Certain components of Springer fibers: algorithms, examples and applications

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Abstract. The Springer fiber associated to closed K-orbits in generalized flag varieties is determined for the real reductive group $GL(N, \mathbf{R})$. Additional examples for other groups are given. It is shown how to use this information to compute associated cycles of representations attached to these orbits.

Introduction

An important invariant of a Harish-Chandra module is its associated cycle. There are no known methods for computing associated cycles in any generality. For example, associated cycles are not known for all discrete series representations. The purpose of this article is to compute associated cycles for cohomologically parabolically induced representations of $GL(N, \mathbf{R})$ and to illustrate how similar methods are used to compute the associated cycles of discrete series representations for several other classical groups. The answer is quite simple for $GL(N, \mathbf{R})$, but is somewhat complicated for other groups. The method is to compute certain components of Springer fibers in an explicit enough form that a theorem of J.-T. Chang can be applied. Therefore, our results are about the geometry of Springer fibers. Our study of the Springer fibers is elementary in nature.

Suppose $G_{\mathbf{R}}$ is a linear real reductive group and G is its complexification. We consider the pair (G, K) where K is the fixed point group of the complexification of a Cartan involution of $G_{\mathbf{R}}$. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the (complexified) Cartan decomposition of the Lie algebra of G. The associated cycle of a Harish-Chandra module is a formal non-negative integer combination of the closures of K-orbits in $\mathcal{N}_\theta \equiv \mathcal{N} \cap \mathfrak{p}$, $\mathcal{N}$ being the nilpotent cone in $\mathfrak{g}$. See [**14**, Page 322] for a definition of the associated cycle. Now suppose that $\mathfrak{F}$ is a generalized flag variety for G and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}^- \in \mathfrak{F}$. The cotangent bundle of $\mathfrak{F}$ may be realized as the homogeneous bundle $T^*\mathfrak{F} = G \underset{Q}{\times} \mathfrak{u}^-$. The moment map for the natural action of G on $T^*\mathfrak{F}$ is $\mu(g, \xi) = \mathrm{Ad}(g)\xi$. Assume that $\mathfrak{q}$ is θ-stable, so $\mathcal{Q} = K \cdot \mathfrak{q}$ is a closed orbit in $\mathfrak{F}$. The conormal bundle to $\mathcal{Q}$ in $\mathfrak{F}$ may be written as a homogeneous bundle for K,

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$T^*_{\mathcal{Q}}\mathfrak{F} = K \underset{Q\cap K}{\times} (\mathfrak{u}^- \cap \mathfrak{p})$. We set

$$\gamma_{\mathcal{Q}} = \mu|_{T^*_{\mathcal{Q}}\mathfrak{F}}.$$

The image of $\gamma_{\mathcal{Q}}$ lies in $\mathcal{N}_\theta$ and is, in fact, the closure of a single K-orbit. If $f \in \mathfrak{u}^- \cap \mathfrak{p}$ we say that f *is generic in* $\mathfrak{u}^- \cap \mathfrak{p}$ when image$(\gamma_{\mathcal{Q}})= \overline{K \cdot f}$. For such an f we will refer to $\gamma_{\mathcal{Q}}^{-1}(f)$ as the Springer fiber for $\mathcal{Q}$. It is a union of irreducible components of the Springer fiber $\mu^{-1}(f) \subset T^*\mathfrak{F}$.

A theorem of J.-T. Chang states that for an irreducible representation attached to a closed orbit $\mathcal{Q}$ in $\mathfrak{F}$ the associated cycle is $m\cdot(\overline{K \cdot f})$ where f is generic in $\mathfrak{u}^-\cap\mathfrak{p}$ and m is the dimension of a space of sections of a sheaf on $\gamma_{\mathcal{Q}}^{-1}(f)$. Our approach is to take a closed orbit $\mathcal{Q} = K \cdot \mathfrak{q}$ and construct a generic element f in $\mathfrak{u}^- \cap \mathfrak{p}$. We do this in a way that allows us to describe the fiber $\gamma_{\mathcal{Q}}^{-1}(f)$. This description is explicit enough to compute the space of sections, thus computing the associated cycle.

For $G_{\mathbf{R}} = GL(N, \mathbf{R})$ we carry out this construction for the closed orbits $\mathcal{Q} = K \cdot \mathfrak{q}$, for each θ-stable parabolic subalgebra. This is done inductively. We first construct f_0, then reduce to a lower rank general linear group. In the smaller group we construct f_1, then again reduce to a smaller group. This is continued to give $f = f_0 + f_1 + f_1 + \cdots f_{m-1}$. We then show that all generic elements in $\mathfrak{u}^- \cap \mathfrak{p}$ are of the form $q \cdot f, q \in Q$. This will imply that $\gamma_{\mathcal{Q}}^{-1}(f) = \{\mathfrak{q}\}$. An important point is that the method is very elementary and may be adapted to other classical groups where the fiber $\gamma_{\mathcal{Q}}^{-1}(f)$ is considerably more complicated. This is the case for $G_{\mathbf{R}} = U(p, q), Sp(2n, \mathbf{R})$ and $O(p, q)$. For these groups we focus on the full flag variety $\mathfrak{B}$ and a closed orbit $K \cdot \mathfrak{b}, \mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$. For $G_{\mathbf{R}} = U(p, q)$ we recall the results of [**1**], where a generic f is constructed and the fiber is explicitly described. We show why the structure of $\gamma_{\mathcal{Q}}^{-1}(f)$ is more complicated that in the case of $G_{\mathbf{R}} = GL(N, \mathbf{R})$. Then using embeddings of $Sp(2n, \mathbf{R})$ and $O(p, q)$ into $U(p, q)$, we illustrate, in several non-trivial examples, how the method applies to compute $\gamma_{\mathcal{Q}}^{-1}(f)$ in these cases. In the final section we show how to compute the associated cycles using our descriptions of $\gamma_{\mathcal{Q}}^{-1}(f)$ along with Chang's theorem.

For classical groups computation of the image of $\gamma_{\mathcal{Q}}$ appears in the literature. A combinatorial algorithm for finding the K-orbit $K \cdot f$ of a generic element f is given in [**11**] and [**12**]. The image of $\gamma_{\mathcal{Q}}$ is described in terms of matrices in [**16**]. The significance of our procedure for finding a generic f is that it allows us to describe the fiber $\gamma_{\mathcal{Q}}^{-1}(f)$. Associated cycles are computed in [**2**] and [**4**] for holomorphic discrete series representations and for discrete series representations of groups of real rank one; the computation in these cases uses Chang's theorem along with a good description of $\gamma_{\mathcal{Q}}^{-1}(f)$. From a very different point of view the equivalent problem of computing character polynomials was carried out in [**7**] for holomorphic discrete series of $SU(p, q)$. In [**17**] the related notion of isotropy representation is studied for discrete series representations. Associates cycles for unitary highest weight modules have been computed in [**9**] using the theta correspondence.

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1. Generic elements and the Springer fiber for $GL(N, \mathbf{R})$

In this section we shall prove two theorems. The first, Theorem 1.11, gives a description of the generic elements in each theta-stable parabolic subalgebra. The second theorem, Theorem 1.15, gives the structure of the Springer fiber for $\mathcal{Q}$.

Let G be $GL(N, \mathbf{C})$ and let $\mathfrak{g} = \mathfrak{gl}(N, \mathbf{C})$ be its Lie algebra. As described in the introduction, we are concerned with the pair (G, K) with K the fixed points of the complexification of a Cartan involution. Thus, K is the orthogonal group defined by some nondegenerate symmetric bilinear form. We shall choose the symmetric form $(\, , \,)$ having matrix

$$S := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \tag{1.1}$$

with respect to the standard basis $\{e_j\}$. The involution $\theta : \mathfrak{g} \to \mathfrak{g}$ defined by $\theta(X) = -\mathrm{Ad}(S)(X^t)$ is the differential of $\Theta : G \to G$ given by $\Theta(g) = (\mathrm{Ad}(S)(g^t))^{-1}$. The fixed point group of Θ is the complex orthogonal group

$$K \equiv O(N, \mathbf{C}) = \{g \in G : g^t S g = S\}.$$

Thus $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition of $\mathfrak{g}$ into ± 1 eigenspaces of θ. Note that $\mathfrak{p}$ is the the vector space of $N \times N$ complex matrices that are symmetric with respect to the anti-diagonal.

In the first three subsections we gather some well-known facts and set some notation.

1.1. Nilpotent orbits.

The adjoint action of $G = GL(N, \mathbf{C})$ on the nilpotent cone

$$\mathcal{N} \equiv \{Y \in \mathfrak{g} : Y^N = 0\}$$

has a finite number of orbits. The Jordan form gives a one-to-one correspondence between these orbits and tableau[1] of size N. For $Y \in \mathfrak{g}$ the number of rows in the corresponding tableau is the number of Jordan blocks in the Jordan normal form of Y; the number of boxes in each row is the size of the corresponding Jordan block.

It is useful to state this slightly differently. By the Jacobson-Morozov Theorem, given $Y \in \mathcal{N}$ there exist $H, X \in \mathfrak{g}$ so that

$$[X, Y] = H, [H, X] = 2X \text{ and } [H, Y] = -2Y. \tag{1.2}$$

Therefore, $\mathrm{span}_{\mathbf{C}}\{X, H, Y\}$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbf{C})$; we denote this subalgebra by $\mathfrak{sl}(2)_Y$. Then $\mathbf{C}^N$ is a representation of $\mathfrak{sl}(2)_Y$ and has a decomposition $\mathbf{C}^N = \oplus V_i$ into irreducible subrepresentations. Then (after ordering the constituents so that $\dim(V_i) \geq \dim(V_{i+1})$) the tableau associated to $G \cdot Y$ has $\dim(V_i)$ boxes in the i^{th} row.

[1]By tableau of size N we mean N boxes arranged in rows where each row has no more boxes than the preceding row.

We now state a fact that will be used in Section 1.4. Suppose that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}^-$ is a parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{q}$ is the stabilizer of a flag

$$\{0\} = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{M-1} \subsetneq W_M = \mathbf{C}^N.$$

Let $d_i = \dim(W_i/W_{i-1}), i = 1, \dots, M$. Then $N = \sum d_i$ is a partition of N, and thus determines a tableau. The tableau of the dual partition (obtained by switching rows and columns) corresponds to a nilpotent orbit $\mathcal{O}$. The following proposition is easily verified. (See [**5**, Section 7.2].)

Proposition 1.1. *With $\mathfrak{q}$ and $\mathcal{O}$ as above $\mathcal{O} \cap \mathfrak{u}^-$ is dense in $\mathfrak{u}^-$ and is a single Q-orbit.*

This orbit $\mathcal{O}$ is referred to as a *Richardson orbit* for Q, due to the general results in [**10**]. It is a fact that $\dim(\mathcal{O}) = 2\dim(\mathfrak{u}^-)$.

Now we describe the K-orbits in $\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{p}$. Suppose $Y \in \mathcal{N}_\theta$. Then there exists an $X \in \mathfrak{p}$ and $H \in \mathfrak{k}$ satisfying (1.2) (see for example [**5**, Thm. 9.4.2]). Consider the decomposition $\mathbf{C}^N = \oplus V_i$ of $\mathbf{C}^N$ as a representation of $\mathfrak{sl}(2, \mathbf{C})_Y$. The proofs of the following are exercises in linear algebra.

Lemma 1.2. *In the decomposition of $\mathbf{C}^N$ into irreducible $\mathfrak{sl}(2, \mathbf{C})_Y$ representations the V_i may be chosen to be mutually orthogonal.*

Let us assume $\mathbf{C}^N = \oplus V_i$ with the V_i mutually orthogonal.

Lemma 1.3. *Suppose V is one of the irreducible constituents and $\dim(V) = d$. Then V has a basis $\{v_1, v_2, \dots, v_d\}$ so that v_1 is a highest weight vector,*

$$v_j = Y^{j-1} v_1, j = 1, ..., d, \text{ and } (v_k, v_l) = \delta_{k,d-l+1}, 1 \le k, l \le d.$$

We associate to the orbit $K \cdot Y$ the partition $N = \sum \dim(V_i)$ and the corresponding tableau as above. It follows easily from this discussion that there is a one-to-one correspondence between K-orbits in $\mathcal{N}_\theta$ and the tableau. It is also follows that each G-orbit in $\mathcal{N}$ meets $\mathfrak{p}$ in a single K-orbit. This is a fact that is special to $GL(N, \mathbf{R})$.

1.2. Generalized flag varieties and $\gamma_{\mathfrak{q}}^{-1}(f)$. The parabolic subalgebras of interest to us are the θ-stable parabolics. We begin by describing them. There will be slight differences for cases of N even and N odd.

Let $\mathfrak{h}$ be a fundamental Cartan subalgebra, that is, $\mathfrak{t} \equiv \mathfrak{h} \cap \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$.

Lemma 1.4. *There is a basis $\{\epsilon_i : i = 1, \dots, N\}$ of $\mathfrak{h}^*$ with the following properties. The set of roots in $\mathfrak{g}$ is $\Delta = \Delta(\mathfrak{h}, \mathfrak{g}) = \{\epsilon_j - \epsilon_k : j \neq k\}$ and*

$$\theta(\epsilon_j) = -\epsilon_{N-j+1}.$$

Proof. The Cartan subalgebra $\mathfrak{h}$ may be chosen to be the subalgebra of diagonal matrices. Now the lemma is easy to verify. $\square$

We will write $\Lambda = \sum \Lambda_i \epsilon_i \in \mathfrak{h}^*$ as $(\Lambda_1, \ldots, \Lambda_N)$. Each such Λ defines a parabolic subalgebra by

$$\begin{aligned} \mathfrak{q}(\Lambda) &= \mathfrak{l} + \mathfrak{u}^- \\ &= \big(\mathfrak{h} + \sum_{\langle \Lambda, \alpha \rangle = 0} g^{(\alpha)}\big) + \sum_{\langle \Lambda, \alpha \rangle < 0} g^{(\alpha)}. \end{aligned}$$

It is clear that $\mathfrak{q}(\Lambda)$ is a θ-stable parabolic if and only if $\theta(\Lambda) = \Lambda$. By Lemma 1.4 this is the case precisely when

$$\Lambda = \begin{cases} (\lambda, -\lambda'), & \text{if } N \text{ is even} \\ (\lambda, 0, -\lambda'), & \text{if } N \text{ is odd} \end{cases} \tag{1.3}$$

where $\lambda' = (\lambda_n, \ldots, \lambda_1)$ for $\lambda = (\lambda_1, \ldots, \lambda_n)$.

