basis vectors is by

$$e_2 \to e_7 \to e_3 \to e_9 \to e_5 \to e_{17} \to e_{10} \to 0$$
$$e_1 \to e_8 \to e_{14} \to e_{18} \to e_{12} \to e_{16} \to e_{11} \to 0.$$

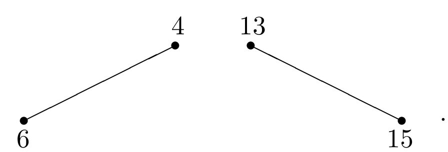
In general

$$f_0(e_{a_i}) = \begin{cases} \pm e_{a_{i+1}}, & i = 1, 2, \dots, N-1 \\ 0, & i = N \end{cases} \tag{2.2}$$

and

$$f_0(e_{b_i}) = \begin{cases} \pm e_{b_{i+1}}, & i = 1, 2, \dots, N-1 \\ 0, & i = N. \end{cases} \tag{2.3}$$

In our example, the smaller doubled array, with string drawn in, is



The action of f_1, up to a multiple of ± 1, on the standard basis vectors is by

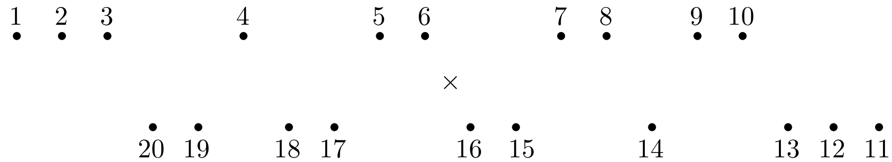
$$e_6 \to e_4 \to 0 \text{ and } e_{13} \to e_{15} \to 0.$$

It follows immediately that the signed tableau may be read from the doubled array. In the example, the signed tableau is

+	-	+	-	+	-	+
+	-	+	-	+	-	+
+	-					
-	+					

.

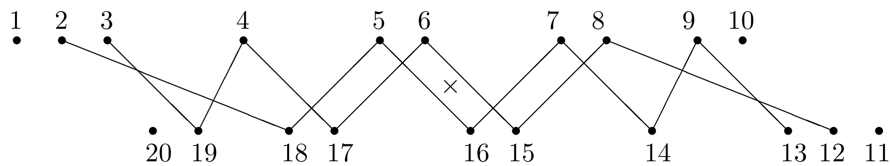
Type D. The doubled array is formed by reflecting the array about a point just to the right of the array. Then renumbering by labeling the dots along the first row with $1, 2, \dots, 2p$, from left to right. Label the dots along the bottom row with $2n, 2n-1, \dots, 2p = 2, 2p+1$. In the example, we reflect about the '$\times$' to get



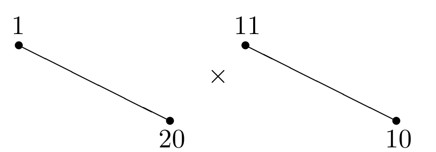
As in the type C case, the string is also reflected to get a pair of strings. This is described slightly differently than in the type C case. Let $a'_1, \dots, a'_{\ell+1}$ be the new labels of the dots labelled by $a_1, \dots, a_{\ell+1}$ in the original array, and similarly let $b'_1, \dots, b'_{N-\ell-1}$ be the new labels of the dots originally labeled by $b_1, \dots, b_{N-\ell-1}$. Then extend to $a'_1, \dots, a'_N$ and $b'_1, \dots, b'_N$ by setting

$$\begin{aligned} a'_i &= \tau(b'_{N-i+1}), \quad \text{for } i = \ell+2, \dots, N \\ b'_i &= \tau(a'_{N-i+1}), \quad \text{for } i = N-\ell, \dots, N. \end{aligned}$$

Then the string is represented by the sequences of dots labeled by the a'_i and the b'_i. Then the formulas of (2.2) and (2.3) for f_0 hold, with the a_i (resp., b_i) replaced by the a'_i (resp., b'_i). The array with the string indicated is



The smaller doubled array is



The tableau has two rows of length 8 and two of length 2; all begin with a '$-$' sign.

2.4. **Parabolic subgroups.** A sequence of parabolic subgroups Q_{2j} of K_{2j} will play an important role for us. These subgroups are analogous to the Q_j used in [1] to describe $\gamma_{\mathrm{Q}}^{-1}(f)$.

Let $\mathfrak{q} \subset \mathfrak{k}$ be the parabolic subalgebra defined by the set of roots in Δ_c^+ that are simple for Δ^+ (the positive system defined by λ). Write R for the roots in the span of these simple roots, then

$$\begin{aligned}\mathfrak{q} &= \mathfrak{l} + \mathfrak{u}^-, \\ \mathfrak{l} &= \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}^{(\alpha)} \\ \mathfrak{u}^- &= \sum_{\alpha \in \Delta_c^+ \setminus R} \mathfrak{g}^{(-\alpha)}.\end{aligned}$$

Note that $\mathfrak{q}$ contains the Borel subalgebra $\mathfrak{b} \cap \mathfrak{k}$, so is a parabolic subalgebra of $\mathfrak{k}$. Let $Q = N_K(\mathfrak{q})$ and write $Q = LU^-$.

Lemma 2.4. *L normalizes $\mathfrak{n}^- \cap \mathfrak{p}$.*

Proof. There is a parabolic subalgebra of $\mathfrak{g}$ defined by the same set of simple roots that define $\mathfrak{q}$. This parabolic subalgebra is $\mathfrak{l} + (\mathfrak{u}^- + \mathfrak{n}^- \cap \mathfrak{p})$. Since $L \subset K$ and normalizes $\mathfrak{u}^- + \mathfrak{n}^- \cap \mathfrak{p}$, L also normalizes $\mathfrak{n}^- \cap \mathfrak{p}$. ☐

Both B and Q may be described in terms of the doubled array. This is done by specifying flags having stabilizers B and Q. We first consider B. The subspaces E_i in $\mathbf{C}^{2n}$ given by

$$E_i = \text{span}_{\mathbf{C}}\{e_k : k \text{ is among the labels of the } i \text{ dots farthest to the right}\},$$

for $i = 1, 2, \ldots, 2n$, define a full flag

$$\{0\} \subset E_1 \subset \cdots \subset E_{2n-1} \subset E_{2n} = \mathbf{C}^{2n}. \tag{2.5}$$

The partial flag

$$\{0\} \subset E_1 \subset \cdots \subset E_n \tag{2.6}$$

is an isotropic flag. Note that $E_i^\perp = E_{2n-i}$.

Lemma 2.7. *The stabilizer of either the flag (2.5) or the flag (2.6) is B.*

Similarly, a flag of length $2\ell+1$ (resp., $2\ell+2$) in type C (resp., type D) is defined by

$$\begin{aligned} F_i = \text{span}_{\mathbf{C}}\{e_k :\ & k \text{ is the label of a dot in one} \\ & \text{of the } i \text{ blocks farthest to the right}\},\end{aligned}$$

for $i = 1, 2, \ldots, 2\ell+1$ (resp., $2\ell+2$). (Note that $2\ell+1$ (resp., $2\ell+2$) is the number of blocks in the doubled array.) The corresponding isotropic flags are

$$\begin{aligned} &\{0\} \subset F_1 \subset \cdots \subset F_\ell, \text{ for type C, and} \\ &\{0\} \subset F_1 \subset \cdots \subset F_{\ell+1}, \text{ for type D.}\end{aligned} \tag{2.8}$$

Lemma 2.9. *The stabilizer of either the flag (F_i) of length $2\ell+1$ (for type C) or $2\ell+2$ (for type D), or the corresponding isotropic flag, is Q.*

The Levi subgroup L of Q is isomorphic to

$$\begin{aligned} &GL(n_1) \times \cdots GL(n_s) \times Sp(2n_{s+1}), \text{ type C, and} \\ &GL(n_1) \times \cdots GL(n_s) \times GL(n_{s+1}), \text{ type D,} \end{aligned} \tag{2.10}$$

where $n_1, \dots, n_{s+1}$ are the sizes of the blocks in the (original) array.

We set $Q_0 = Q$. The parabolic subgroups Q_{2j} of K_{2j}, $j = 1, \dots, m$, are defined in the same manner, that is, by considering the array with the dots of the first j strings omitted.

Remark 2.11. One easily verifies that

$$Q_{2m} \cdots Q_2 Q_0 \cdot \mathfrak{b} \subset \gamma_{\mathfrak{Q}}^{-1}(f). \tag{2.12}$$

This may be done by showing

$$Q_0 Q_2 \cdots Q_{2m} \cdot f \subset \mathfrak{n}^- \cap \mathfrak{p}$$

as follows. By induction

$$Q_2 \cdots Q_{2m} \cdot (f - f_0) \subset \mathfrak{n}_2^- \cap \mathfrak{p},$$

since $f - f_0$ is the generic element for the pair (G_2, K_2). Now

$$\begin{aligned} Q_0 Q_2 \cdots Q_{2m} \cdot f &\subset Q_0(f_0 + Q_2 \cdots Q_{2m} \cdot (f - f_0)) \\ &\subset Q_0(f_0 + \mathfrak{n}_2^- \cap \mathfrak{p}), \text{ by induction,} \\ &\subset Q_0 \cdot (\mathfrak{n}^- \cap \mathfrak{p}), \text{ since } \mathfrak{n}_2^- \cap \mathfrak{p} \subset \mathfrak{n}^- \cap \mathfrak{p}, \\ &\subset \mathfrak{n}^- \cap \mathfrak{p}, \text{ by Lemma 2.4.} \end{aligned}$$

The results of [1] and [2] suggest that equality might hold in (2.12). However, this is not the case. For instance, in the type C example considered earlier, if σ_{68} is the reflection in the root $\epsilon_6 - \epsilon_8$, then $\sigma_{68} \in N_K(f, \mathfrak{n}^- \cap \mathfrak{p})$, but $\sigma_{68} \cdot \mathfrak{b}$ is not in the left-hand side of (2.12). The description of $\gamma_{\mathfrak{Q}}^{-1}(f)$ is much more subtle and requires the introduction of several more subgroups; this is contained in the following section.

3. The structure of the fiber

A description of $\gamma_{\mathfrak{Q}}^{-1}(f)$ is given for any closed K-orbit $\mathfrak{Q}$ in $\mathfrak{B}$ and corresponding generic element f. The statement is contained in Theorem 3.3. This description is in terms of (i) several reductive subgroups of K that are naturally defined in terms of the array and (ii) some special one-parameter subgroups of $Z_K(f)$.

3.1. **Subgroups of K_{2j}.** Recall that subgroups

$$G = G_0 \supset G_2 \supset \cdots \supset G_{2m}$$

have already been defined, as have $K_{2j} = K \cap G_{2j}$. The pairs (G_{2j}, K_{2j}) are of the same type as (G, K). Reductive subgroups $L_1, \widehat{L}_1$ and $\widehat{\widehat{L}}_1$ will be defined below.

Applying the same definition to the pair (G_{2j}, K_{2j}) results in reductive subgroups $L_{2j+1}, \widehat{L}_{2j+1}$ and $\widehat{\widehat{L}}_{2j+1}$ in K_{2j}, for any $j = 0, 1, \ldots, m$.

Let $S_1 = \{a_1, \ldots, a_\ell\}$, the set of labels of the first ℓ dots in the first string, and set

$$\begin{aligned} W_1 &= \operatorname{span}_{\mathbf{C}}\{e_k : k \in S_1 \cup \tau(S_1)\} \text{ and} \\ V_1 &= \operatorname{span}_{\mathbf{C}}\{e_k : k \notin S_1 \cup \tau(S_1)\}. \end{aligned}$$

Then $\mathbf{C}^{2n} = W_1 \oplus V_1$, a direct sum of orthogonal subspaces of $\mathbf{C}^{2n}$. Define, in analogy with the definition of G_2,

$$G_1 = \{g \in G : g(V_1) \subset V_1 \text{ and } g|_{W_1} = Id\},$$

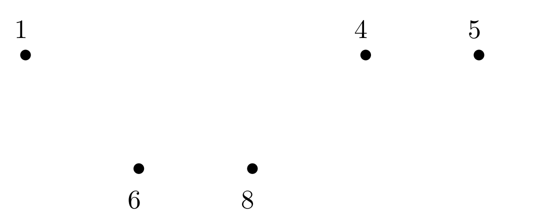
and set $K_1 = K \cap G_1$. Then $\mathfrak{h}_1 := \mathfrak{h} \cap \mathfrak{g}_1$ is a Cartan subalgebra of $\mathfrak{g}_1$ and $\lambda|_{\mathfrak{h}_1}$ is regular. Therefore, $\lambda|_{\mathfrak{h}_1}$ defines a positive system of roots Δ_1^+ in $\Delta(\mathfrak{h}_1, \mathfrak{g}_1)$ (by $\langle \lambda|_{\mathfrak{h}_1}, \alpha\rangle > 0$). This gives a Borel subalgebra $\mathfrak{b}_1 = \mathfrak{h}_1 + \mathfrak{n}_1^- = \mathfrak{h}_1 + \sum_{\alpha\in\Delta_1^+} \mathfrak{g}^{(-\alpha)}$ $(= \mathfrak{b} \cap \mathfrak{g}_1)$. An array for Δ_1^+ is obtained from the one for Δ^+ by omitting the dots labeled by $a_1, \ldots, a_\ell$. We refer to the blocks in this array for Δ_1^+ as *1-blocks*.