One easily sees that the Weyl group of K acts on $\{\Lambda : \theta(\Lambda) = \Lambda\}$ by all permutations and sign changes of the coordinates of λ. Thus, each θ-stable parabolic subalgebra is conjugate to some $\mathfrak{q}(\Lambda)$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

Suppose that $Q = LU^-$ is a parabolic subgroup of G and $\mathfrak{F}$ is the generalized flag variety of parabolic subalgebras conjugate to $\mathfrak{q}$. Then, $\mathfrak{F} \simeq G/Q$. It follows from [**15**] and [**8**] that there is just one closed K-orbits in $\mathfrak{F}$.

We remark that the situation is slightly different when the group G is replaced by $SL(N, \mathbf{C})$.

Now assume that $Q = Q(\Lambda)$ and $\mathfrak{q} = \mathfrak{q}(\Lambda)$ for some Λ satisfying (1.3). We assume that $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Then the unique closed orbit in $\mathfrak{F}$ is $\mathcal{Q} \equiv K \cdot \mathfrak{q}$.

Let

$$\begin{aligned} \gamma_{\mathcal{Q}} &: T^*_{\mathcal{Q}}(\mathfrak{F}) \to \mathcal{N}_\theta \\ \gamma_{\mathcal{Q}}(k, Y) &= \mathrm{Ad}(k)Y \end{aligned}$$

be as in the introduction. The image of $\gamma_{\mathcal{Q}}$ is a the closure of a single K-orbit in $\mathcal{N}_\theta$.

Definition 1.5. We say that $f \in \mathfrak{u}^- \cap \mathfrak{p}$ is *generic* in $\mathfrak{u}^- \cap \mathfrak{p}$ if $\mathcal{O} = K \cdot f$ is dense in $\mathrm{im}(\gamma_{\mathcal{Q}})$.

Note that if f is generic in $\mathfrak{u}^- \cap \mathfrak{p}$ then $K \cdot f$ is the orbit of greatest dimension that meets $\mathfrak{u}^- \cap \mathfrak{p}$.

As indicated in the introduction, our main interest is in explicitly describing $\gamma_{\mathcal{Q}}^{-1}(f)$ when f is generic in $\mathfrak{u}^- \cap \mathfrak{p}$.

It is useful to view the Springer fiber associated to $\mathcal{Q}$ as a subvariety of the flag variety $\mathcal{Q} \simeq K/Q \cap K$. This is done as follows. Let

$$N(f, \mathfrak{u}^- \cap \mathfrak{p}) = \{k \in K : k \cdot f \in \mathfrak{u}^- \cap \mathfrak{p}\}.$$

Then

$$
\begin{aligned}
\gamma_{\mathcal{Q}}^{-1}(f) &= \{(k,\xi) \in K \underset{Q\cap K}{\times} (\mathfrak{u}^- \cap \mathfrak{p}) : k\cdot\xi = f\} \\
&= \{(k, k^{-1}\cdot f) : k^{-1} \in N(f, \mathfrak{u}^- \cap \mathfrak{p})\} \\
&\simeq \{k\cdot\mathfrak{q} \in \mathcal{Q} : k^{-1} \in N(f, \mathfrak{u}^- \cap \mathfrak{p})\} \\
&= N(f, \mathfrak{u}^- \cap \mathfrak{p})^{-1}\cdot\mathfrak{q}.
\end{aligned} \tag{1.4}
$$

1.3. Weight vectors in $\mathfrak{p}$. To prove our main result we will need to work with weight vectors in $\mathfrak{p}$. We give a lemma that describes the $\mathfrak{t}$-weight vectors in $\mathfrak{p}$ and their actions on $\mathbf{C}^N$.

Let $\mu_i = \epsilon_i|_{\mathfrak{t}}$. The weights in $\mathfrak{p}$ are

$$
\begin{aligned}
&\{\pm(\mu_j \pm \mu_k) : 1 \le j \le k \le n\}, \text{ for } N = 2n \text{ and} \\
&\{\pm(\mu_j \pm \mu_k) : 1 \le j \le k \le n\} \cup \{\pm\mu_j : 1 \le j \le n\}, \text{ for } N = 2n+1.
\end{aligned}
$$

Lemma 1.6. *If $N = 2n$ then there are weight vectors $X_{\pm(i+j)}$, $i \ge j$, and X_{i-j}, $i \ne j$, $\mathfrak{p}$ so that:*

$$X_{\pm(i+j)} \text{ has weight } \pm(\mu_i + \mu_j) \text{ and } X_{i-j} \text{ has weight } \mu_i - \mu_j, \tag{1.5}$$

$$
\begin{aligned}
X_{i-j}e_k &= \delta_{k,j}e_i + \delta_{k,N-i+1}e_{N-j+1} \\
X_{i+j}e_k &= \delta_{k,N-j+1}e_i + \delta_{k,N-i+1}e_j \\
X_{2\cdot i}e_k &\equiv X_{i+i}e_k = \delta_{k,N-i+1}e_i \\
X_{-(i+j)}e_k &= \delta_{k,i}e_{N-j+1} + \delta_{k,j}e_{N-i+1}, \\
X_{-2\cdot i}e_k &\equiv X_{-(i+i)}e_k = \delta_{k,i}e_{N-i+1}.
\end{aligned} \tag{1.6}
$$

If $N = 2n+1$ then there are weight vectors $X_{\pm(i+j)}$, $i \ge j$, and X_{i-j}, $i \ne j$, and $X_{\pm i}$, $1 \le i \le n$ so that: (1.5) and (1.6) hold and

$$X_{\pm j} \text{ has weight } \pm\mu_j \text{ and} \tag{1.7}$$

$$
\begin{aligned}
X_i e_k &= \delta_{k,n+1}e_i + \delta_{k,N-i+1}e_{n+1} \\
X_{-i}e_k &= \delta_{k,n+1}e_{N-i+1} + \delta_{k,i}e_{n+1}.
\end{aligned}
$$

Proof. Consider the case $N = 2n$. We may take $\{e_j\}$ to be the standard basis vectors and we may write the weight vectors in terms of $E_{j,k}$ (the matrix with a one in the (j,k)-place and zeros elsewhere). We get

$$
\begin{aligned}
X_{i-j} &= E_{i,j} + E_{N-j+1,N-i+1} \\
X_{i+j} &= E_{i,N-j+1} + E_{j,N-i+1} \\
X_{-(i+j)} &= E_{N-i+1,j} + E_{N-j+1,i} \\
X_{-(i+i)} &= E_{N-i+1,i}.
\end{aligned}
$$

The first part follows from this. The case $N = 2n+1$ is similar. $\square$

1.4. The Springer fiber for $\mathcal{Q}$. Let us fix once and for all $\Lambda \in \mathfrak{h}^*$ satisfying $\theta(\Lambda) = \Lambda$. Then Λ determines λ as in (1.3), which we may assume satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. We let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}^-$ be the θ-stable parabolic subalgebra $\mathfrak{q}(\Lambda)$ as in Subsection 1.2. This in turn specifies a flag variety $\mathfrak{F} = G \cdot \mathfrak{q}$ and a closed K-orbit $\mathcal{Q} = K \cdot \mathfrak{q}$ in $\mathfrak{F}$.

Before describing how to find a nice generic element in $\mathfrak{q}$ in general, we consider four examples.

Example 1. Suppose $N = 2n$ and $\lambda = (n, n-1, \ldots, 2, 1)$. Then $\mathfrak{q}$ is a Borel subalgebra. Set

$$f = X_{2-1} + X_{3-2} + \cdots + X_{n-(n-1)} + X_{-2\cdot n}.$$

Then $f \in \mathfrak{u}^- \cap \mathfrak{p}$. To describe the linear transformation f we will use the following convenient notation. For any linear transformation T suppose that $T(u) = v, T(v) = w, \ldots$, then we will write $T : u \to v \to w, \ldots$, etc. We will use this notation throughout when T is nilpotent, in which case it is particularly descriptive. Therefore, Lemma 1.6 tells us that

$$f : e_1 \to e_2 \to e_3 \to \cdots \to e_{2n-1} \to e_{2n} \to 0.$$

Thus, $f^N = 0$ and $f^{N-1} \neq 0$ and the tableau of $K \cdot f$ is

$$\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \quad\cdots\quad & \ \\ \hline \end{array}\,.$$

It follows that f is a principal nilpotent in $\mathfrak{g}$, so $G \cdot f$ is the G orbit of greatest dimension. We conclude that $K \cdot f$ is the K-orbit in $\mathcal{N}_\theta$ of greatest dimension (by [**5**, Rem. 9.5.2]), and therefore f is generic in $\mathfrak{u}^- \cap \mathfrak{p}$.

Example 2. Consider $N = 2n + 1$. Let $\lambda = (n, n-1, \ldots, 2, 1)$. Set

$$f = X_{2-1} + X_{3-2} + \cdots + X_{n-(n-1)} + X_{-n}.$$

Then

$$f : e_1 \to e_2 \to e_3 \to \cdots \to e_{2n} \to e_{2n+1} \to 0.$$

It follows that f is a principal nilpotent element in $\mathfrak{g}$, so is generic in $\mathfrak{u}^- \cap \mathfrak{p}$. Again, the tableau has just one row.

Example 3. Consider $N = 2n + 1$ and $\lambda = (n-1, \ldots, 2, 1, 0)$. Then

$$f = X_{2-1} + X_{3-2} + \cdots + X_{(n-1)-(n-2)} + X_{-(n-1)}.$$

is in $\mathfrak{u}^- \cap \mathfrak{p}$ and

$$\begin{aligned} &e_1 \to e_2 \to \cdots \to e_{n-1} \to e_{n+1} \to e_{n+3} \to e_{n+4} \cdots \to e_{2n} \to e_{2n+1} \to 0 \\ &e_n \to 0, \quad e_{n+2} \to 0. \end{aligned}$$

The tableau is

$$\begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \quad\cdots\quad & \ & \ \\ \hline \ & \multicolumn{5}{c}{} \\ \cline{1-1} \ & \multicolumn{5}{c}{} \\ \cline{1-1} \end{array}\,. \tag{1.8}$$

To see that f is generic in $\mathfrak{u}^- \cap \mathfrak{p}$ note that the parabolic subalgebra $\mathfrak{q}$ is given as the stabilizer of the flag

$$\begin{aligned}
&\{0\} = F_{2n-1} \subsetneq F_{2n-2} \subsetneq \cdots \subsetneq F_1 \subsetneq F_0 = \mathbf{C}^N : \\
&\quad F_i = \mathrm{span}_{\mathbf{C}}\{e_{i+1}, e_{i+2} \dots, e_N\}, \text{ for } i = 0, 1, \dots, n-1, \\
&\quad F_{n+j-1} = (F_{n-j})^{\perp} = \mathrm{span}_{\mathbf{C}}\{e_{n+j+2}, e_{n+j+3} \dots, e_N\}, \text{ for } j = 1, \dots, n.
\end{aligned}$$

So the partition $N = \sum \dim(F_{i-1}/F_i) = 1 + \cdot + 1 + 3 + 1 + \cdot + 1$ has dual giving the partition of (1.8). Now Proposition 1.1 tells us that if $Y \in \mathfrak{u}^-$ then $Y \in \overline{G \cdot f}$. In particular, for any $Y \in \mathfrak{u}^- \cap \mathfrak{p}$, $\dim_{\mathbf{C}}(K \cdot Y) \le \dim_{\mathbf{C}}(K \cdot f)$ ([**5**, Rem. 9.5.2]), so f is generic in $\mathfrak{u}^- \cap \mathfrak{p}$.

Example 4. Let $N = 2n$ and $\lambda = (n-1, \dots, 2, 1, 0)$. Then

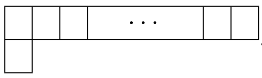
$$f = X_{2-1} + X_{3-2} + \cdots + X_{n-(n-1)} + X_{-(n+(n-1))}.$$

is in $\mathfrak{u}^- \cap \mathfrak{p}$ and

$$\begin{aligned}
&f \,:\, e_1 \to e_2 \to \cdots \to e_{n-1} \to e_n + e_{n+1} \to 2e_{n+1} \to 2e_{n+2} \to \cdots \to 2e_{2n} \to 0 \\
&\text{and } e_n - e_{n+1} \to 0.
\end{aligned}$$

The tableau is

(1.9)



To see that f is generic in $\mathfrak{u}^- \cap \mathfrak{p}$ note that the parabolic subalgebra $\mathfrak{q}$ is the stabilizer of the flag

$$\begin{aligned}
&\{0\} = F_{2n-1} \subsetneq F_{2n-2} \subsetneq \cdots \subsetneq F_1 \subsetneq F_0 = \mathbf{C}^N : \\
&\quad F_i = \mathrm{span}_{\mathbf{C}}\{e_{i+1}, e_{i+2} \dots, e_N\}, \text{ for } i = 0, 1, \dots, n-1, \\
&\quad F_{n+j-1} = (F_{n-j})^{\perp} = \mathrm{span}_{\mathbf{C}}\{e_{n+j+2}, e_{n+j+3} \dots, e_N\}, \text{ for } j = 1, \dots, n.
\end{aligned}$$

The argument for f being generic is the same as in *Example 3.*

We now return to our arbitrary θ-stable parabolic subalgebra $\mathfrak{q}$ defined by Λ as above and give a construction of a generic element f in $\mathfrak{u}^- \cap \mathfrak{p}$. This will be accomplished by an inductive procedure; first f_0 will be specified (roughly as in the examples), then a reduction will be made to a lower rank general linear group where f_1 will be specified, etc. Then $f = f_0 + f_1 + \cdots + f_{m-1}$ will be our generic element.

Write $\lambda = (\lambda_1, \dots, \lambda_n)$ as

$$\lambda = (\underbrace{a, \dots, a}_{d_1}, \underbrace{b, \dots, b}_{d_2}, \dots, \underbrace{c, \dots, c}_{d_l}), \text{ with } a > b. \cdots > c \ge 0. \tag{1.10}$$

To facilitate the description of f we will refer to the indices of each set of equal coordinates of λ as *blocks*. There are l blocks.

Specifying f_0 depends on two things. It depends on whether or not $\lambda_n = c$ is zero and if $N = 2n$ or $N = 2n+1$.

Case 1. Suppose $\lambda_n = c \neq 0$. For $i = 1, \dots, l$ let k_i be the last index in the i^{th} block (counting from left to right).

When $N = 2n$ define

$$f_0 = (\sum_{i=1}^{l-1} X_{k_{i+1}-k_i}) + X_{-2\cdot k_l}.$$

When $N = 2n + 1$ define

$$f_0 = (\sum_{i=1}^{l-1} X_{k_{i+1}-k_i}) + X_{-k_l}.$$

It will be useful for us to extend the sequence $k_1, k_2, \dots, k_l$ to $k_1, \dots, k_l, \dots, k_\ell$ with

$$\ell = \begin{cases} 2l, & \text{if } N = 2n \\ 2l+1 & \text{if } N = 2n+1 \end{cases}$$

This is done by setting $k_{\ell-i+1} = N - k_i + 1$, for $i = 1, 2, \dots, l$ and, in the odd case, $k_{l+1} = n + 1\,(= k_l + 1)$.