Now we are in position to define L_1. Let R_1 be the roots in the span of the compact roots that are simple for Δ_1^+. Let

$$\mathfrak{q}_1 = \mathfrak{l}_1 + \mathfrak{u}_1^- = \big(\mathfrak{h}_1 + \sum_{\alpha\in R_1} \mathfrak{g}^\alpha\big) + \big(\sum_{\alpha\in\Delta_c^+\setminus R_1} \mathfrak{g}^{(-\alpha)}\big).$$

The corresponding parabolic subgroup is $Q_1 = L_1 U_1^-$.

In the earlier type C example the array for Δ_1^+ is



and $L_1 \simeq GL(1) \times GL(2) \times Sp(4)$. In general,

$$\begin{aligned} &L_1 \simeq GL(m_1) \times GL(m_2) \times \cdots \times GL(m_r) \times Sp(2m_{r+1}), \text{ for type C, and} \\ &L_1 \simeq GL(m_1) \times GL(m_2) \times \cdots \times GL(m_r) \times GL(m_{r+1}), \text{ for type D,} \end{aligned}$$

where m_i is the size of the i^{th} 1-block (counting from left to right in the array).

In type D the groups Q_{2j} and Q_{2j+1} are the 'correct' groups for our purposes. However, in type C the situation is more complicated; it turns out that $Q_m \cdots Q_1 Q_0 \cdot \mathfrak{b}$ is *not* contained in $\gamma_{_Q}^{-1}(f)$. To remedy this situation we need to consider two subgroups $\widehat{L}_1$ and $\widehat{\widehat{L}}_1$ of L_1, and also subgroups $\widehat{L}_{2j+1}$ and $\widehat{\widehat{L}}_{2j+1}$ of L_{2j+1}.

The following definitions apply only to the type C case.

(A) Definition of $\widehat{L}_1$. Define

$$\widehat{L}_1 = \{k \in L_1 : k(e_{a_{\ell+1}}) = e_{a_{\ell+1}} \text{ and } k(e_{\tau(a_{\ell+1})}) = e_{\tau(a_{\ell+1})}\}.$$

Then the roots in $\Delta(\widehat{\mathfrak{l}}_1)$ are

$$\begin{aligned} \epsilon_i - \epsilon_j &: i, j \text{ in the same 1-block, } i, j \neq a_{\ell+1} \text{ and} \\ \pm(\epsilon_i + \epsilon_j) &: i, j \text{ in the last 1-block, } i, j \neq a_{\ell+1}. \end{aligned} \tag{3.1}$$

With, $m_1, \ldots, m_{r+1}$ as above,

$$\widehat{L}_1 \simeq GL(m_1) \times GL(m_2) \times \cdots \times GL(m_r) \times Sp(2(m_{r+1} - 1)).$$

(B) Definition of $\widehat{\widehat{L}}_1$. Let B_1 be the set of labels of the dots in the last 1-block. Then $\widehat{\widehat{L}}_1$ is the Levi subgroup of the parabolic subgroup of

$$\{k \in L_1 : k(e_i) = e_i \text{ for all } i \notin B_1 \cup \tau(B_1)\}$$

having roots

$$\Delta(\widehat{\widehat{\mathfrak{l}}}_1) = \{\epsilon_i - \epsilon_j : i, j \in B_1 \setminus \{a_{\ell+1}\}\} \cup \{\pm(\epsilon_i + \epsilon_{\ell+1}) : i \in B_1 \setminus \{a_{\ell+1}\}\}.$$

Therefore,

$$\widehat{\widehat{L}}_1 \simeq GL(m_{r+1}) \subset Sp(2m_{r+1}).$$

3.2. **One-parameter subgroups of $Z_K(f)$.** Consider the subgroups

$$\begin{aligned} G_2' &= \{g \in G : g(W_2) \subset W_2 \text{ and } g|_{V_2} = Id\} \\ K_2' &= K \cap G_2'. \end{aligned}$$

Then (G_2', K_2') is a pair of the same type as (G, K). Note that G_2 and G_2' are mutually commuting subgroups of G. Also, $f_0 \in \mathfrak{g}_2' \cap \mathfrak{p}$ and the tableau of $f_0 \in \mathfrak{g}_2'$ has exactly two rows (both of length N).

Recall that in §1.4, for the 2-row case, we constructed independent elements Z_i of $\mathfrak{z}_{\mathfrak{k}}(f)$. Applying this construction to $f_0 \subset \mathfrak{g}_2'$ we get $Z_i \in \mathfrak{z}_{\mathfrak{k}_2'}(f_0) \subset \mathfrak{z}_{\mathfrak{k}}(f)$. These Z_i are described as follows.

Write $v_1 = e_{a_1}$ and $v_1' = e_{b_1}$, the highest weight vectors of the two $\mathfrak{sl}(2)_{f_0}$ constituents of W_2. Let $v_l = f^{l-1}(v_1)$, $(= \pm e_{a_l})$ and $v_l' = f^{l-1}(v_1')$, $(= \pm e_{b_l})$. Then, in the four cases, $Z_i(v_l) = 0$, for all l, and

$$Z_i(v_l') = \begin{cases} v_{l+2i-1}, & \text{for type C, } N \text{ even or type D, } N \text{ odd,} \\ v_{l+2i-2}, & \text{for type C, } N \text{ odd or type D, } N \text{ even,} \end{cases}$$

where we understand $v_k = 0$ when $k \geq N + 1$.

Let $r = \ell - [\frac{N}{2}]$ in type C and $r = \ell - [\frac{N+1}{2}] + 1$ in type D.

Lemma 3.2. *The following hold.*
(1) $Z_k \in \mathfrak{q}$ if and only if $k \geq r+1$.
(2) $Z_1, \dots, Z_r$ are independent modulo $\mathfrak{q}$.