Then in both even and odd cases of N, using the notation of the four examples,

$$f \,:\, e_{k_1} \to e_{k_2} \to \cdots e_{k_{\ell-1}} \to e_{k_\ell} \to 0 \text{ and } e_k \to 0 \text{ for } k \notin \{k_i\}.$$

Set

$$V' = \mathrm{span}_{\mathbf{C}}\{e_{k_i} : i = 1, \dots, \ell\}$$

and let $V = (V')^\perp$. Observe that $(\ ,\ )$ is nondegenerate on both V' and V, and $\mathbf{C}^N = V \oplus V'$. Define

$$\begin{aligned} G_1' &= \{g \in G : g(V') \subset V' \text{ and } g|_V = I_V\} \text{ and} \\ G_1 &= \{g \in G : g(V) \subset V \text{ and } g|_{V'} = I_{V'}\}. \end{aligned} \tag{1.11}$$

Observe that G_1' and G_1 are mutually commuting Θ-stable subgroups of G, each isomorphic to a general linear group.

Since $f_0 \in \mathfrak{g}_1' \cap \mathfrak{p}$, we may choose $e_0 \in \mathfrak{g}_1' \cap \mathfrak{p}$ and $h_0 \in \mathfrak{g}_1' \cap \mathfrak{k}$ so that $\{e_0, h_0, f_0\}$ is a standard basis for a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbf{C})$. Let us call this subalgebra $\mathfrak{sl}(2)_{f_0}$. In fact, h_0 may be chosen to lie in $\mathfrak{h} \cap \mathfrak{g}_1'$. Therefore, the standard basis vectors e_k are h_0-weight vectors. It is immediately seen that V' is an irreducible $\mathfrak{sl}(2)_{f_0}$-representation and e_{k_1} is a highest weight vector; this is essentially Examples 1 and 2.

Case 2. $\lambda_n = c = 0$. Again let k_i be the last index in the i^{th}-block. Let $\ell = 2l - 1$ and $k_{\ell-i+1} = N - k_i + 1$, for $i = 1, 2, \dots, l - 1$.

First consider the case $N = 2n + 1$. Let Define

$$f_0 = (\sum_{i=1}^{l-2} X_{k_{i+1}-k_i}) + X_{-k_{l-1}}.$$

Then

$$f \,:\, e_{k_1} \to e_{k_2} \to \cdots\cdots e_{k_{\ell-1}} \to e_{k_\ell} \to 0 \text{ and } e_k \to 0 \text{ for } k \notin \{k_i\}.$$

Let

$$V' = \mathrm{span}_{\mathbf{C}}\{e_{k_i} : i = 1, \dots, \ell\}$$

Now consider $N = 2n$. This case is a little more involved. Define

$$f_0 = (\sum_{i=1}^{l-1} X_{k_{i+1}-k_i}) + X_{-(k_l+k_{l-1})}.$$

Then

$$f : e_{k_1} \to \cdots \to e_{k_{l-1}} \to e_{k_l} + e_{k_l+1} \to 2e_{k_{l+1}} \to \cdots \to 2e_{k_\ell} \to 0$$
$$e_{k_l} - e_{k_l+1} \to 0 \text{ and } e_k \to 0 \text{ for } k \notin \{k_i\}.$$

We set

$$V' = \mathrm{span}_{\mathbf{C}}(\{e_{k_i}, e_{k_{\ell-i+1}} : i = 1, \ldots, l-1\} \cup \{e_{k_l} + e_{k_l+1}\}).$$

In either case $N = 2n$ or $2n+1$, let $V = (V')^\perp$ and define G_1' and G_1 as in (1.11). Choosing $\mathfrak{sl}(2)_{f_0}$ as in Case 1 we see that V' is irreducible.

Note that in the four cases $\ell = \dim(V')$ is given by

$$\ell = \begin{cases} 2l+1, & \text{if } \lambda_n \neq 0 \text{ and } N = 2n+1 \\ 2l, & \text{if } \lambda_n \neq 0 \text{ and } N = 2n \\ 2l-1, & \text{if } \lambda_n = 0 \text{ and } N = 2n \text{ or } 2n+1. \end{cases} \tag{1.12}$$

In both Cases 1 and 2, G_1 is a general linear group of lower rank than G, and (G_1, K_1), $K_1 = K \cap G_1$, is a pair of the same type as (G, K). The Lie algebra $\mathfrak{g}_1$ contains the θ-stable parabolic subalgebra $\mathfrak{q}_1 = \mathfrak{q} \cap \mathfrak{g}_1$. Write $\mathfrak{q}_1 = \mathfrak{l}_1 + \mathfrak{u}_1^-$. This parabolic $\mathfrak{q}_1$ is defined by $\Lambda_1 = \Lambda|_{\mathfrak{h}\cap\mathfrak{g}_1}$.

Now choose $f_1 \in \mathfrak{u}_1^- \cap \mathfrak{p}$ inside $\mathfrak{g}_1$ by the same procedure that was used to choose $f_0 \in \mathfrak{u}^- \cap \mathfrak{p}$. Define G_2' and G_2 in G_1 in the same way that G_1' and G_1 were defined in G. Continue by choosing $f_2 \in \mathfrak{u}_2^- \cap \mathfrak{p}$ inside $\mathfrak{g}_2$, etc. The procedure ends when $\Lambda|_{\mathfrak{h}\cap\mathfrak{g}_m} = 0$. Finally, we take

$$f = f_0 + f_1 + f_2 + \cdots + f_{m-1}. \tag{1.13}$$

Before stating and proving the main result, we shall give a description of the parabolic subalgebra $\mathfrak{q}$ as the stabilizer of a flag in $\mathbf{C}^N$. First consider the case when $\lambda_n = c \neq 0$. Let $F_0 = \mathbf{C}^N$,

$$F_i = \mathrm{span}_{\mathbf{C}}\{e_k : k > k_i\}, \text{ for } i = 1, \ldots, l$$

and

$$F_{l+i} = \begin{cases} (F_{l-i+1})^\perp,\ i = 1, \ldots, l+1, & \text{if } N = 2n+1, \\ (F_{l-i})^\perp,\ i = 1, \ldots, l, & \text{if } N = 2n \end{cases}.$$

Now suppose that $\lambda_n = c = 0$. Let $F_0 = \mathbf{C}^N$,

$$F_i = \mathrm{span}_{\mathbf{C}}\{e_k : k > k_i\}, \text{ for } i = 1, \ldots, l-1$$

and

$$F_{l+i-1} = (F_{l-i})^\perp, i = 1, \ldots, l, \text{ for } N = 2n \text{ or } N = 2n+1.$$

Therefore we have defined a flag

$$\{0\} = F_\ell \subsetneq F_{\ell-1} \subsetneq F_{\ell-2} \subsetneq \cdots \subsetneq F_1 \subsetneq F_0 = \mathbf{C}^N,$$

where ℓ is as in (1.12)

It is easy to see that $\mathfrak{q} = \{X \in \mathfrak{g} : X(F_i) \subset F_i, \text{for } i = 1, \dots, \ell\}$ and

$$\mathfrak{u}^- = \{X \in \mathfrak{g} : X(F_{i-1}) \subset F_i, \text{for } i = 1, \dots, \ell\}. \tag{1.14}$$

REMARK 1.7. This flag may also be defined as follows. Consider

$$H_\Lambda = \begin{pmatrix} \lambda_1 &&&&& \\ & \ddots &&&& \\ && \lambda_n &&& \\ &&& -\lambda_n && \\ &&&& \ddots & \\ &&&&& -\lambda_1 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 &&&&&& \\ & \ddots &&&&& \\ && \lambda_n &&&& \\ &&& 0 &&& \\ &&&& -\lambda_n && \\ &&&&& \ddots & \\ &&&&&& -\lambda_1 \end{pmatrix},$$

for $N = 2n$ or $2n+1$. Then $F_{\ell-i}$ $(i = 1, \dots, \ell)$ is the sum of the eigenspaces of H_Λ for the i smallest eigenvalues.

It follows from the tableau of $K \cdot f$ and the description of the flag that Proposition 1.1 implies that f is Richardson in $\mathfrak{u}^-$.

REMARK 1.8. At this point we may conclude that the statement of Theorem 1.15 holds by appealing to the well-known fact that the moment map μ is birational for $GL(N, \mathbf{C})$. This follows, for example, from [**5**, Section 7.2] and the description of the the centralizers of nilpotent elements in $\mathfrak{gl}(N, \mathbf{C})$ given in [**5**, Theorem 6.1.3] However, we shall give an independent proof. This is for two reasons; we will obtain slightly more information (Theorem 1.11) and the method of proof applies to other classical groups.

REMARK 1.9. There is a basis $\{e'_k\}$ of $\mathbf{C}^N$ with the following properties:

(a) V' is the span of $\{e'_{k_i} : i = 1, 2, \dots, \ell\}$ and this basis is as in Lemma 1.3,

(b) V is spanned by $\{e_k : k \notin \{k_i\}\}$,

(c) Setting $k_0 = 0$, $\{e_k : k_{i-1} < k < k_i\}$ maps to a basis in $F_{i-1} \cap V / F_i \cap V$.

To see this take $e'_k = e_k$ unless $\lambda_n = c = 0$ and $N = 2n$. When $\lambda_n = c = 0$ and $N = 2n$ set

$$e'_{k_i} = \begin{cases} e_{k_i}/\sqrt{2}, & \text{if } i \neq l \\ (e_{k_l} + e_{k_l+1})/\sqrt{2}, & \text{if } i = l \end{cases}$$

$$e'_k = \begin{cases} (e_{k_l} - e_{k_l+1})/\sqrt{-2}, & \text{if } k = k_l + 1 \\ e_k, & \text{if } k \neq k_l + 1 \text{ or } k_i (\text{any } i) \end{cases}.$$

PROPOSITION 1.10. *Let f be as in (1.13) and suppose that $Y \in \mathfrak{u}^- \cap \mathfrak{p}$ and $f \in \overline{K \cdot Y}$. Then there exists $q \in Q \cap K$ so that*

(a) $Y_1 \equiv (q \cdot Y) - f_0 \in \mathfrak{u}_1^- \cap \mathfrak{p}$ *and*

(b) $f_1 + \cdots f_{m-1} \in \overline{K \cdot Y_1}$.

PROOF. Let $Y \in \mathfrak{u}^- \cap \mathfrak{p}$ with $f \in \overline{K \cdot Y}$. Consider a standard triple $\{X, H, Y\}$ with the property that $X \in \mathfrak{p}$ and $H \in \mathfrak{h}$. Let $\mathfrak{sl}(2)_Y$ be the span of $\{X, H, Y\}$. It follows from [**5**, Lem. 6.2.2] that

$$\operatorname{rank}(f^j) \leq \operatorname{rank}(Y^j), \text{ for all } j = 1, 2, \dots. \tag{1.15}$$

Since $f_0^{\ell-1} \neq 0$ (because $\ell = \dim(V')$) it follows that $Y^{\ell-1} \neq 0$. On the other hand, since $Y \in \mathfrak{u}^-$ it follows from (1.14) that $Y^\ell = 0$. We conclude that $\mathbf{C}^N$ has an irreducible $\mathfrak{sl}(2)_Y$-subrepresentation of dimension ℓ. Let us call this subrepresentation W'. By Lemma 1.2 we may assume that $(\ ,\ )$ is nondegenerate on W'. Let $W = (W')^\perp$. Let $w_1', w_2', \dots, w_\ell'$ be a basis as in Lemma 1.3. Note that $w_j' \in F_j \setminus F_{j-1}$.
<u>Claim</u>. *For each* $i, F_i = (F_i \cap W') \oplus (F_i \cap W)$, *therefore* $\dim(F_{i+1} \cap W/F_i \cap W) = \dim(F_{i+1}/F_i) - 1$.
Let us prove the claim. Let $v \in F_i$. Write $v = w' + w \in W' \oplus W$. Therefore, $v = \sum_{j=1}^{\ell} \alpha_j w_j' + w$. Since $v \in F_i$, $Y^{\ell-i+1}v = 0$. It follows that

$$0 = \sum_{j=1}^{\ell} \alpha_j Y^{\ell-i} w_j' + Y^{\ell-i} w.$$

But W' and W are Y-stable, so

$$0 = \sum_{j=1}^{\ell} \alpha_j Y^{\ell-i+1} w_j' = \sum_{j=i+1}^{\ell} \alpha_j w_j'.$$

Therefore, $\alpha_{i+1} = \alpha_{i+2} = \cdots = \alpha_\ell = 0$. We may now conclude that $\sum_{j=1}^{\ell} \alpha_j w_j' \in F_i \cap W'$, and therefore $w \in F_i \cap W$. This proves the claim.

It follows from the claim that there is an ordered basis $u_1, u_2, \dots, u_N$ of $\mathbf{C}^N$ so that

(i) $u_{k_i} = w_i', i = 1, 2, \dots, \ell$
(ii) $\{u_k : k_{i-1} < k < k_i\}$ maps to a basis in $F_{i-1} \cap W/F_i \cap W$ and
(ii) $(u_j, u_{N-k+1}) = \delta_{jk}$.

Define a linear transform $q : \mathbf{C}^N \to \mathbf{C}^N$ by $q(u_k) = e_k'$, for all $k = 1, 2, \dots, N$. By (ii) and (iii) $q \in Q \cap K$. Now we will prove statement (a) of the proposition by showing that $Y_1 = q \cdot Y - f_0$ satisfies $Y_1|_{V'} = 0$ and $Y_1(V) \subset V$. By (i),

$$\begin{aligned} \big((q \cdot Y) - f_0\big)(e_{k_i}') &= qY(w_i') - e_{k_{i+1}}' \\ &= qw_{i+1}' - e_{k_{i+1}}' = 0 \end{aligned}$$

and for $k \notin \{k_i\}$

$$(q \cdot Y - f_0)(e_k') = qYu_k - 0 \in q(W) \subset V.$$

Therefore $Y_1 = q \cdot Y - f_0 \in \mathfrak{g}_1$. But $q \in K$, so $Y_1 \in \mathfrak{p}$. We now conclude that $Y_1 \in \mathfrak{u}_1^- \cap \mathfrak{p}$, proving (a).

For statement (b) note that

$$\operatorname{rank}((f - f_0)^j) = \operatorname{rank}(f^j) - (\ell - j) \text{ and}$$
$$\operatorname{rank}(Y_1^j) = \operatorname{rank}(Y^j) - (\ell - j)$$

and apply (1.15) and [**5**, Lem. 6.2.2]. $\square$

THEOREM 1.11. *Let $\mathfrak{q}$ be any θ-stable parabolic subalgebra of $\mathfrak{g} = \mathfrak{gl}(N, \mathbf{C})$. Then any two generic elements in $\mathfrak{q}$ are conjugate under $Q \cap K$.*

PROOF. This is a consequence of the following lemma.

LEMMA 1.12. *Let $Y \in \mathfrak{u}^- \cap \mathfrak{p}$ and f be as in (1.13). If $f \in \overline{K \cdot Y}$, then there exists $q \in Q \cap K$ so that $q \cdot Y = f$.*

PROOF. The proof is by induction on N. By Proposition 1.10 there is $q \in Q \cap K$, $Y_1 = (q \cdot Y) - f_0 \in \mathfrak{u}_1^- \cap \mathfrak{p}$ and $f_1 + \cdots + f_{m-1} \in \overline{K \cdot Y_1}$. Since G_1 is a lower rank general linear group, the inductive hypothesis says that there exists $q_1 \in Q_1 \cap K \subset Q \cap K$ so that $q_1 \cdot Y_1 = f_1 + \cdot + f_{m-1} = f - f_0$. Therefore,

$$\begin{aligned} f &= q_1 \cdot Y_1 + f_0 \\ &= q_1 \cdot (Y_1 + f_0), \text{ since } f_0 \in \mathfrak{g}_1' \text{ commutes with } \mathfrak{g}_1, \\ &= q_1 q \cdot Y. \end{aligned}$$

□

The theorem now follows since the lemma implies that any two generic elements in $\mathfrak{q}$ are $Q \cap K$-conjugate to f. □

COROLLARY 1.13. *f is generic in $\mathfrak{u}^- \cap \mathfrak{p}$.*

We now turn to the fiber of $\gamma_{\mathcal{Q}}$. As described in (1.4)

$$\gamma_{\mathcal{Q}}^{-1}(f) = (N(f, \mathfrak{u}^- \cap \mathfrak{p}))^{-1} \cdot \mathfrak{q}.$$

Suppose that k is in $N(f, \mathfrak{u}^- \cap \mathfrak{p})$. Then $k \cdot f$ lies in $\mathfrak{u}^- \cap \mathfrak{p}$ and is generic. By Theorem 1.11 there is a $q \in Q \cap K$ so that $k \cdot f = q \cdot f$. Therefore, $q^{-1}k \in Z_K(f)$, the centralizer of f in K, so

$$\gamma_{\mathcal{Q}}^{-1}(f) = Z_K(f)\,(Q \cap K) \cdot \mathfrak{q} = Z_K(f) \cdot \mathfrak{q}.$$

LEMMA 1.14. *$\gamma_{\mathcal{Q}}^{-1}(f)$ is finite.*

PROOF. Since f is Richardson in $\mathfrak{u}^-$, $\dim(G \cdot f) = 2\dim(\mathfrak{u}^-)$. Therefore we have $\dim(K \cdot f) = \dim(\mathfrak{u})$. Now

$$\begin{aligned} \dim(\gamma_{\mathcal{Q}}^{-1}(f)) &= \dim(T^*_{\mathcal{Q}}\mathfrak{F}) - \dim(K \cdot f) \\ &= \dim(\mathfrak{F}) - \dim(\mathfrak{u}^-) \\ &= 0 \end{aligned}$$

□

THEOREM 1.15. *Let $\mathfrak{q}$ be any θ-stable parabolic in $\mathfrak{g}$ and $\mathcal{Q} = K \cdot \mathfrak{q}$ the corresponding closed orbit in $\mathfrak{F}$. Then $\gamma_{\mathcal{Q}}^{-1}(f) = \{\mathfrak{q}\}$.*

PROOF. Since the unipotent part of $Z_K(f)$ is connected it is enough to show that the reductive part of $Z_K(f)$ is contained in Q. We consider $Z_K(f)_{red} = Z_K(\mathfrak{sl}(2)_f)$, the $\mathfrak{sl}(2)_f$-intertwining operators of $\mathbf{C}^N$.

We prove that $Z_K(f)_{red} \subset L \subset Q$ by induction on N. Let $V(\ell)$ be the isotypic subspace of $\mathbf{C}^N$ for the ℓ-dimensional irreducible representation of $\mathfrak{sl}(2)_f$. Then $\mathbf{C}^N = V(\ell) \oplus (V(\ell))^\perp$. Define

$$\begin{aligned}\tilde{G}' &= \{g \in G : g(V(\ell)) \subset V(\ell), g|_{(V(\ell))^\perp} = I_{(V(\ell))^\perp}\}\\ \tilde{G} &= \{g \in G : g((V(\ell))^\perp) \subset (V(\ell))^\perp, g|_{V(\ell)} = I_{V(\ell)}\}.\end{aligned}$$

Since $(V(\ell))^\perp$ contains no irreducible subrepresentation of dimension ℓ, $Z_K(f)_{red} \subset \tilde{G}' \times \tilde{G}$. In fact

$$Z_K(f)_{red} = (\tilde{G}' \cap Z_K(f)_{red}) \times (\tilde{G} \cap Z_K(f)_{red}).$$

The first factor is contained in L since each intertwining map must preserve h-weight spaces in $V(\ell)$ (thus preserve the flag, see 1.7). The second factor is in $L \cap \tilde{G} \subset L$ by induction. $\square$

2. Indefinite unitary groups

Let $G_{\mathbf{R}} = U(p,q)$ be the group of linear transformations g satisfying $gI_{p,q}\overline{g}^t = I_{p,q}$, where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}. \tag{2.1}$$

Let $n = p+q$. The complexification of $G_{\mathbf{R}}$ is $G = GL(n,\mathbf{C})$ and $\theta = \mathrm{Ad}(I_{p,q}) : \mathfrak{g} \to \mathfrak{g}$ is the complexification of a Cartan involution of $\mathfrak{g}_{\mathbf{R}}$. The complexified Cartan decomposition is $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where

$$\mathfrak{k} = \Big\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \,:\, A \in \mathfrak{gl}(p,\mathbf{C}) \text{ and } D \in \mathfrak{gl}(q,\mathbf{C}) \Big\}$$

and

$$\mathfrak{p} = \Big\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \,:\, B, C^t \in M_{pq}(\mathbf{C}) \Big\}.$$

The group

$$K = \Big\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \,:\, a \in GL(p,\mathbf{C}) \text{ and } d \in GL(q,\mathbf{C}) \Big\}$$

is the fixed point group of the involution Θ of G given by conjugation by $I_{p,q}$. Thus, we consider the pair

$$(G,K) = (GL(n,\mathbf{C}), GL(p,\mathbf{C}) \times GL(q,\mathbf{C})). \tag{2.2}$$

We will show how to find generic elements in θ-stable Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ and compute the Springer fibers associated to the closed K-orbits $\mathfrak{Q} = K \cdot \mathfrak{b}$ in $\mathfrak{B}$. These results are proved in [**1**]. The method used there is similar to the method used in Section 1.4. The result for $(G,K) = (GL(n,\mathbf{C}), GL(p,\mathbf{C}) \times GL(q,\mathbf{C}))$ is however quite different from the case of $(GL(N,\mathbf{C}), O(N,\mathbf{C}))$ described in the preceding section. The purpose of this section is two-fold. First, we illustrate these differences by means of a few examples. Second, the cases of $G_{\mathbf{R}} = Sp(2n,\mathbf{R})$ and $G_{\mathbf{R}} = O(p,q)$ in the following sections are more easily understood in terms of this case along with an embedding of these groups into indefinite unitary groups.

To describe the K-orbits in $\mathcal{N}_\theta$ two well-known lemmas, analogous to Lemmas 1.2 and 1.3, will apply. To state these lemmas let $\langle\, ,\, \rangle$ be the hermitian form having matrix $I_{p,q}$ with respect to the standard basis of $\mathbf{C}^n$. Suppose Y is in $\mathcal{N}_\theta \setminus \{0\}$ and $\{X, H, Y\}$ is a standard triple with $X \in \mathfrak{p}$ and $H \in \mathfrak{k}$. Let $\mathfrak{sl}(2)_Y = \mathrm{span}_{\mathbf{C}}\{X, H, Y\}$. Write the decomposition of $\mathbf{C}^n$ into irreducible $\mathfrak{sl}(2)_Y$-representations as $\mathbf{C}^n = \oplus V_i$.

LEMMA 2.1. *In the decomposition of $\mathbf{C}^n$ into irreducible $\mathfrak{sl}(2)_Y$-representations we may assume that each V_i is $I_{p,q}$-invariant and the V_i's are mutually orthogonal with respect to $\langle\, ,\, \rangle$ (and therefore the hermitian form is nondegenerate on each V_i).*

LEMMA 2.2. *If V is any one of the irreducible constituents in $\mathbf{C}^n$ (as in the previous lemma), then there is a basis $v_1, v_2, \ldots, v_\ell$ so that each v_i is an H-weight vector,*

$$v_i = Y^{i-1}v_1 \ \textit{and}\ \langle v_j, v_{\ell-k+1}\rangle = \delta_{jk}.$$

Necessarily, each v_i is and eigenvector for $I_{p,q}$ of eigenvalue ± 1 and these eigenvalues alternate in the sense that $I_{p,q}v_i = (-1)^i v_i$, for all i, or $I_{p,q}v_i = (-1)^{i-1}v_i$, for all i.

A signed tableau is associated to $K \cdot Y$ as follows. Arranging the decomposition $\mathbf{C}^n = \oplus V_i$ as in the first lemma and so that $\dim(V_i) \geq \dim(V_{i+1})$ for all i, the tableau has $\dim(V_i)$ boxes in the i^{th} row. A plus or minus sign is placed in the first block of the i^{th} row according to the sign of the eigenvalue of $I_{p,q}$ on the lowest weight vector in V_i. Then the remaining boxes are filled with $+$ or $-$ signs so that the signs alternate along each row. There is a one-to-one correspondence between K-orbits in $\mathcal{N}_\theta$ and such tableaux up to permutation of equal size rows.

Now let us describe the closed K-orbits in the full flag variety $\mathfrak{B}$. Let $\mathfrak{h}$ be the diagonal Cartan subalgebra in $\mathfrak{g}$; $\mathfrak{h}$ is also a Cartan subalgebra in $\mathfrak{k}$. Fix the positive system of compact roots

$$\Delta_c^+ = \{\epsilon_i - \epsilon_j \ :\ 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq n\}. \tag{2.3}$$

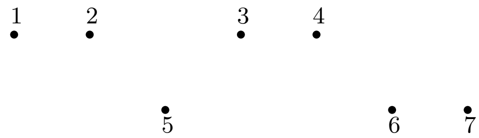
Suppose $\Lambda \in \mathfrak{h}^*$ is Δ_c^+-dominant and regular. Then the positive system $\Delta^+ = \{\alpha : \langle \Lambda, \alpha\rangle > 0\}$ in $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$ contains Δ_c^+. This Δ^+ defines a Borel subalgebra by

$$\mathfrak{b}(\Lambda) = \mathfrak{b} = \mathfrak{h} + \mathfrak{n}^- \ \text{ with}$$
$$\mathfrak{n}^- = \sum_{\langle \Lambda, \alpha\rangle < 0} \mathfrak{g}^{(\alpha)}$$

and $\mathcal{Q} = K \cdot \mathfrak{b}$ is a closed K-orbit in $\mathfrak{B}$. In fact, the closed K-orbits in $\mathfrak{B}$ are in one-to-one correspondence with positive systems of roots in $\Delta(\mathfrak{h}, \mathfrak{g})$ that contain Δ_c^+ ([**15**]). Therefore, each closed orbit is determined by a Δ_c^+-dominant and Δ-regular $\Lambda \in \mathfrak{h}^*$. The first p coordinates of such a Λ are decreasing as are the last q coordinates. Let us fix such a Λ and Borel subalgebra $\mathfrak{b}(\Lambda) = \mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$.

We will give an algorithm for finding a generic element in $\mathfrak{n}^- \cap \mathfrak{p}$ and describe the structure of $\gamma_{\mathcal{Q}}^{-1}(f)$. The discussion will be somewhat informal and will be accompanied by an example.

The example we will use is in $U(4, 3)$. Take $\Lambda = (7, 6, 4, 3 \mid 5, 2, 1)$. Then we associate to Λ the positive system as described above. We also associate to Λ the following array.

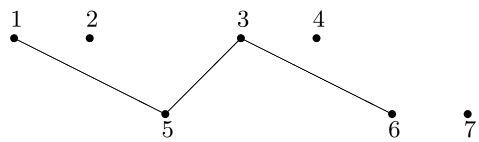


This array is formed by placing dots in one of two rows . The first dot (counting from left to right) is labelled with the index of the greatest coordinate of Λ. It is placed in the upper row if this coordinate is one of the first p coordinates of Λ and in the lower row otherwise. The next dot is labelled with the index of the of the next greatest coordinate of Λ, and is placed in the upper or lower row in the same manner as the first dot. This is continued until n dots are placed.

A few observations are useful. If i and j are labels of dots in the array, then $\epsilon_i - \epsilon_j$ is in Δ^+ if and only if i appears to the left of j. Also, $\epsilon_i - \epsilon_j$ is a compact (respectively, noncompact) root if both i and j are in the same row (respectively, in different rows). The simple roots in Δ^+ are $\epsilon_i - \epsilon_j$ with i and j the labels of consecutive dots (with i to the left of j). In the example the simple roots are $\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_5, \epsilon_5 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4 - \epsilon_6$ and $\epsilon_6 - \epsilon_7$. The compact simple roots are $\epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4$ and $\epsilon_6 - \epsilon_7$.

Before giving our description of a generic element in $\mathfrak{n}^- \cap \mathfrak{p}$ we introduce one piece of terminology. A *block* is a subset of $\{1, 2, \dots, n\}$ that is a maximal with respect to being labels of dots (i) that are consecutive and (ii) lie in just one of the two rows. Therefore, there are four blocks in the example; they are $\{1, 2\}, \{5\}, \{3, 4\}$ and $\{6, 7\}$.