Proof. The lemma will be proved for type C; the type D case is essentially the same. We first show that if $k = 1, 2, \dots, \ell - [\frac{N}{2}]$, then $Z_k \notin \mathfrak{q}$, that is, Z_k does not preserve the isotropic flag (F_i) of (2.8). Consider $b_{N-\ell+1}$. This is the label of the dot immediately to the right of the central block of the doubled array. Therefore, $v'_{N-\ell+1} = \pm e_{b_{N-\ell+1}} \in F_\ell$. If $k \leq \ell - [\frac{N}{2}]$, then

$$\begin{aligned} &N - \ell + 2k \leq \ell, \text{ if } N \text{ is even,} \\ &N - \ell + 2k - 1 \leq \ell, \text{ if } N \text{ is odd.} \end{aligned}$$

In particular $a_{N-\ell+2k}$ and $a_{N-\ell+2k-1}$ label dots left of center. Now

$$Z_k(v'_{N-\ell+1}) = \begin{cases} v_{N-\ell+2k}, & N \text{ even,} \\ v_{N-\ell+2k-1}, & N \text{ odd,} \end{cases}$$

so, $Z_k(v'_{N-\ell+1}) \notin F_\ell$. Therefore, $Z_k \notin \mathfrak{q}$.

For the converse, we assume $k \geq \ell - [\frac{N}{2}] + 1$ and show that Z_k preserves the isotropic flag (F_i). Again we consider the label $b_{N-\ell+1}$. We use the following.
Claim: Suppose that $X \in \mathfrak{g}'_2 \simeq \mathfrak{sp}(N)$ vanishes on each v_i and there is an $s \in \mathbf{Z}_{\geq 0}$ so that $X(v'_i) = v_{i+s}$, for each i. Then the following are equivalent
(A) X preserves the isotropic flag (F_i).
(B) $b_{N-\ell+1}$ labels a dot to the left of the dot labeled by $a_{N-\ell+s+1}$.
If (A) holds then $X(v'_{N-\ell+1}) = v_{N-\ell+s+1} \in F_\ell$, so (B) follows. Conversely, suppose that (B) holds. We need to check that $b_{N-\ell+j}$ is left of $a_{N-\ell+s+j}$, for $j = 1, 2, \dots$ (since the $v_{N-\ell+j}$ are precisely the v_i's in F_ℓ). However, this is clear from (B) since the string labeled by $b_1, \dots, b_N$ passes through *every* block right of the central block. The claim is now proved.

Now assume that N is even. It suffices to show that $a_{N-\ell+2k}$ is right of $b_{N-\ell+1}$(by taking $s = 2k-1$). But we are assuming that $k \geq \ell - [\frac{N}{2}] + 1$, that is, $2k \geq 2\ell - N + 2$. Therefore, $N - \ell + 2k \geq \ell + 2$, so $a_{N-\ell+2k}$ is to the right of $a_{\ell+2}$, which is in the same block as $b_{N-\ell+1}$.

When N is odd we need to show that $a_{N-\ell+2k-1}$ is to the right of $b_{N-\ell+1}$. We are assuming that $2k \geq 2\ell - (N-1) + 2$, therefore $N - \ell + 2k - 1 \geq \ell + 2$. Again $a_{N-\ell+2k-1}$ is right of the central block, so is right of $b_{N-\ell+1}$. □

For each $k = 1, \dots, r$, let U_k be the one-parameter subgroup for Z_k. Note that each Z_k is nilpotent and the Z_k's mutually commute (in fact, $Z_k Z_{k'} = 0$, for all k, k'). Let

$$Z^{(0)} = U_1 U_2 \cdots U_r.$$

The same construction applied to $f' = f - \sum_{i=0}^{j-1} f_i$ in G_{2j} gives a subgroup $Z^{(j)}$ of $Z_K(f)$. In particular, $f_j \in \mathfrak{g}'_{2j+2}$ is a two-row nilpotent, which gives elements of the centralizer $Z_1^j, Z_2^j, Z_3^j, \dots$ as described in §1.4. Then $Z^{(j)} = \prod_i U_i^j$, where U_i^j is the one-parameter subgroup for Z_i^j. Since each G'_{2j} commutes with all $G_{2k}, k \geq j$, the subgroups $Z^{(j)}$ mutually commute.

3.3. **The main theorem.** Let $\mathcal{Q}$ be a closed K-orbit in $\mathfrak{B}$ and let f be the generic element constructed by the algorithm of §2.1. Set

$$M_j = \begin{cases} \widehat{L}_{2j+1}\widehat{\widehat{L}}_{2j+1}L_{2j}, & \text{in type C} \\ L_{2j+1}L_{2j}, & \text{in type D.} \end{cases}$$

Then our first main theorem is the following description of $\gamma_{\mathcal{Q}}^{-1}(f)$.

Theorem 3.3. *The closure of*

$$\big(\prod_{j=0}^{m} Z^{(j)}\big) M_m M_{m-1} \cdots M_1 M_0 \cdot \mathfrak{b} \tag{3.4}$$

is $\gamma_{\mathcal{Q}}^{-1}(f)$.

The proof is given in the next section.

4. Proof of Theorem 3.3

The proof of Theorem 3.3 is given in three steps. We first prove the inclusion

$$\big(\prod_{j=0}^{m} Z^{(j)}\big) M_m \cdots M_1 M_0 \cdot \mathfrak{b} \subset \gamma_{\mathcal{Q}}^{-1}(f). \tag{4.1}$$

Then it is shown that both sides of (4.1) have the same dimension. In §4.4 the fact that $\gamma_{\mathcal{Q}}^{-1}(f)$ is irreducible is established.

Particular parabolic subgroups of $\widehat{L}_1$ and $\widehat{\widehat{L}}_1$ will play a role. Let $\widehat{R}_1 = \widehat{L}_1 \cap Q$. Then $\widehat{R}_1$ is a parabolic subgroup of $\widehat{L}_1$ having Levi factor $\widehat{L}_1 \cap L$. We write $\widehat{N}_1^-$ for the nilradical of $\widehat{R}_1$. Therefore a Levi decomposition of $\widehat{R}_1$ is $(\widehat{L}_1 \cap L)\widehat{N}_1^-$. Similarly, $\widehat{\widehat{R}}_1 = \widehat{\widehat{L}}_1 \cap Q$ is a parabolic subgroup of $\widehat{\widehat{L}}_1$ and $\widehat{\widehat{R}}_1 = (\widehat{\widehat{L}}_1 \cap L)\widehat{\widehat{N}}_1^-$. The roots in $\Delta(\hat{\hat{\mathfrak{n}}}_1)$ are

$$\begin{aligned} &\epsilon_i - \epsilon_j \,:\, i < j \text{ in the last 1-block but in different blocks, } i, j \neq a_{\ell+1}, \\ &\epsilon_i + \epsilon_{a_{\ell+1}} \,:\, i \text{ in the last 1-block but not in the last block.} \end{aligned} \tag{4.2}$$