Our generic f will be determined by first choosing f_0, then forming a smaller array and choosing f_1, etc. Begin by connecting a dot in the first block with one in the second block. Then connect this to a dot in the third block, etc. One such choice for the example is



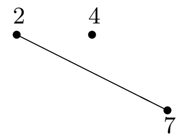
Write $\{k_1, k_2, \dots, k_\ell\}$ for the labels of dots in each block that have been connected. These are listed from left to right, i.e., k_i is the label of a dot in the i^{th} block. Set

$$f_0 = \sum_{i=1}^{\ell-1} E_{k_{i+1},k_i} = E_{6,3} + E_{3,5} + E_{5,1}.$$

Here $E_{i,j}$ is the matrix with 1 in the i, j place and 0 elsewhere, an $\epsilon_i - \epsilon_j$-root vector. Observe that

$$f_0 \,:\, e_{k_1} \to e_{k_2} \to \cdots \to e_{k_\ell} \to 0 \text{ and } e_k \to 0, \text{ for } e_k \notin \{k_i\}.$$

Now omit the dots labelled by the k_i and form a new (smaller) array and repeat. In the example we get



and $f_1 = E_{7,2}$. Note that $f_1 : e_2 \to e_7 \to 0$ and $e_k \to 0$ for all $k \neq 2$.

For the example $f = f_0 + f_1$. In general, this procedure of choosing an f_i by connecting dots in consecutive blocks, then deleting dots in the array will end after, say, m repetitions. Then

$$f = f_0 + f_1 + \cdots f_{m-1}.$$

One easily sees that the standard basis vectors are h-weight vectors for $\mathfrak{sl}(2)_f = \mathrm{span}_{\mathbf{C}}\{e, h, f\}$. Therefore the tableau of $K \cdot f$ is easily read off from the diagram. In the example the tableau is

$-$	$+$	$-$	$+$
$-$	$+$		
$+$			

The fact that the f constructed above is generic in $\mathfrak{n}^- \cap \mathfrak{p}$ is proved by essentially the same method used in Section 1.4. However, the situation here is much more complicated. In fact, Proposition 1.10 *does not hold*; it is clear that $B \cap K \cdot f$ consists of generic elements in $\mathfrak{n}^- \cap \mathfrak{p}$ (since $B \cap K$ normalizes $\mathfrak{n}^- \cap \mathfrak{p}$), but typically this is not all of the generic elements. To see this, let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}^-$ be the parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ determined by the property that $\Delta(\mathfrak{l}) = \mathrm{span}\{\text{compact simple roots}\} \cap \Delta$. Then $Q \cap K$ normalizes $\mathfrak{n}^- \cap \mathfrak{p} = \mathfrak{u}^- \cap \mathfrak{p}$, so $Q \cap K \cdot f$ consists of generic elements. The real issue is that $Q \cap K \cdot f$ is still not all of the generic elements (in general).

It takes a little preparation to give the correct analogue of Proposition 1.10. In our inductive procedure for finding f we have chosen f_0, thus determining the set $S_0 = \{k_1, k_2, \ldots, k_\ell\}$. Let

$$V' = \mathrm{span}_{\mathbf{C}}\{e_{k_i} \,:\, i = 1, 2, \ldots, \ell\}$$
$$V = \mathrm{span}_{\mathbf{C}}\{e_k \,:\, k \notin S_0\}$$

and define

$$G_1 = \{g \in G \,:\, g(V) \subset V \text{ and } g|_{V'} = I_{V'}\}.$$

Note that $G_1 \simeq GL(n - \ell, \mathbf{C})$, a lower rank general linear group, and (G_1, K_1), with $K_1 = K \cap G_1$, is a pair as in (2.2). Then f_1 is chosen in $\mathfrak{n}^- \cap \mathfrak{p} \cap \mathfrak{g}_1$, etc. Inside $\mathfrak{g}_1$ we may define $\mathfrak{q}_1$ (as $\mathfrak{q}$ was defined inside $\mathfrak{g}$). The key observation is that $\mathfrak{q}_1 \not\subseteq \mathfrak{q} \cap \mathfrak{g}_1$. This is illustrated by our example. When $1, 5, 3$ and 6 are omitted from the array, dots 2 and 4 'collapse' to form a single block. Therefore the root vectors for $\pm(\epsilon_2 - \epsilon_4)$ are in $\mathfrak{l}_1$, but the root vector for $\epsilon_2 - \epsilon_4$ is not in $\mathfrak{q}$. Similarly define $G_2, Q_2, \ldots, G_m, Q_m$, and write $Q_i = L_i U_i^-$. Now, the analogue of Proposition 1.10 is the following ([**1**, Sections 3 and 4]).

Proposition 2.3. *The set of generic elements in $\mathfrak{n}^- \cap \mathfrak{p}$ is*

$$QQ_1 \cdots Q_{m-1} \cdot f.$$

From this it can be shown that

$$\gamma_{_\mathcal{Q}}^{-1}(f) = L_m L_{m-1} \cdots L_1 L \cdot \mathfrak{b} \subset \mathcal{Q}.$$

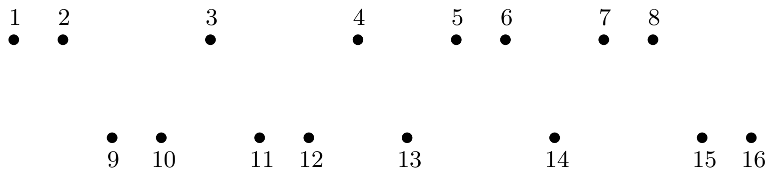
In our example we see that $\gamma_{_\mathcal{Q}}^{-1}(f) = L_2 L_1 L \cdot \mathfrak{b} = L_1 L \cdot \mathfrak{b}$. It is clear from the collapse that this properly contains $L \cdot \mathfrak{b} = Q \cap K \cdot \mathfrak{b}$.

We end this section with two examples, both of which we will return to in later sections.

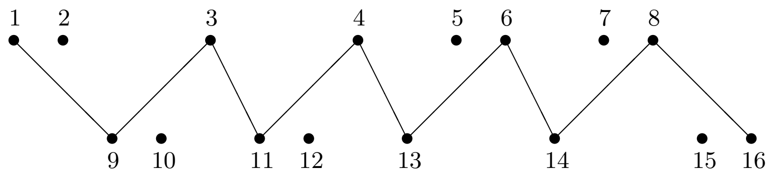
EXAMPLE 2.4. Consider $G_{\mathbf{R}} = U(8,8)$ and

$$\Lambda = (16, 15, 12, 9, 7, 6, 4, 3 \mid 14, 13, 11, 10, 8, 5, 2, 1).$$

Then the diagram is



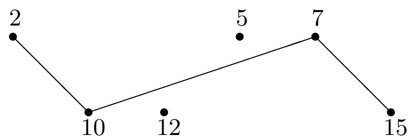
Connecting the dots in consecutive blocks gives



and

$$f_0 = E_{16,8} + E_{8,14} + E_{14,6} + E_{6,13} + E_{13,4} + E_{4,11} + E_{11,3} + E_{3,9} + E_{9,1}. \tag{2.4}$$

Omitting the dots labelled by the k_i and connecting the blocks gives



and

$$f_1 = E_{15,7} + E_{7,10} + E_{10,2}.$$

Finally, $f_2 = E_{5,12}$.

Thus, $f = f_0 + f_1 + f_2$ and the tableau is

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
- & + & - & + & - & + & - & + & - & + \\
\hline
- & + & - & + \\
\cline{1-4}
+ & - \\
\cline{1-2}
\end{array}.$$

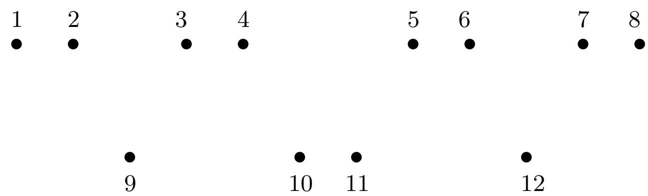
For the fiber, we see that

$$\Delta(\mathfrak{l}) = \{\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_5 - \epsilon_6), \pm(\epsilon_7 - \epsilon_8), \pm(\epsilon_9 - \epsilon_{10}), \pm(\epsilon_{11} - \epsilon_{12}), \pm(\epsilon_{15} - \epsilon_{16})\},$$
$$\Delta(\mathfrak{l}_1) = \{\pm(\epsilon_5 - \epsilon_7), \pm(\epsilon_{10} - \epsilon_{12})\}$$

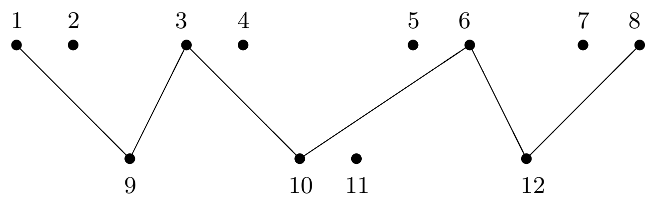
and L_2 and L_3 are contained in $H = \exp(\mathfrak{h})$ (so do not contribute to the fiber). Therefore,

$$\gamma_{\mathfrak{Q}}^{-1}(f) = L_1 L \cdot \mathfrak{b}.$$

Example 2.5. Let $G_{\mathbf{R}} = U(8,4)$ and $\Lambda = (12, 11, 9, 8, 5, 4, 2, 1 \mid 10, 7, 6, 3)$. This gives the array



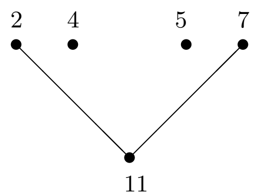
Connecting adjacent blocks gives



and

$$f_0 = E_{9,1} + E_{3,9} + E_{10,3} + E_{6,10} + E_{12,6} + E_{8,12}.$$

Omitting the indices $1, 9, 3, 10, 6, 12, 8$ gives the smaller array



and

$$f_1 = E_{11,2} + E_{7,11}.$$

Set $f = f_0 + f_1$. Then

$$\begin{aligned} f : e_1 &\to e_9 \to e_3 \to e_{10} \to e_6 \to e_{12} \to e_8 \to 0, \\ e_2 &\to e_{11} \to e_7 \to 0 \\ e_4 &\to 0,\ e_5 \to 0. \end{aligned}$$

The tableau for $K \cdot f$ is

+	−	+	−	+	−	+
+	−	+				
+						
+						

.

3. Real symplectic groups

In this section we consider the real symplectic group and show how to find generic elements in $\mathfrak{n}^- \cap \mathfrak{p}$ for a θ-stable Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ and we show how to compute the fiber $\gamma_{\mathcal{Q}}$. This works much as in the $U(p,q)$ case and we do not provide proofs of the statements given.

Consider $G = Sp(2n, \mathbf{C}) = \{G \in GL(2n, \mathbf{C}) : gJg^t = J\}$ where the matrix J is

$$J = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix},$$

with S as in (1.1). Then

$$G_{\mathbf{R}} = U(n,n) \cap Sp(2n, \mathbf{C})$$

is the split real form of G, i.e., $G_{\mathbf{R}}$ is a real symplectic group. The involution $g \mapsto (\overline{g}^t)^{-1}$ is a Cartan involution of $G_{\mathbf{R}}$. As this coincides with conjugation by $I_{n,n}$ on $G_{\mathbf{R}}$, the Cartan involution extends to $\Theta(g) = I_{n,n} g I_{n,n}$ for $g \in G$. The fixed point group of Θ is

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & -Sa^tS \end{pmatrix} \right\} \simeq GL(n, \mathbf{C}).$$

Thus we consider the pair $(G, K) = (Sp(2n, \mathbf{C}), Gl(n, \mathbf{C}))$.

In our realization of $Sp(2n, \mathbf{C})$, the Lie algebra is given in block form as follows. Let $\eta : \mathfrak{gl}(n, \mathbf{C}) \to \mathfrak{gl}(n, \mathbf{C})$ be defined by $\eta(A) = -\mathrm{Ad}(S)A^t$. Note that $\eta(A) = A$ (resp. $\eta(A) = -A$) means that A is skew symmetric (resp., symmetric) with respect to the anti-diagonal. We may write

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & \eta(A) \end{pmatrix} : A, B, C \in \mathfrak{gl}(n, \mathbf{C}), \eta(B) = -B \text{ and } \eta(C) = -C \right\}.$$

One sees right away that

$$\mathfrak{h} = \left\{ \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & -t_n & & \\ & & & & \ddots & \\ & & & & & -t_1 \end{pmatrix} \right\}$$

is a Cartan subalgebra of both $\mathfrak{k}$ and $\mathfrak{g}$. Since $\theta = \mathrm{Ad}(I_{n,n})$, one also easily sees that

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} : \eta(B) = -B \text{ and } \eta(C) = -C \right\}.$$

Letting $\mu_i, i = 1, 2 \dots, n$, be as in Subsection 1.3 the roots in $\mathfrak{g}$ are

$$\pm(\mu_i - \mu_j),\ 1 \le i < j \le n, \text{ the compact roots,}$$

$$\pm(\mu_i + \mu_j),\ 1 \le i \le j \le n, \text{ the non-compact roots.}$$

We fix once and for all the positive system of compact roots

$$\Delta_c^+ = \{\mu_i - \mu_j,\, 1 \le i < j \le n\}.$$

Our expression for a generic f is a sum of noncompact root vectors. We choose root vectors as follows.

$$\begin{aligned} X_{i+j} &= X_{\mu_i+\mu_j} = E_{i,2n-j+1} + E_{j,2n-i+1} \\ X_{-(i+j)} &= X_{-(\mu_i+\mu_j)} = E_{2n-j+1,i} + E_{2n-i+1,j} \end{aligned} \tag{3.1}$$

The closed K-orbits in $\mathfrak{B}$ are in one-to-one correspondence with Weyl chambers in $\mathfrak{h}$ that are Δ_c^+-dominant. Therefore, the closed K-orbits in $\mathfrak{B}$ are parameterized by the Weyl group conjugates of $(n, n-1, \ldots, 2, 1)$ that are Δ_c^+-dominant. If λ is such a Weyl group conjugate then λ determines the Borel subalgebra

$$\mathfrak{b}(\lambda) = \mathfrak{b} = \mathfrak{h} + \mathfrak{n}^- \text{ with } \mathfrak{n}^- = \sum_{\langle\lambda,\alpha\rangle<0} \mathfrak{g}^{(\alpha)}. \tag{3.2}$$

The corresponding closed K-orbit in $\mathfrak{B}$ is $\mathcal{Q} = K \cdot \mathfrak{b}$.