4.1. **A lemma.**

Lemma 4.3. *$Q_2\widehat{L}_1\widehat{\widehat{L}}_1 \subset L_2\widehat{L}_1\widehat{\widehat{L}}_1 Q$ in type C and $Q_2L_1 \subset L_2L_1Q$ in type D.*

Proof. First consider type C. Since $Q_2 = L_2U_2^-$, the statement follows once we show that $U_2^-\widehat{L}_1\widehat{\widehat{L}}_1 \subset \widehat{L}_1\widehat{\widehat{L}}_1 U_0^-$. We have

$$U_2^- \subset U_1^- \subset U_0^- \text{ and } \widehat{L}_1, \widehat{\widehat{L}}_1 \subset L_1.$$

Since L_1 normalizes U_1^-, for any $\hat{\ell}_1 \in \widehat{L}_1$ and $\hat{\hat{\ell}}_1 \in \widehat{\widehat{L}}_1$

$$\hat{\ell}_1^{-1}U_2^-\hat{\ell}_1 \subset U_1^- \text{ and } \hat{\hat{\ell}}_1^{-1}U_1^-\hat{\hat{\ell}}_1 \subset U_1^-. \tag{4.4}$$

We conclude that

$$U_2^-\,\hat{\ell}_1\hat{\hat{\ell}}_1 \subset \hat{\ell}_1\hat{\hat{\ell}}\,U_0^-.$$

Now consider type D. It suffices to show that $U_2^-L_1 \subset L_1Q$. But L_1 normalizes U_1^-, which contains U_2^-, therefore, for $\ell_1 \in L_1$

$$U_2^-\ell_1 \subset \ell_1U_1^- \subset \ell_1Q.$$

□

Corollary 4.5. *For any $j = 0, 1, \ldots, m$,*

$$\begin{aligned}
&Q_{2j+2}\widehat{L}_{2j+1}\widehat{\widehat{L}}_{2j+1} \subset L_{2j+2}\widehat{L}_{2j+1}\widehat{\widehat{L}}_{2j+1}Q_{2j}, \text{ for type C},\\
&Q_{2j+2}L_{2j+1} \subset L_{2j+2}L_{2j+1}Q_{2j}, \text{ for type D}.
\end{aligned}$$

Proof. This is exactly the lemma applied to G_{2j}. □

Corollary 4.6. *If*

$$M_j' = \begin{cases} \widehat{L}_{2j+1}\widehat{\widehat{L}}_{2j+1}Q_{2j}, & \text{for type C}\\ L_{2j+1}Q_{2j}, & \text{for type D,}\end{cases}$$

then

$$M_j \cdots M_1M_0 \cdot \mathfrak{b} = M_j' \cdots M_1'M_0' \cdot \mathfrak{b},$$

for any $j = 0, 1, \ldots, m$.

4.2. **The inclusion.** By (1.1), in order to prove

$$\Big(\prod_{j=0}^{m} Z^{(j)}\Big)M_{m+1}M_m \cdots M_1M_0 \cdot \mathfrak{b} \subset \gamma_{\Omega}^{-1}(f) \tag{4.7}$$

it suffices to prove that

$$\begin{aligned}
&L_0\widehat{\widehat{L}}_1\widehat{L}_1L_2 \cdots \widehat{\widehat{L}}_{2m-1}\widehat{L}_{2m-1}L_{2m} \cdot f \subset \mathfrak{n}^- \cap \mathfrak{p}, \text{ in type C, and}\\
&L_0L_1L_2 \cdots L_{2m} \cdot f \subset \mathfrak{n}^- \cap \mathfrak{p}, \text{ for type D.}
\end{aligned} \tag{4.8}$$

The proof is by induction on the rank of the group G. In the case when the rank is one, $f = 0$ and the statement clearly holds.

Considering first the type C case, assume that the statement holds for all pairs $(Sp(2n'), Sp(2p') \times Sp(2q'))$ with $n' < n$. Since (G_2, K_2) is such a lower rank pair (or $m = 1$ and we are done), we have

$$L_2\widehat{\widehat{L}}_3\widehat{L}_3L_4\cdots\widehat{L}_{2m-1}L_{2m}\cdot f' \subset \mathfrak{n}_2^- \cap \mathfrak{p},$$

where $f' = f - f_0$. Therefore, the left-hand side of (4.8) is contained in

$$L_0\widehat{\widehat{L}}_1\widehat{L}_1(f_0 + \mathfrak{n}_2^- \cap \mathfrak{p}),$$

since G_2 centralizes f_0.

Since $\mathfrak{n}_2^- \subset \mathfrak{n}_1^- \subset \mathfrak{n}^-$, $\widehat{L}_1, \widehat{\widehat{L}}_1 \subset L_1$, and L_1 normalizes $\mathfrak{n}_1^- \cap \mathfrak{p}$, we have

$$\widehat{\widehat{L}}_1\widehat{L}_1(\mathfrak{n}_2^- \cap \mathfrak{p}) \subset \mathfrak{n}_1^- \cap \mathfrak{p} \subset \mathfrak{n}^- \cap \mathfrak{p}.$$

Since L_0 normalizes $\mathfrak{n}^- \cap \mathfrak{p}$ we conclude that

$$L_0\widehat{\widehat{L}}_1\widehat{L}_1 \cdot (\mathfrak{n}_2^- \cap \mathfrak{p}) \subset \mathfrak{n}^- \cap \mathfrak{p}.$$

It follows that we only need to verify that

$$\widehat{\widehat{L}}_1\widehat{L}_1 \cdot f_0 \subset \mathfrak{n}^- \cap \mathfrak{p}. \tag{4.9}$$

Consider the labels $a_1, \dots, a_l, a_{l+1}, b_{N-l-1}, \dots, b_1$ of the first string through the array, as in §2.1. Then

$$f_0 = \Big(\sum_{i=1}^{l} X_{a_{i+1}-a_i}\Big) + X_{-(a_{l+1}+b_{N-l-1})} + \Big(\sum_{j=1}^{N-l-1} X_{b_{j+1}-b_j}\Big).$$

The first term is centralized by L_1, so is centralized by $\widehat{\widehat{L}}_1\widehat{L}_1$. The last term lies in $\mathfrak{n}_1^- \cap \mathfrak{p}$, which is normalized by L_1, so also by $\widehat{\widehat{L}}_1\widehat{L}_1$. We therefore need to check that

$$\widehat{\widehat{L}}_1\widehat{L}_1 \cdot X_{-(a_{l+1}+b_{N-l-1})} \subset \mathfrak{n}^- \cap \mathfrak{p}. \tag{4.10}$$