It is a very important observation for us that if

$$(\hat{G}, \hat{K}) = (GL(2n, \mathbf{C}), GL(n, \mathbf{C}) \times GL(n, \mathbf{C}))$$

is the pair from Section 2, then

$$G \subset \hat{G} \text{ and } K = \hat{K} \cap G.$$

This embedding of (G, K) into $(\hat{G}, \hat{K})$ has several very nice properties. If we let $\hat{\mathfrak{h}}$ be the diagonal Cartan subalgebra and $\hat{\Delta}_c^+$ the fixed system of positive compact roots in $\hat{\mathfrak{g}}$ (as in Section 2) then

$$\Delta_c^+ = \{\alpha|_{\mathfrak{h}} \,:\, \alpha \in \hat{\Delta}_c^+\}.$$

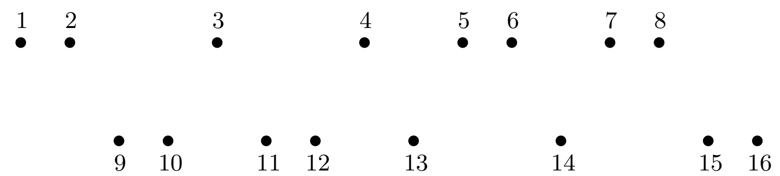
In addition, given a Δ_c^+-dominant $\lambda \in \mathfrak{h}^*$ and corresponding Borel subalgebra $\mathfrak{b}(\lambda)$ of $\mathfrak{g}$, there exists $\Lambda \in \hat{\mathfrak{h}}^*$ that is $\hat{\Delta}_c^-$-dominant and $\Lambda|_{\mathfrak{h}} = \lambda$. For this we may take $\Lambda = \frac{1}{2}(\lambda, -\lambda')$, $\lambda' = (\lambda_n, \ldots, \lambda_2, \lambda_1)$.

The K-orbits in $\mathcal{N}_\theta$ are parameterized by signed tableau: odd length rows occur an even number of times, the signs along each row alternate and for each odd row beginning with a $+$ (resp. $-$) sign there is another row of the same length beginning with a $-$ (resp., $+$) sign. Much like the cases already considered, the row lengths correspond to the dimensions of irreducible $\mathfrak{sl}(2, \mathbf{C})$ representations on $\mathbf{C}^{2n}$ and the signs beginning rows correspond to the eigenvalues of $I_{n,n}$ on lowest weight vectors.

For a given Δ_c^+-dominant Weyl group conjugate λ of $(n, \ldots, 2, 1)$, and corresponding closed K-orbit $\mathcal{Q} = K \cdot \mathfrak{b}$ in $\mathfrak{B}$, we describe how to find a *nice* generic f in $\mathfrak{n}^- \cap \mathfrak{p}$. As an example consider $G = Sp(16, \mathbf{C})$ and $\lambda = (8, 7, 4, 1, -2, -3, -5, -6)$. Then

$$\Lambda = (8, 7, 4, 1, -2, -3, -5, -6 \,|\, 6, 5, 3, 2, -1, -4, -7, -8)$$

gives a diagram (as described in Example 2.4).



Note that the diagram is symmetric about its center of mass; this symmetry is equivalent to the property that the positive system for $\hat{\mathfrak{g}}$ restricts to a positive system for $\mathfrak{g}$.

Choose the f_i as we did in Section 2 with the following additional requirement. The choice of dots to be connected should be symmetric in the sense that for blocks in the left half of the diagram dots farthest to the left in each block should be connected and in the right half of the diagram dots farthest to the right in each block should be connected. We arrive at the exact same $f = f_0 + f_1 + f_2$ as in Example 2.4. Note that this is a sum of root vectors in $\hat{\mathfrak{g}}$. However. the symmetry condition implies that each $f_i \in \mathfrak{g}$ and may easily be written in terms of root vectors in $\mathfrak{g}$. Writing $X_{\pm(i\pm j)}$ for root vectors for $\pm(\mu_i + \mu_j)$ we see that

$$\begin{aligned} f_0 &= X_{-(4+4)} + X_{4+6} + X_{-(6+3)} + X_{3+8} + X_{-(8+1)}, \\ f_1 &= X_{7+7} + X_{-(7+2)}, \\ f_2 &= X_{5+5}. \end{aligned}$$

Note that in the expression for f_0 (for example) as in (2.4) the first and the last terms in (2.4) combine to give a root vector in $\mathfrak{g}$ by (3.1). Similarly for the second and the second to the last, etc. The middle $\hat{\mathfrak{g}}$-root vector is always some $X_{\pm(i+i)}$. The tableau for f is the same as in Example 2.4 and the generic element in $\mathfrak{n}^- \cap \mathfrak{p}$ is also generic in $\hat{\mathfrak{n}}^- \cap \hat{\mathfrak{p}}$.

The set of generic elements is described much as it is in the case of $G_{\mathbf{R}} = U(p,q)$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}^-$ be defined by $\Delta(\mathfrak{l})$ being the the roots in the span of the compact simple roots. With $\hat{G}_1, \hat{G}_2, \ldots$ and $\hat{Q}_1, \hat{Q}_2, \ldots$ defined as in Section 2, set $Q_i = \hat{Q}_i \cap G_i$, for $i = 1, 2, \ldots, m$. It can be shown that $Q\,Q_1 Q_2 \cdots Q_{m-1} \cdot f$ is the set of generic elements in $\mathfrak{n}^-$. From this it can be shown that the Springer fiber for $\mathcal{Q}$ is

$$\gamma_{\mathcal{Q}}^{-1}(f) = L_m \cdots L_2 L_1 L \cdot \mathfrak{b}.$$

In the example

$$\gamma_{\mathcal{Q}}^{-1}(f) = L_1 L \cdot \mathfrak{b} \tag{3.3}$$

with L_1 the copy of $GL(2.\mathbf{C})$ having roots $\pm(\mu_5 - \mu_7)$ (along with a torus).

The description of the generic element f and $\gamma_{\mathcal{Q}}^{-1}(f)$ given for the above example extends in a straightforward way to an arbitrary closed orbit in the flag variety for $\mathfrak{sp}(2n, \mathbf{C})$.

In Section 5 we will illustrate how to compute the multiplicity from this description of $\gamma_{\mathcal{Q}}^{-1}(f)$.

4. Indefinite orthogonal groups

As an example of the computation of the Springer fiber for a closed orbit $\mathcal{Q}$ in $\mathfrak{B}$ for the indefinite orthogonal groups we will work with $O(8,4)$. No attempt will be made to explain the algorithm that applies to all orthogonal groups for either finding a generic f or for computing $\gamma_{\mathcal{Q}}^{-1}(f)$. We will however pick an example and explicitly construct the generic element in $\mathfrak{n}^- \cap \mathfrak{p}$ and describe the fiber; reasonably complete proofs will be given. With considerable effort the proofs can be made to work for general indefinite orthogonal groups. The techniques of Section 1 will be used.

4.1. Notation and generalities for $O(2p, 2q)$. Our realization of $O(2p, 2q)$ will not be the usual realization. The reason is that it is very convenient to use a realization with an embedding into an indefinite unitary group that allows us to easily adapt the method of Section 2.

Let $\hat{G}_{\mathbf{R}} = U(2p, 2q)$ be defined by the hermitian form having matrix

$$I_{2p,2q} = \begin{pmatrix} I_{2p} & 0 \\ 0 & -I_{2q} \end{pmatrix}$$

with respect to the standard basis of $\mathbf{C}^{2n}$ (with $n = p + q$). The corresponding pair is $(\hat{G}, \hat{K}) = (GL(2n, \mathbf{C}), GL(2p, \mathbf{C}) \times GL(2q, \mathbf{C}))$. Now let $G = O(2n, \mathbf{C})$ be defined by the symmetric form $(\ ,\ )$ having matrix

$$\begin{pmatrix} S_{2p} & 0 \\ 0 & S_{2q} \end{pmatrix}$$

with respect to the standard basis of $\mathbf{C}^{2n}$. Here S_{2p} (resp., S_{2q}) is the matrix of (1.1) of size $2p \times 2p$ (resp., $2q \times 2q$). Write $\eta_p(A) = -\mathrm{Ad}(S_{2p})(A^t)$ and $\eta_p(D) = -\mathrm{Ad}(S_{2p})(D^t)$. Then

$$\mathfrak{g} = \Big\{ \begin{pmatrix} A & B \\ -S_{2q}B^tS_{2p} & D \end{pmatrix} \ : \ B \in M_{2p\times 2q}(\mathbf{C}), \eta_p(A) = A, \eta_p(D) = D \Big\}$$

Note that $\eta_p(A) = A$ means that A is skew symmetric with respect to the anti diagonal. The complexified Cartan involution of $\mathfrak{g}$ is $\theta = \mathrm{Ad}(I_{2p,2q})$ and

$$\mathfrak{k} = \Big\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \ : \ \eta_p(A) = A, \eta_p(D) = D \Big\}.$$

The subalgebra of diagonal matrices

$$\mathfrak{h} \cong diag(t_1, \ldots, t_p, -t_p, \ldots, -t_1 \mid t_{p+1}, \ldots, t_{p+q}, -t_{p+q}, \ldots, -t_{p+1})$$

is a Cartan subalgebra of $\mathfrak{g}$. Define $\mu_i \in \mathfrak{h}^*, i = 1, 2, \ldots, n$, by

$$\mu_i(diag(t_1, \ldots, t_p, -t_p, \ldots, -t_1 \mid t_{p+1}, \ldots, t_{p+q}, -t_{p+q}, \ldots, -t_{p+1})) = t_i.$$

The roots of $\mathfrak{h}$ in $\mathfrak{g}$ are

$$\pm(\mu_i \pm \mu_j),\ 1 \le i < j \le p \text{ or } p+1 \le i < j \le p+q \text{ (the compact roots)}$$
$$\pm(\mu_i \pm \mu_j),\ 1 \le i \le p < j \le p+q \text{ (the noncompact roots).}$$

Let us fix a positive system of compact roots by

$$\Delta_c^+ = \{\mu_i \pm \mu_j \ : \ 1 \le i < j \le p \text{ or } p+1 \le i < j \le p+q\}.$$

As in the previous section, take $\hat{\mathfrak{h}}$ to be the diagonal Cartan subalgebra in $\hat{\mathfrak{g}} = \mathfrak{gl}(2n, \mathbf{C})$. Then $\Delta^+(\hat{\mathfrak{h}}, \hat{\mathfrak{g}}) = \{\pm(\epsilon_i - \epsilon_j)\ ,\ 1 \le i < i \le 2n\}$ and let $\hat{\Delta}_c^+$ be the fixed system of positive compact roots as in Section 2. Then it follows that

$$\Delta_c^+ = \{\alpha|_{\mathfrak{h}}\ ,\ \alpha \in \hat{\Delta}_c^+\}.$$

If $\lambda = (\lambda_1, \ldots, \lambda_n) = \sum \lambda_i \mu_i$ is Δ_c^+-dominant, then there exists $\Lambda \in \hat{\mathfrak{h}}^*$ that is $\hat{\Delta}_c^+$-dominant and $\lambda = \Lambda|_{\mathfrak{h}}$. It follows that if $\mathfrak{b}(\lambda) = \mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ is the Borel subalgebra determined by a Δ-regular, Δ_c^+-dominant λ ($\mathfrak{n}^- = \sum_{\langle\lambda,\mathfrak{a}\rangle<0} \mathfrak{g}^{(\alpha)}$) then there is a Borel subalgebra $\hat{\mathfrak{b}}$ (defined by some Λ) so that $\mathfrak{b} = \hat{\mathfrak{b}} \cap \mathfrak{g}$.

The K-orbits in $\mathcal{N}_\theta$ are described in much the same manner as for the other groups. Let $Y \in \mathcal{N}_\theta$ and form a standard triple $\{X, H, Y\}$ with $X \in \mathfrak{p}$ and $H \in \mathfrak{h}$, and let $\mathfrak{sl}(2)_Y = \mathrm{span}_{\mathbf{C}}\{X, H, Y\}$. Write $\mathbf{C}^{2n} = \oplus V_i$ for a decomposition of $\mathbf{C}^{2n}$ into irreducible $\mathfrak{sl}(2)_Y$-representations.

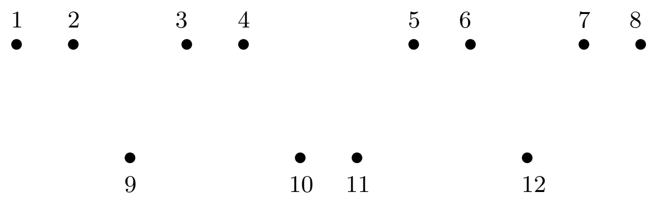
Lemma 4.1. *Let Y and $\mathfrak{sl}(2)_Y$ be as above. It may be assumed that the V_i in the decomposition of $\mathbf{C}^{2n}$ are pairwise orthogonal with respect to $(\, , \,)$ and each V_i is $I_{2p,2q}$-invariant. For each constituent V_i, a basis $\{v_1, \dots, v_\ell\}$ may be chosen so that each v_i is an H-weight vector, $v_i = Y^{i-1}v_1$ and $(v_j, v_{\ell-k+1}) = \delta_{jk}$.*

The signed tableau for $K \cdot Y$ has rows with $\dim(V_i)$ boxes; the number of rows of a given even length is even. The i^{th} row begins with the sign of the eigenvalue of the lowest weight vector of V_i and signs are placed in the other boxes so as to alternate along each row. There are an even number of rows having a given even number of boxes, and half begin with $+$ and half with $-$.

The last piece of general information about the indefinite orthogonal groups that we need is a description of roots vectors. Since we want to determine a particular $f \in \mathfrak{n}^- \cap \mathfrak{p}$ in terms of root vectors, only noncompact root vectors need to be described. Take $1 \le i \le p < j \le p+q$ and set

$$\begin{aligned} X_{i-j} &\equiv X_{\mu_i-\mu_j} = E_{i,p+j} - E_{2n+p-j+1,2p-i+1} \\ X_{-(i-j)} &\equiv X_{-(\mu_i-\mu_j)} = E_{p+j,i} - E_{2p-i+1,2n+p-j+1} \\ X_{i+j} &\equiv X_{\mu_i+\mu_j} = E_{i,2n+p-j+1} - E_{p+j,2p-i+1} \\ X_{-(i+j)} &\equiv X_{-(\mu_i+\mu_j)} = E_{2n+p-j+1,i} - E_{2p-i+1,p+j}. \end{aligned} \tag{4.1}$$

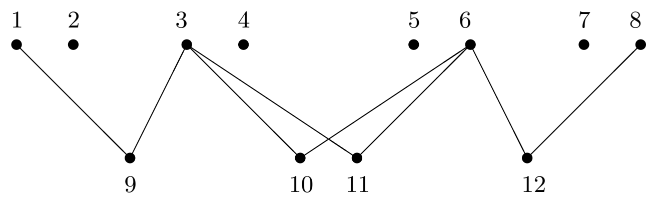
4.2. The example. Let $G_{\mathbf{R}} = O(8,4)$ and $\lambda = (6,5,3,2,4,1)$. Then $\Lambda \in \hat{\mathfrak{h}}$ is given by $\Lambda = (6,5,3,2,-2,-3,-5,-6 \mid 4,1,-1,-4)$. This gives a positive system in $\Delta(\hat{\mathfrak{h}}, \hat{\mathfrak{g}})$ containing $\hat{\Delta}_c^+$; the corresponding array (as described in Section 2) is



We adjust Example 2.5 by setting

$$f_0 = E_{9,1} + E_{3,9} + E_{10,3} + E_{11,3} - E_{6,10} - E_{6,11} - E_{12,6} - E_{8,12},$$

which we represent by



(One easily checks that this is conjugate under $\hat{L}$ to the f_0 chosen in Example 2.5.) Using (4.1) we may rewrite f_0 in terms of root vectors for $\mathfrak{g}$ as

$$f_0 = X_{5-1} + X_{3-5} + X_{6-3} + X_{-(6+3)}.$$

Similarly

$$\begin{aligned} f_1 &= E_{10,2} - E_{11,2} + E_{7,10} - E_{7,11} \\ &= X_{6-2} - X_{-(6+2)}. \end{aligned}$$

Set $f = f_0 + f_1$. It is easy to check (from (4.1)) that

$$\begin{aligned} f : e_1 &\to e_9 \to e_3 \to e_{10} + e_{11} \to -2e_6 \to 2e_{12} \to -2e_8 \to 0, \\ e_2 &\to e_{10} - e_{11} \to 2e_7 \to 0. \end{aligned}$$

The tableau for $K \cdot f$ is

+	−	+	−	+	−	+
+	−	+				
+						
+						

.