We first show that

$$\widehat{L}_1 X_{-(a_{\ell+1}+b_{N-\ell-1})} \subset \mathfrak{w},$$

where

$$\mathfrak{w} := \operatorname{span}_{\mathbf{C}}\{X_{-(a_{\ell+1}+k)} \in \mathfrak{p} : k \text{ is in a 1-block}\}.$$

This is done by showing that $\mathfrak{w}$ is $\widehat{\mathfrak{l}}_1$-stable. We check that brackets by root vectors in $\widehat{\mathfrak{l}}_1$ (see (3.1)) with those in $\mathfrak{w}$ are in $\mathfrak{w}$:

$$[X_{k-i}, X_{-(a_{\ell+1}+k)}] = X_{-(a_{\ell+1}+i)} \in \mathfrak{w},$$
$$[X_{\pm(i+j)}, X_{-(a_{\ell+1}+k)}] = 0.$$

Now we show that $\widehat{\widehat{L}}_1\mathfrak{w} \subset \mathfrak{n}^- \cap \mathfrak{p}$. Since Q normalizes $\mathfrak{n}^- \cap \mathfrak{p}$ (and $\widehat{\widehat{R}}_1 \subset Q$) it suffices to check that $\widehat{\widehat{N}}_1\mathfrak{w} \subset \mathfrak{n}^- \cap \mathfrak{p}$. Of the root vectors in $\hat{\hat{\mathfrak{n}}}_1$, only $X_{i+a_{\ell+1}}$ fails to commute with $\mathfrak{w}$. But

$$[X_{i+a_{\ell+1}}, X_{-(a_{\ell+1}+k)}] = X_{i-k},$$

which is in $\mathfrak{n}^- \cap \mathfrak{p}$, since i is in the last 1-block and k is in an earlier 1-block.

This completes the proof of (4.7) for type C.

Now suppose (G, K) is of type D. The inductive hypothesis says that

$$L_2L_3 \cdots L_{2m} \cdot f' \subset \mathfrak{n}_2^- \cap \mathfrak{p},$$

for $f' = f - f_0$. Since $\mathfrak{n}_2^- \subset \mathfrak{n}_1^-$, and $\mathfrak{n}_1^-$ is normalized by L_1, we have

$$L_1L_2 \cdots L_m \cdot f' \subset \mathfrak{n}_1^- \cap \mathfrak{p} \subset \mathfrak{n}^- \cap \mathfrak{p}.$$

Therefore, (4.8) will be proved once we show

$$L_1 \cdot f_0 \subset \mathfrak{n}^- \cap \mathfrak{p}, \tag{4.11}$$

since L_0 normalizes $\mathfrak{n}^- \cap \mathfrak{p}$.

Write

$$f_0 = \Big(\sum_{i=1}^{l} X_{a_{i+1}-a_i}\Big) + X_{-(a_{l+1}+b_{N-l-1})} + \Big(\sum_{j=1}^{N-l-1} X_{b_{j+1}-b_j}\Big).$$

Note that L_1 centralizes the first term and the last two terms are in $\mathfrak{n}_1^- \cap \mathfrak{p}$, which is normalized by L_1. Therefore, (4.11) holds and the proof of the inclusion (4.7) is complete.

4.3. **Calculation of the dimension.** We now prove

$$\dim\Big(\big(\prod_{j=0}^{m} Z^{(j)}\big)M_m \cdots M_1M_0 \cdot \mathfrak{b}\Big) = \dim(\gamma_{_Q}^{-1}(f)). \tag{4.12}$$

Let us recall a few facts about the $Z^{(j)}$ from §3.2. The Lie algebra of $Z^{(j)}$ is spanned by some $Z_1^j, Z_2^j, \dots$, each a sum of root vectors of the form $X_{\pm(i\pm k)}$ with i, k in the string in the array for G_{2j}. Therefore, since each $Z^{(j)}$ commutes with G_{2j+2}, it follows that

$$\big(\prod_{j=0}^{m} Z^{(j)}\big)M_m \cdots M_1M_0 \cdot \mathfrak{b} = (Z^{(m)}M_m) \cdots (Z^{(1)}M_1)(Z^{(0)}M_0) \cdot \mathfrak{b}. \tag{4.13}$$

Each Z_i^j may be decomposed under the direct sum $\mathfrak{u}_j \oplus \mathfrak{q}_j$. Write this decomposition as $Z_i^j = X_i^j + X_i^{j\prime}$. Then X_i^j is a sum of root vectors in $\mathfrak{u}_j \subset \mathfrak{n}$. We observe that if each $Z^{(j)}$ in either (4.12) or (4.13) is replaced by the product of the one parameter subgroups for the X_i^j, then the space is unchanged. This follows from Lemma 4.3.

A standard result about reductive algebraic groups states that if $\alpha_1, \dots, \alpha_d$ is an ordering of the positive roots, then

$$\varphi : \mathfrak{n} \to N \cdot \mathfrak{b}$$
$$\varphi(\sum t_i X_{\alpha_i}) = \exp(t_1 X_{\alpha_1}) \cdots \exp(t_d X_{\alpha_d}) \cdot \mathfrak{b}$$

is an algebraic isomorphism. See for example [8, 8.2.1]. We apply this fact by finding a subspace of $\mathfrak{n}$ of dimension at least $\dim(\gamma_{\mathcal{Q}}^{-1}(f))$ that maps (via φ) into (4.13). Having done this we will be able to conclude that

$$\dim\Big((\prod_{j=0}^{m} Z^{(j)}) M_m \cdots M_1 M_0 \cdot \mathfrak{b}\Big) \geq \dim(\gamma_{\mathcal{Q}}^{-1}(f)).$$

Then, by the inclusion (4.7), the dimension formula (4.12) will be proved.

Again we handle the two types of pairs (G, K) separately, the type D case being somewhat easier than the type C case. Assume first that (G, K) is of type C. We begin by specifying an ordering of roots. For each $j = 0, 1, \dots, m$ consider root vectors occurring in

(1) the expressions for all $X_1^j, X_2^j, \dots,$
(2) $\hat{\mathfrak{n}}_{2j+1} \setminus \mathfrak{g}_{2j+2}$,
(3) $\hat{\hat{\mathfrak{n}}}_{2j+1} \setminus \mathfrak{g}_{2j+2}$,
(4) $\mathfrak{l}_{2j} \cap \mathfrak{n} \setminus \hat{n}_{2j+1} \cap \hat{\hat{\mathfrak{n}}}_{2j+1} \cap \mathfrak{g}_{2j+2}$.

Write the corresponding roots as $\alpha_1^j, \alpha_2^j, \dots$ with those in (1) first, then those in (2), etc.

Lemma 4.14. *The roots $\alpha_i^j, j = 0, 1, \dots, m$ and all i, are distinct.*

Proof. Since each $\alpha_i^j \in \Delta(\mathfrak{g}_{2j}) \setminus \Delta(\mathfrak{g}_{2j+2})$ it is clear that no $\alpha_i^j = \alpha_i^{j'}$, unless $j = j'$.

Consider $j = 0$. We write down the form of roots of each of the types (1)-(4). Let S be the set of labels of dots in the first string in the array.

(1) $\epsilon_a \pm \epsilon_b$, $a \leq b, a, b \in S$, and a, b in different blocks. Note that if a, b were in the same block, then the root vector would be in $\mathfrak{l}$, so would not appear in the expression for X_i^0.
(2) $\epsilon_i - \epsilon_j$, $i < j$, with i, j in the same 1-block and $j \in \{b_1, \dots, b_{N-\ell-1}\}$. Note that neither i nor j can be among $a_1, \dots, a_{\ell+1}$ (or the root vector would not be in $\widehat{\mathfrak{l}}_1$), and both cannot be in $\{1, \dots, n\} \setminus S$ (or the root vector would be in $\mathfrak{g}_2$).
(3) $\epsilon_i + \epsilon_{a_{\ell+1}}$, i in the last 1-block, $i \neq a_{\ell+1}$. Note that all $\epsilon_i - \epsilon_j$ occurring in $\hat{\hat{\mathfrak{n}}}_1$ are in $\mathfrak{g}_2$.
(4) $\epsilon_i - \epsilon_j$, $i < j$, with i, j in the same block and $j \in \{a_1, \dots, a_{\ell+1}\}$ and $2\epsilon_{a_{\ell+1}}$. Note that at least one of i, j must be in S (or the root vector would be in $\mathfrak{g}_2$)

and if j is among $b_1, \dots, b_{N-\ell-1}$ then the root vector is in $\hat{\mathfrak{n}}_1$. Also, $\epsilon_i + \epsilon_{a_{\ell+1}}$ occurs in $\hat{\hat{\mathfrak{n}}}_1$, for $i \neq a_{\ell+1}$.

A similar description applies for arbitrary j.

It is now clear that no two of $\alpha_1^j, \alpha_2^j, \dots$ are equal. □

It follows that we may order the roots appearing in (1)-(4), as j ranges over $0, 1, \dots, m$, by

$$\alpha_1^m, \alpha_2^m, \dots, \alpha_1^{m-1}, \alpha_2^{m-1}, \dots, \alpha_1^0, \alpha_2^0 \dots.$$

Writing $\mathfrak{v}_j$ for the span of the root vectors appearing in (1)-(4) for a given j, we have that φ restricted to $\mathfrak{v}_1 \oplus \dots \oplus \mathfrak{v}_m$ is an isomorphism onto its image. Letting $\mathfrak{v}_j'$ be the span of the $X_1^j, X_2^j, \dots$ and the root vectors appearing in (2)-(4), the restriction of φ to $\mathfrak{v}_0' \oplus \dots \oplus \mathfrak{v}_m'$ is still an isomorphism (onto its image), because the root vectors in the expression for $X_1^j, X_2^j, \dots$ mutually commute. But

$$\varphi(\mathfrak{v}_0' \oplus \dots \oplus \mathfrak{v}_m') \subset (Z^{(m)} M_m) \cdots (Z^{(1)} M_1)(Z^{(0)} M_0) \cdot \mathfrak{b}.$$

Since $\gamma_{\mathfrak{Q}}^{-1}(f) \subset \mu^{-1}(f)$, to complete the proof of (4.12) it now suffices to prove the following.

Claim: $\sum_{j=0}^m \dim(\mathfrak{v}_j') = \dim(\mu^{-1}(f))$.

This is accomplished by applying induction on the rank of G. By the inductive hypothesis we have

$$\sum_{j=1}^m \dim(\mathfrak{v}_j') = \dim(\mu_2^{-1}(f')),$$

where $f = f_0 + f'$ and μ_2 is the moment map of the cotangent bundle of the flag variety for G_2. We therefore need to prove that

$$\dim(\mathfrak{v}_0') = \dim(\mu^{-1}(f)) - \dim(\mu_2^{-1}(f')). \tag{4.15}$$

The left-hand side is computed first. By Lemma 3.2(2) the dimension of $Z^{(0)}$ (i.e, the number of X_i^0's) is $\ell - \left[\frac{N}{2}\right]$. Now we must count the roots appearing in (2)-(4) of the proof of Lemma 4.14. In (2) there are $n - N - (s_1 - 1)$ possibilities for i, where s_1 is the number of dots in the last 1-block. In (3) there are $s_1 - 1$ possibilities for i. In (4) the number of possibilities is $(n - (\ell + 1)) + 1 = n - \ell$.
Adding these we get $\dim(\mathfrak{v}_0') = 2n - \left[\frac{3N}{2}\right]$.

Now compute the right hand side of (4.15) using the dimension formula

$$\begin{aligned} \dim(\mu^{-1}(f)) &= \dim(\mathfrak{B}) - \frac{1}{2}(\dim(G \cdot f)) \\ &= \frac{1}{2}\big(\dim(Z_G(f)) - \operatorname{rank}(G)\big) \end{aligned} \tag{4.16}$$

for a Springer fiber; see for example [7, §6.7]. Recall that the rank of G_2 is $n - N$. Also, the tableaux of f' is obtained from that of f by deleting the first two rows.

Let

$$\delta = \begin{cases} 1, & \text{if } N \text{ is odd} \\ 0, & \text{if } N \text{ is even.} \end{cases}$$

Applying the formula of (1.10) we get

$$\begin{aligned}
&\dim(Z_G(f)) - \dim(Z_{G_2}(f')) \\
&\qquad = \frac{1}{2}\Big(\sum_{i=1}^{N}(c_i^2 - (c_i-2)^2)\Big) + \delta, \\
&\qquad = \sum_{i=1}^{N}(2c_i - 2) + \delta \\
&\qquad = 2(2n - N) + \delta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\dim(\mu^{-1}(f)) - \dim(\mu_2^{-1}(f')) \\
&\qquad = 2n - N + \frac{\delta}{2} - \frac{N}{2} \\
&\qquad = 2n - \Big[\frac{3N}{2}\Big].
\end{aligned}$$

This proves (4.15), and therefore (4.12) in type C.