Let

$$\begin{aligned} V_1 &= \operatorname{span}_{\mathbf{C}}\{e_1, e_9, e_3, e_{10} + e_{11}, e_6, e_{12}, e_8\}, & V_3 &= \operatorname{span}_{\mathbf{C}}\{e_4\}, \\ V_2 &= \operatorname{span}_{\mathbf{C}}\{e_2, e_{10} - e_{11}, e_7\}, & V_4 &= \operatorname{span}_{\mathbf{C}}\{e_5\}. \end{aligned}$$

Then $\mathbf{C}^{12} = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ as $\mathfrak{sl}(2)_f$-representations.

In order to show that f is generic in $\mathfrak{n}^- \cap \mathfrak{p}$ (and compute $\gamma_{\mathrm{Q}}^{-1}(f)$) we need to give Q as the stabilizer of a flag. Note that $\Delta(\mathfrak{l}) = \{\pm(\mu_1 - \mu_2), \pm(\mu_3 - \mu_4)\}$ and Q is the stabilizer of

$$\{0\} = F_7 \subsetneq F_6 \subsetneq F_5 \subsetneq F_4 \subsetneq \mathbf{C}^{12}$$

where

$$\begin{aligned} F_7 &= \{0\}, & F_5 &= \operatorname{span}_{\mathbf{C}}\{e_{12}, e_7, e_8\}, \\ F_6 &= \operatorname{span}_{\mathbf{C}}\{e_7, e_8\}, & F_4 &= \operatorname{span}_{\mathbf{C}}\{e_5, e_6, e_{12}, e_7, e_8\}. \end{aligned}$$

This is an isotropic flag. This flag may be extended to a longer flag by setting $F_3 = (F_4)^\perp$, $F_2 = (F_5)^\perp$, $F_1 = (F_6)^\perp$ and $F_0 = (F_7)^\perp = \mathbf{C}^{12}$. Then Q is also the subgroup of G that stabilizes

$$\{0\} = F_7 \subsetneq F_6 \subsetneq F_5 \subsetneq F_4 \subsetneq F_3 \subsetneq F_2 \subsetneq F_1 \subsetneq F_0 = \mathbf{C}^{12}. \tag{4.2}$$

It may checked that

$$\mathfrak{u}^- = \{Y \in \mathfrak{g} \ : \ Y(F_j) \subset F_{j+1}, j = 0, 1, \ldots, 6\} \tag{4.3}$$

Define

$$G_1 = \{g \in G \ : \ g|_{V_1} = I_{V_1} \text{ and } g(V_1^\perp) \subset V_1^\perp\}.$$

Then $(G_1, K_1) = (O(5, \mathbf{C}), O(4, \mathbf{C}) \times O(1, \mathbf{C}))$ and we define Q_1 to be the parabolic subalgebra of G_1 stabilizing the flag

$$\{0\} = F_3' \subsetneq F_2' \subsetneq F_1' \subsetneq F_0' = V_1^\perp, \tag{4.4}$$

with

$$\begin{aligned} F_3' &= \{0\}, & F_1' &= \mathrm{span}_{\mathbf{C}}\{e_{10}-e_{11}, e_5, e_7\},\\ F_2' &= \mathrm{span}_{\mathbf{C}}\{e_5, e_7\}, & F_4' &= \mathrm{span}_{\mathbf{C}}\{e_2, e_4, e_{10}-e_{11}, e_5, e_7\}. \end{aligned}$$

Note that $\Delta(\mathfrak{l}_1) = \{\pm(\epsilon_2 - \epsilon_4)\}$.

PROPOSITION 4.2. *Each generic element in* $\mathfrak{n}^- \cap \mathfrak{p}$ *is of the form* $qq_1 \cdot f$, *for some* $q \in Q \cap K$ *and* $q_1 \in Q_1 \cap K$.

PROOF. Let Y be generic in $\mathfrak{n}^- \cap \mathfrak{p}$. Therefore $f \in \overline{K \cdot Y}$. Then

$$\mathrm{rank}(f^j) \leq \mathrm{rank}(Y^j), j = 1, 2, \dots. \tag{4.5}$$

Since $f^6 \neq 0$, Y^6 is non-zero. By (4.3), $Y^7 = 0$. Therefore, $\mathbf{C}^{12}$ decomposes as $\mathfrak{sl}(2)_Y$-representation as $W_1 \oplus W_1^\perp$, with W_1 an irreducible seven dimensional subrepresentation. One can show that

$$F_i = (F_i \cap W_1) \oplus (F_i \cap W_1^\perp), \text{ for } i = 0, 1, \dots, 7. \tag{4.6}$$

The argument for this is essentially the same as the argument proving the claim in Proposition 1.10. A basis of $\{w_1, \dots, w_7\}$ of W_1 may be chosen as in Lemma 4.1, and it follows from (4.3) and (4.6) that

$$w_i \in F_{i-1} \cap W_1 \setminus F_i, i = 1, \dots, 7.$$

It follows from (4.6) that $\dim(F_{i-1} \cap W_1^\perp / F_i \cap W_1^\perp)$ is one for $i = 1, 3, 4, 5, 7$, and is zero for $i = 2, 6$. Therefore a basis $\{w_8, w_9, \dots, w_{12}\}$ may be found so that

$$\begin{aligned} &w_8 \in F_0 \cap W_1^\perp \setminus F_1 \cap W_1^\perp\\ &w_9 \in F_2 \cap W_1^\perp \setminus F_3 \cap W_1^\perp\\ &w_{10} \in F_3 \cap W_1^\perp \setminus F_4 \cap W_1^\perp\\ &w_{11} \in F_4 \cap W_1^\perp \setminus F_5 \cap W_1^\perp\\ &w_{12} \in F_6 \cap W_1^\perp \end{aligned}$$

and $(w_8, w_{12}) = (w_9, w_{11}) = (w_{10}, w_{10}) = 1$ and $(w_j, w_k) = 0$, for other j, k, $8 \leq j \leq k \leq 12$.

We now reorder and scale the basis of weight vectors in the $\mathfrak{sl}(2)_f$-representation $\mathbf{C}^{12} = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ by

$$\begin{aligned} &e_1' = \frac{e_1}{\sqrt{-2}} & &e_2' = \frac{e_9}{\sqrt{2}} & &e_3' = \frac{e_3}{\sqrt{-2}} & &e_4' = \frac{e_{10}+e_{11}}{\sqrt{2}}\\ &e_5' = \sqrt{-2}\, e_6 & &e_6' = \sqrt{2}\, e_{12} & &e_7' = \sqrt{-2}\, e_8 & &e_8' = \frac{e_2}{\sqrt{2}}\\ &e_9' = \frac{e_{10}-e_{11}}{\sqrt{-2}} & &e_{10}' = \sqrt{2}\, e_7 & &e_{11}' = e_{11} & &e_{12}' = e_5 \end{aligned}$$

Therefore $\{e_1', \dots, e_7'\}$ is a basis for V_1 as in Lemma 4.1, and similarly for the others.

The linear transform $q : \mathbf{C}^{12} \to \mathbf{C}^{12}$ defined by $q(w_i) = e_i'$, $i = 1, 2, \dots, 12$,

(i) preserves $(\ ,\ )$, so is in G,

(ii) preserves the $I_{2p,2q}$-eigenspaces, so commutes with $I_{2p,2q}$ and is therefore in K, and

(iii) stabilizes the flag (F_i), so lies in Q.

In addition, for $v \in V_1$, $q^{-1}f_0(v) = Yq^{-1}(v)$. Therefore, $(q \cdot Y - f_0)|_{V_1} = 0$.

Since $W_1^\perp$ is Y-stable we see that $q \cdot Y - f_0 : V_1^\perp \to V_1^\perp$. We conclude that $Y_1 \equiv q \cdot Y - f_0 \in \mathfrak{g}_1$. Also, $Y_1 \in \mathfrak{n}^- \cap \mathfrak{p} = \mathfrak{u}^- \cap \mathfrak{p}$.

Claim: There exist $q_1 \in Q_1 \cap K$ so that $q_1 \cdot Y_1 = f_1$. Once this is accomplished the proposition will be proved because

$$\begin{aligned} f_1 &= q_1 \cdot Y_1 \\ &= q_1(q \cdot Y - f_0) \\ &= q_1 q \cdot Y - f_0, \text{ since } \mathfrak{g}_1 \text{ commutes with } f_0, \end{aligned}$$

therefore $Y = q^{-1} q_1^{-1} \cdot f$.

To prove the claim, consider the $\mathfrak{sl}(2)_Y$-representation on $W_1^\perp$. From (4.5) we see that $W_1^\perp$ contains an irreducible subrepresentation of dimension at least three. However, the $I_{2p,2q}$-eigenvalue $+1$ (resp., -1) on $W_1^\perp$ has multiplicity four (resp., one). Therefore $W_1^\perp$ contains an irreducible subrepresentation of dimension exactly 3, since the eigenvalues must alternate from one weight vector to the next within an irreducible constituent. Again by considering the eigenvalues of $I_{2p.2q}$ we conclude that there are two copies of the trivial representation in $W_1^\perp$. Let's write $W_1^\perp = W_2 \oplus W_3 \oplus W_4$, with $\dim(W_2) = 3$. Now choose a basis $\{u_8, \dots, u_{12}\}$ of $W_1^\perp$ so that u_8, u_9, u_{10} are as in Lemma 4.1, and $u_{11} \in F_0' \cap W_3 \setminus F_1'$ and $u_{12} \in F_2' \cap W_4$, and $(u_{11}, u_{12}) = 1$ and $(u_{11}, u_{11}) = (u_{12}, u_{12}) = 0$.

Define $q_1 : \mathbf{C}^{12} \to \mathbf{C}^{12}$ by $q_1|_{V_1} = I_{V_1}$ and $q_1(u_i) = e_i', i = 8, 9, \dots, 12$. Then by the same reasoning as above, $q_1 \in Q_1 \cap K$. One easily checks that $q_1 \cdot Y_1 - f_1 = 0$. $\square$

It follows that $N(f, \mathfrak{n}^- \cap \mathfrak{p}) = QQ_1 Z_K(f)$, and $\gamma_{_Q}^{-1}(f) = Z_K(f) Q_1 Q \cdot \mathfrak{b}$. One can also show that the centralizer may be omitted from this expression for the fiber and Q (resp., Q_1) can be replaced by L (resp., L_1). We arrive at

$$\gamma_{_Q}^{-1}(f) = L_1 L \cdot \mathfrak{b}.$$

Here $\Delta(\mathfrak{l}_1) = \{\pm(\mu_2 - \mu_4)\}$.

5. Computation of the multiplicity polynomial

The associated cycle of a Harish-Chandra module is a formal linear combination of closures of K-orbits in $\mathcal{N}_\theta$ with non-negative integer coefficients. Writing the associated cycle of a Harish-Chandra module X as $\mathrm{AC}(X) = \sum m_i \overline{\mathcal{O}_i}$, the associated variety of X is $\cup \overline{\mathcal{O}_i}$ and m_i is referred to as the *multiplicity* of $\overline{\mathcal{O}_i}$ in the associated cycle. Suppose that X is an irreducible Harish-Chandra module with regular integral infinitesimal character. It is well-known that X fits into a coherent family $\{X(\lambda) : \lambda \in \Lambda_{wt}\}$ (with Λ_{wt} the weight lattice in the dual of a Cartan subalgebra $\mathfrak{h}^*$); see [**13**, Lem. 7.2.6 and Cor. 7.3.23]. It is also well-known that the associated cycles of the $X(\lambda)$ have the form

$$\mathrm{AC}(X(\lambda)) = \sum m_i(\lambda) \overline{\mathcal{O}_i}$$

and each $m_i(\lambda)$ extends to a W-harmonic homogeneous polynomial on $\mathfrak{h}^*$.

Our goal is to illustrate how to use our description of $\gamma_{_Q}^{-1}(f)$ to compute the multiplicity polynomials for various representations of classical groups. The main tool is a theorem of J.-T. Chang that relates the multiplicity to $\gamma_{_Q}^{-1}(f)$. This theorem applies to representations associated to closed K-orbits in generalized flag varieties, that is, to cohomologically parabolically induced representations (the $A_{\mathfrak{q}}$-lambda representations).

To make precise the connection between the associated cycles of the cohomologically parabolically induced representations and the Springer fibers we begin with the discrete series representations. Suppose that $G_{\mathbf{R}}$ is a real reductive group with nonempty discrete series. Then $G_{\mathbf{R}}$ has a compact Cartan subgroup $H_{\mathbf{R}}$. Complexification gives a pair (G,K) with the property that there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is contained in $\mathfrak{k}$. As in previous sections we fix a positive system of compact roots Δ_c^+. The closed orbits are in one-to-one correspondence with positive systems Δ^+ that contain Δ_c^+. Letting

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-,\ \mathfrak{n}^- = \sum_{\alpha\in\Delta^+} \mathfrak{g}^{(-\alpha)},$$

the closed orbit corresponding to Δ^+ is $\mathcal{Q} = K\cdot\mathfrak{b}$. For each closed orbit $\mathcal{Q}$ there is a family of discrete series representations; these are in a coherent family $\{X_{\mathcal{Q}}(\lambda) : \lambda\in\Lambda'_{wt}\}$, where Λ'_{wt} is a translate of the weight lattice in $\mathfrak{h}^*$. Each discrete series representation lies in

$$\bigcup_{\mathcal{Q}\text{ closed}} \{X_{\mathcal{Q}}(\lambda) \,:\, \lambda\in\Lambda'_{wt}\}.$$

We may now state Chang's theorem ([**3**]).

Theorem 5.1. *Given $\Delta^+\supset\Delta_c^+$ and the corresponding closed K-orbit $\mathcal{Q}=K\cdot\mathfrak{b}$ in $\mathfrak{B}$, let f be generic in $\mathfrak{n}^-\cap\mathfrak{p}$. Then*

$$\mathrm{AC}(X_{\mathcal{Q}}(\lambda)) = m_{\mathcal{Q}}(\lambda)\,\overline{K\cdot f}$$

with

$$m_{\mathcal{Q}}(\lambda) = \dim\big(H^0(\gamma_{\mathcal{Q}}^{-1}(f),\mathcal{O}(\tau))\big).$$

The sheaf $\mathcal{O}(\tau)$ is the restriction to $\gamma_{\mathcal{Q}}^{-1}(f)$ of the sheaf of regular sections of the homogeneous line bundle $\mathcal{L}_\tau\to K\cdot\mathfrak{b}$ corresponding to $\tau=\lambda+\rho-2\rho_c$ (using the standard notation of $\rho=\frac{1}{2}\sum_{\Delta^+}\alpha$ and $\rho_c=\frac{1}{2}\sum_{\Delta_c^+}\alpha$).

It is important to note that $\gamma_{\mathcal{Q}}^{-1}(f)$ is typically not homogeneous, in particular the Borel-Weil Theorem does not immediately apply to compute the space of sections.

Now we turn to an example of the computation of the associated cycle for a discrete series representation. We will consider the example of Section 3 in $G_{\mathbf{R}} = Sp(2n,\mathbf{R})$ and use the formula for the fiber given in (3.3).

The cohomology space $H^0(\gamma_{\mathcal{Q}}^{-1}(f),\mathcal{O}(\mathcal{L}_\tau))$ may be computed using the Borel-Weil Theorem as follows. Let $W_{-\tau}$ be the irreducible K-representation of lowest weight $-\tau$ and let $w_{-\tau}$ be a lowest weight vector. The Borel-Weil Theorem states that $W_{-\tau}^*\simeq H^0(Z,\mathcal{O}(\mathcal{L}_\tau))$. This isomorphism is given in terms of matrix coefficients by

$$v\mapsto\varphi_v,\ v\in W_{-\tau}^*$$
$$\varphi_v(k) = \langle v, kw_{-\tau}\rangle.$$

Using a small amount of algebraic geometry one may conclude that for λ sufficiently dominant

$$\dim\left(H^0(\gamma^{-1}(f), \mathcal{O}_\tau)\right) = \dim\left(\operatorname{span}_{\mathbf{C}}\{k^{-1}w_{-\tau} \,:\, k \in N(f, \mathfrak{n}^- \cap \mathfrak{k})\}\right). \tag{5.1}$$

Now our description of $\gamma^{-1}(f)$ says that the multiplicity is

$$\dim\left(\operatorname{span}_{\mathbf{C}}\{k \cdot w_{-\tau} \,:\, k \in L_1 L\}\right).$$

See [**6**, §6.1-6.3] and [**1**, Section 6] for a discussion of this method.

Now we will compute the multiplicity polynomial for the family of discrete series representations $\{X_{\mathcal{Q}}(\lambda)\}$ associated to the closed orbit of our example in Section 3. Let $\tau = (\tau_1, \ldots, \tau_8)$ and let $U_{-\tau}$ be the irreducible finite dimensional L-representation of lowest weight $-\tau$. Then $U_{-\tau} \simeq \operatorname{span}_{\mathbf{C}}\{L \cdot w_{-\tau}\}$. To compute $\operatorname{span}_{\mathbf{C}}\{L_1 L \cdot w_{-\tau}\}$ first decompose $U_{-\tau}$ under $L_1 \cap L$ (a torus). Then each weight vector is annihilated by $\mathfrak{l}_1 \cap \mathfrak{u}^-$ (since L normalizes $\mathfrak{u}^-$), therefore $\operatorname{span}_{\mathbf{C}}\{L \cdot w_{-\tau}\}$ is the direct sum of irreducible L_1-representations having these lowest weight vectors. Let $H = \exp(\mathfrak{h})$. Then

$$U_{-\tau}|_H = -\sum_{a=0}^{\tau_1-\tau_2} \sum_{b=0}^{\tau_5-\tau_6} \sum_{c=0}^{\tau_7-\tau_8} (\tau_1 - a, \tau_2 + a, \tau_3, \tau_4, \tau_5 - b, \tau_6 + b, \tau_7 - c, \tau_8 + c).$$

Since $\Delta(\mathfrak{l}_1) = \{\pm(\epsilon_5 - \epsilon_7)\}$ the dimension of $\operatorname{span}_{\mathbf{C}}\{L_1 L \cdot w_{-\tau}\}$ is

$$\begin{aligned}
&\sum_{a=0}^{\tau_1-\tau_2} \sum_{b=0}^{\tau_5-\tau_6} \sum_{c=0}^{\tau_7-\tau_8} (\tau_5 - \tau_7 + 1 - b + c) \\
&= (\tau_1 - \tau_2 + 1) \sum_{b=0}^{\tau_5-\tau_6} \sum_{c=0}^{\tau_7-\tau_8} (\tau_5 - \tau_7 + 1 - b + c) \\
&= (\tau_1 - \tau_2 + 1)\big((\tau_5 - \tau_6 + 1)(\tau_7 - \tau_8 + 1)(\tau_5 - \tau_7 + 1) \\
&\quad - \frac{(\tau_5 - \tau_6 + 1)(\tau_5 - \tau_6)}{2}(\tau_7 - \tau_8 + 1) + (\tau_5 - \tau_6 + 1)\frac{(\tau_7 - \tau_8 + 1)(\tau_7 - \tau_8)}{2}\big) \\
&= \frac{1}{2}(\tau_1 - \tau_2 + 1)(\tau_5 - \tau_6 + 1)(\tau_7 - \tau_8 + 1)(\tau_5 + \tau_6 - \tau_7 - \tau_8 + 2).
\end{aligned}$$

In terms of the parameter λ of the family of discrete series representations $\{X_{\mathcal{Q}}(\lambda)\}$ (using $\tau = \lambda + \rho - 2\rho_c = \lambda + (1, 2, -2, 0, -1, 0, 0, 1)$) we get

$$m_{\mathcal{Q}}(\lambda) = \frac{1}{2}(\lambda_1 - \lambda_2)(\lambda_5 - \lambda_6)(\lambda_7 - \lambda_8)(\lambda_5 + \lambda_6 - \lambda_7 - \lambda_8).$$

We point out that in general (for $G_{\mathbf{R}} = Sp(2n, \mathbf{R})$) the calculation of $m_{\mathcal{Q}}(\lambda)$ is not much more involved than the above example. The restrictions of irreducible L_i-representations to $L_i \cap L_{i+1}$ are *always* carried out using the very simple branching rule for restricting representations from $GL(p+1, \mathbf{C})$ to $GL(p, \mathbf{C}) \times GL(1, \mathbf{C})$.

More generally, now let $G_{\mathbf{R}}$ be any real reductive group and $\mathfrak{F}$ a generalized flag variety for G. If $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}^-$ is a θ-stable parabolic subalgebra in $\mathfrak{F}$, then $\mathcal{Q} = K \cdot \mathfrak{q}$ is a closed orbit in $\mathfrak{F}$. There is a family of cohomologically induced representations associated to $\mathcal{Q}$. Denote the coherent family by $\{X_{\mathcal{Q}}(\lambda) : \lambda \in \Lambda_{wt}\}$. The method of [**2**] and [**3**] computes the associated cycle as follows. Let $\mathfrak{B}$ be the full flag variety and $\pi : \mathfrak{B} \to \mathfrak{F}$ the natural projection. Then $\pi^{-1}(\mathcal{Q})$ is the closure of a single K-orbit $\mathcal{Q}_0$ in $\mathfrak{B}$. We may describe $\mathcal{Q}_0$ by noting that $\pi^{-1}(\mathfrak{q})$ is the flag variety $\mathfrak{B}_L$

for L, and therefore contains a dense open $K \cap L$-orbit. Write this dense orbit as $K \cap L \cdot \mathfrak{b}_L$ and write $\mathfrak{b}_L = \mathfrak{h}_L + \mathfrak{n}_L^-$, where $\mathfrak{h}_L$ is a maximally split Cartan subalgebra of $\mathfrak{l}$. Then $\mathcal{Q}_0 = K \cdot \mathfrak{b}$, with $\mathfrak{b} = \mathfrak{b}_L + \mathfrak{u}^- = \mathfrak{h} + \mathfrak{n}^-$, is dense in $\pi^{-1}(\mathcal{Q})$. One may conclude from the arguments in [**2**] and [**3**] that

$$\mathrm{AC}(X_{\mathcal{Q}}(\lambda)) = \dim\left(H^0(\gamma_{\mathcal{Q}_0}^{-1}(f), \mathcal{O}(\tau))\right) \overline{K \cdot f},$$

when f is generic in $\mathfrak{n}^- \cap \mathfrak{p}$. Here $\gamma_{\mathcal{Q}_0} : \overline{T^*_{\mathcal{Q}_0}\mathfrak{B}} \to \mathcal{N}_\theta$ is the restriction of the moment map μ to $\overline{T^*_{\mathcal{Q}_0}\mathfrak{B}}$ and τ is described below.

The important geometric observations are that f is generic in $\mathfrak{u}^- \cap \mathfrak{p}$ if and only if f is generic in $\mathfrak{n}^- \cap \mathfrak{p}$, and

$$\gamma_{\mathcal{Q}_0}^{-1}(f) = \pi^{-1}(\gamma_{\mathcal{Q}}^{-1}(f)).$$

To see this, note that since $K \cap L \cdot \mathfrak{b}_L$ is open in $\mathfrak{B}_L$, we have $\mathfrak{k} \cap \mathfrak{l} + \mathfrak{b}_L = \mathfrak{l}$. From this it follows that $\mathfrak{n}_L^- \cap \mathfrak{p} = 0$ (since $\mathfrak{l} = \mathfrak{l} \cap \mathfrak{k} + \mathfrak{b}_L$ is the orthogonal complement of $\mathfrak{n}_L^- \cap \mathfrak{p}$ in $\mathfrak{l}$). Therefore, $\mathfrak{n}^- \cap \mathfrak{p} = \mathfrak{u}^- \cap \mathfrak{p}$. Now it is immediate that f is generic in $\mathfrak{n}^- \cap \mathfrak{p}$ if it is generic in $\mathfrak{u}^- \cap \mathfrak{p}$. It also follows that $N(f, \mathfrak{u}^- \cap \mathfrak{p}) = N(f, \mathfrak{n}^- \cap \mathfrak{p})$.

We are now in position to compute the multiplicity polynomials for the cohomologically induced representations for $G_{\mathbf{R}} = GL(N, \mathbf{R})$ using Theorem 1.15. For each θ-stable parabolic subalgebra and corresponding closed K-orbit $\mathcal{Q} = K \cdot \mathfrak{q}$ in a flag variety $\mathfrak{F}$ we have shown that $N(f, \mathfrak{u}^- \cap \mathfrak{p}) = Q \cap K$. Therefore,

$$\gamma_{\mathcal{Q}_0}^{-1}(f) \simeq \overline{Q \cap K \cdot \mathfrak{b}} = \overline{(L \cap K) \cdot \mathfrak{b}} = L \cdot \mathfrak{b} = \mathfrak{B}_L.$$

It can be seen that $\mathcal{O}(\tau)$ is the sheaf of local sections of $\mathcal{L}_\tau \to \mathfrak{B}_L$, the homogeneous bundle $L \underset{B_L}{\times} \mathbf{C}_{\lambda - \rho + 2\rho(\mathfrak{u} \cap \mathfrak{p})}$. Here $2\rho(\mathfrak{u} \cap \mathfrak{p})$ is a weight of $\mathfrak{h}_L \cap \mathfrak{k}$ extended to be zero on $\mathfrak{h}_L \cap \mathfrak{p}$. Therefore, the multiplicity polynomial is

$$m_{\mathcal{Q}}(\lambda) = \dim\left(H^0(\gamma_{\mathcal{Q}_0}^{-1}(f), \mathcal{O}(\mathcal{L}_\tau))\right) = \dim\left(H^0(\mathfrak{B}_L, \mathcal{O}(\mathcal{L}_{\lambda - \rho + 2\rho(\mathfrak{u} \cap \mathfrak{p})}))\right).$$

By the Borel-Weil Theorem and the Weyl dimension formula we get

$$m_{\mathcal{Q}}(\lambda) = \prod_{\alpha \in \Delta^+(\mathfrak{h}_L, \mathfrak{l})} \frac{\langle \lambda - \rho + 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \rho(\mathfrak{l}), \alpha \rangle}{\langle \rho(\mathfrak{l}), \alpha \rangle}.$$

This may be simplified slightly. Referring to (1.10), the structure of the group $L_{\mathbf{R}}$ is

$$L_{\mathbf{R}} = \begin{cases} GL(d_1, \mathbf{C}) \times \cdots \times GL(d_l, \mathbf{C}), & \text{if } c \neq 0, N = 2n \\ GL(d_1, \mathbf{C}) \times \cdots \times GL(d_l, \mathbf{C}) \times GL(1, \mathbf{R}), & \text{if } c \neq 0, N = 2n+1 \\ GL(d_1, \mathbf{C}) \times \cdots \times GL(d_{l-1}, \mathbf{C}) \times GL(2d_l, \mathbf{R}), & \text{if } c = 0, N = 2n \\ GL(d_1, \mathbf{C}) \times \cdots \times GL(d_{l-1}, \mathbf{C}) \times GL(2d_l + 1, \mathbf{R}), & \text{if } c = 0, N = 2n+1. \end{cases}$$

It suffices to express the formula for $m_{\mathcal{Q}}(\lambda)$ in terms of the diagonal Cartan subalgebra $\mathfrak{h}$ (as in Section 1). Note that this nearly coincides with $\mathfrak{h}_L$. Then it is easy to check that $-\rho + 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \rho(\mathfrak{l}) = -\rho(\mathfrak{u}) + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ is

$$\sum_{i=1}^{n} (\epsilon_i - \epsilon_{N-i+1}), \text{ if } c \neq 0,$$
$$\sum_{i=1}^{n-d_l} (\epsilon_i - \epsilon_{N-i+1}), \text{ if } c = 0,$$

which is orthogonal to $\Delta(\mathfrak{l})$. Therefore,

$$m_{\mathcal{Q}}(\lambda) = \prod_{\alpha \in \Delta^+(\mathfrak{h},\mathfrak{l})} \frac{\langle \lambda, \alpha \rangle}{\langle \rho(\mathfrak{l}), \alpha \rangle}.$$

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