Now consider pairs of type D. Again we specify an ordering of roots. For each $j = 0, 1, \ldots, m$ consider root vectors occurring in

(1) the expressions for all $X_1^j, X_2^j, \ldots,$
(2) $\mathfrak{n}_{2j+1} \setminus \mathfrak{g}_{2j+2}$,
(3) $\mathfrak{l}_{2j} \cap \mathfrak{n} \setminus \mathfrak{n}_{2j+1} \cap \mathfrak{g}_{2j+2}$.

Write the corresponding roots as $\alpha_1^j, \alpha_2^j, \ldots$ with those in (1) first, then those in (2), etc.

For $j = 0$ we explicitly write the roots occurring in (1)-(3). Let S be the set of labels in the first string through the array and s_1 the number of dots in the last 1-block.

(1) $\epsilon_a \pm \epsilon_b$, $a \leq b, a, b \in S$, and a, b in different blocks.
(2) $\epsilon_i - \epsilon_j$, $i < j$, with i, j in the same 1-block and $j \in \{b_1, \ldots, b_{N-\ell-1}\} \cup \{a_{\ell+1}\}$.
(3) $\epsilon_i - \epsilon_j$, $i < j$, with i, j in the same block and $j \in \{a_1, \ldots, a_{\ell+1}\}$

A similar description applies for arbitrary j. It is clear that the α_i^j are distinct, and we may order these roots as we did for type C.

Reasoning as before, it suffices to prove that

$$\dim(\mathfrak{v}_0') = \dim(\mu^{-1}(f)) - \dim(\mu_2^{-1}(f')). \tag{4.17}$$

We first determine $\dim(\mathfrak{v}_0')$. By Lemma 3.2, $\dim(Z^{(0)}) = r = \ell - [\frac{N+1}{2}] + 1$. The number of root vectors appearing in (2) is $n - N + (s_1 - 1)$. The number in (3) is $n - \ell - s_1$. This gives $\dim(\mathfrak{v}_0') = 2n - [\frac{3N+1}{2}]$.

The righthand side is computed as in the type C case. Let $\epsilon = 0$ if N is even and $\epsilon = 1$ if N is odd.

$$\begin{aligned} \dim(\mu^{-1}(f)) &- \dim(\mu_2^{-1}(f')) \\ &= \frac{1}{2}\big(\dim(Z_G(f)) - \dim(Z_{G_2}(f'))\big) - \frac{1}{2}\big(\mathrm{rank}(G) - \mathrm{rank}(G_2)\big) \\ &= \frac{1}{2}\big(2(2n-N) - \epsilon - N\big), \text{ by (1.10)}, \\ &= 2n - \frac{3N+\epsilon}{2} \\ &= 2n - [\frac{3N+1}{2}]. \end{aligned}$$

Now (4.12) is proved.

Note that we have shown

$$\dim(\gamma_{\mathcal{Q}}^{-1}(f)) = \dim(\mu^{-1}(f)) \tag{4.18}$$

for types C and D.

4.4. **Proof that f is generic.** Formula (4.12) is now used to prove that the element f constructed by the algorithm is in fact generic. For this we use a general geometric lemma. Suppose that $\mathcal{Q}$ is any K-orbit in the flag variety $\mathfrak{B}$ and $\gamma_{\mathcal{Q}} : \overline{T_{\mathcal{Q}}^*\mathfrak{B}} \to \mathcal{N}_\theta$ is the restriction of the moment map μ of $T^*\mathfrak{B}$.

Lemma 4.19. *Suppose $\gamma_{\mathcal{Q}}(\overline{T_{\mathcal{Q}}^*\mathfrak{B}}) = \overline{K\cdot x}$. If $y \in \overline{K\cdot x} \setminus K\cdot x$, then $\dim(\gamma_{\mathcal{Q}}^{-1}(y)) < \dim(\mu^{-1}(y))$.*

Proof. Since $\overline{T_{\mathcal{Q}}^*\mathfrak{B}}$ is irreducible, $\dim(\overline{T_{\mathcal{Q}}^*\mathfrak{B}} \setminus \gamma_{\mathcal{Q}}^{-1}(K\cdot x)) < \dim(\overline{T_{\mathcal{Q}}^*\mathfrak{B}})$. Consider

$$\begin{aligned} B &:= \overline{K\cdot x} \setminus \overline{K\cdot y} \text{ and} \\ A &:= \overline{T_{\mathcal{Q}}^*\mathfrak{B}} \setminus \gamma_{\mathcal{Q}}^{-1}(B). \end{aligned}$$

The restriction of γ to A is a surjection $A \to \overline{K\cdot y}$ and $\gamma_{\mathcal{Q}}^{-1}(y) \subset A$. Now

$$\begin{aligned} \dim(\gamma_{\mathcal{Q}}^{-1}(y)) &= \dim(A) - \dim(\overline{K\cdot y}) \\ &< \dim(\overline{T_{\mathcal{Q}}^*\mathfrak{B}}) - \dim(\overline{K\cdot y}) \\ &= \dim(\mu^{-1}(y)), \text{ by (4.16)}. \end{aligned}$$

□

Proposition 4.20. *For any closed K-orbit $\mathcal{Q}$ in $\mathfrak{B}$, the element f constructed by the algorithm is generic in $\mathfrak{n}^- \cap \mathfrak{p}$.*

Proof. If not, then by taking $y = f$ in the lemma, $\dim(\gamma_{\mathfrak{Q}}^{-1}(f)) < \dim(\mu^{-1}(f))$. This would contradict (4.18). $\square$

Corollary 4.21. *$\gamma_{\mathfrak{Q}}^{-1}(f)$ is irreducible.*

Proof. In general, if $\mu(\overline{T_{\mathfrak{Q}}^*\mathfrak{B}}) = \overline{K \cdot f}$, then $\mu^{-1}(f) \cap \overline{T_{\mathfrak{Q}}^*\mathfrak{B}}$ contains several irreducible components of $\mu^{-1}(f)$. It is a fact (see [5, Prop. 2.10]) that the component group $A_K(f) = Z_K(f)/Z_K(f)_e$ acts transitively on this set of components. However, for the pairs considered here, it is known that the component group is trivial. Now [2, Prop. 2.1], for example, tells us that $\gamma_{\mathfrak{Q}}^{-1}(f)$ is precisely one irreducible component. $\square$

The proof of Theorem 3.3 is now complete.

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