COMPONENTS OF SPRINGER FIBERS ASSOCIATED TO CLOSED ORBITS FOR THE SYMMETRIC PAIRS $(Sp(2n), Sp(2p) \times Sp(2q))$ AND $(SO(2n), GL(n))$, II

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Introduction

In this article we consider the pairs (G, K) of complex groups

$$\begin{aligned} &(Sp(2n), Sp(2p) \times Sp(2q)), n = p + q, \\ &(SO(2n), GL(n)), \end{aligned} \tag{1}$$

which are referred to as type C and type D, and the corresponding real forms

$$G_{\mathbf{R}} = Sp(p, q),\ SO^*(2n). \tag{2}$$

A method is given to compute associated cycles of discrete series representations of $G_{\mathbf{R}}$. The method in fact computes the associated cycle of any representation in a Harish-Chandra cell which contains a discrete series representation. In each such cell we find a discrete series representation X' for which a relatively elementary argument gives us the associated cycle. We show that the corresponding cell representation is generated by X'. Then the general theory of characteristic cycles, Springer representations and coherent continuation applies to show that the computation of $AC(X')$ gives the associated cycle of any irreducible Harish-Chandra module in the cell of X'. The computation is in the form of an algorithm, which will be outlined below. An important part of the algorithm is based on the results and methods of [4].

Before describing the algorithm we establish a small amount of notation. The subgroup K is the fixed point group of an involution Θ of G. The (complexified) Cartan decomposition of the Lie algebra of G (for θ, the differential of Θ) is written as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is contained in $\mathfrak{k}$.

The principal geometric objects considered are the flag variety $\mathfrak{B}$ for G and the nilpotent cone $\mathcal{N}$ of the Lie algebra $\mathfrak{g}$. We also consider $\mathcal{N}_\theta := \mathcal{N} \cap \mathfrak{p}$. The moment map of the cotangent bundle to $\mathfrak{B}$ is denoted by $\mu : T^*\mathfrak{B} \to \mathcal{N}$. If $\mathfrak{Q} \subset \mathfrak{B}$ is a K-orbit, then the restriction of μ to the conormal bundle maps to $\mathcal{N}_\theta$. The fibers of μ play an important role and are referred to as Springer fibers; we often use the common notation $\mathfrak{B}^f$ for $\mu^{-1}(f)$. A standard fact is that when $\mu(\overline{T^*_{\mathfrak{Q}}\mathfrak{B}}) = \overline{K \cdot f}$, then $\mathfrak{B}^f \cap \overline{T^*_{\mathfrak{Q}}\mathfrak{B}}$ is a union of irreducible components of $\mathfrak{B}^f$. One easily sees that

$\mathfrak{B}^f$ may be identified with the set of Borel subalgebras that contain f, thus may be identified with a subvariety of $\mathfrak{B}$.

Let X be a discrete series representation having the same infinitesimal character as the trivial representation (infinitesimal character equal to ρ). Then X is associated to a closed K-orbit $\mathcal{Q} \subset \mathfrak{B}$, its support in the Beilinson-Bernstein description of irreducible Harish-Chandra modules. There is a coherent family $\{X_\lambda\}_{\lambda\in\Lambda}$, Λ the integral lattice in $\mathfrak{h}^*$, so that $X = X_\rho$. The associated cycle of X_λ, $\lambda \in \Lambda^+$, is

$$AC(X_\lambda) = m_{\mathcal{Q}}(\lambda)\,\overline{\mathcal{O}}$$

where $\mu(T^*_{\mathcal{Q}}\mathfrak{B}) = \overline{\mathcal{O}}$, $\mathcal{O} = K \cdot f \subset \mathcal{N}_\theta$. The definition of the associated cycle of a Harish-Chandra module is given in [18]. A theorem of J.-T. Chang ([6]) states that the multiplicity $m_{\mathcal{Q}}(\lambda)$ of $\overline{\mathcal{O}}$ in $AC(X_\lambda)$ is given by

$$m_{\mathcal{Q}}(\lambda) = \dim(H^0(\mu^{-1}(f) \cap T^*_{\mathcal{Q}}\mathfrak{B}, \mathcal{O}(\lambda + \rho - 2\rho_c))), \tag{3}$$

for some invertible sheaf $\mathcal{O}(\lambda + \rho - 2\rho_c)$. For the groups $U(p,q), Sp(2n,\mathbf{R})$ and $O(p,q)$ algorithms are given in [2] and [3] to compute $AC(X_\lambda)$. The procedure there is to begin with a closed K-orbit $\mathcal{Q}$ in $\mathfrak{B}$ (which corresponds to a positive system of roots), then construct a nice 'generic' element $f \in \mathcal{N}_\theta$ in terms of root vectors. It turns out that $\mu^{-1}(f) \cap T^*_{\mathcal{Q}}\mathfrak{B}$, a component of the Springer fiber (or several components), has a particularly nice form and the Borel-Weil Theorem can be applied to compute $m_{\mathcal{Q}}(\lambda)$. For the real forms (2) considered in the present article, $\mu^{-1}(f) \cap T^*_{\mathcal{Q}}\mathfrak{B}$ does not seem to have a nice form for every closed orbit $\mathcal{Q}$. Therefore $m_{\mathcal{Q}}(\lambda)$ cannot always be computed directly using (3).

The algorithm presented here to compute $AC(X_\lambda)$ goes as follows. As for the other classical groups, begin by finding the (nice) 'generic' element f. This is done in [4]. The next step is to replace $\mathcal{Q} = \text{support}(X)$ by another closed K-orbit $\mathcal{Q}' \subset \mathfrak{B}$ for which $m_{\mathcal{Q}'}(\lambda)$ can be computed in an elementary way using (3) and the Borel-Weil Theorem. The point is that, with $\mathcal{Q}'$ properly chosen, $\mu^{-1}(f) \cap T^*_{\mathcal{Q}'}\mathfrak{B}$ has a very simple form (often homogeneous). This is carried out in §3.

The final step is to compute $m_{\mathcal{Q}}(\lambda)$, now that $m_{\mathcal{Q}'}(\lambda)$ is known. Suppose that X' is the discrete series representation of trivial infinitesimal character with $\mathcal{Q}' = \text{support}(X')$ and $\{X'_\lambda\}_{\lambda\in\Lambda}$ is the coherent family with $X' = X'_\rho$. From the algorithm to compute $\mathcal{Q}'$ from $\mathcal{Q}$ one sees (§3 and [4]) that X and X' have the same associated variety. It follows from a theorem of McGovern ([11]) that X and X' are in the same Harish-Chandra cell $\mathcal{C}$.

In §2 we show that X' (indeed any discrete series representation, for our $G_{\mathbf{R}}$) generates the cell representation $V_{\mathcal{C}}$ as $\mathbf{Q}[W]$-module, that is, $V_{\mathcal{C}} = \mathbf{Q}[W] \cdot X'$. It follows from the W-equivariance of the map sending a representation to its multiplicity polynomial, that

$$m_{\mathcal{Q}}(\lambda) = \sum_{w \in W} a_w m_{\mathcal{Q}'}(w^{-1}\lambda), \text{ for some } a_w \in \mathbf{Q}. \tag{4}$$

The W-equivariance is well-known; it is stated carefully, with references, in §1. There is an algorithm, which has been implemented in the Atlas software, for computing the W-action on $V_{\mathcal{C}}$. Therefore, the coefficients a_w appearing in (4), i.e., the coefficients for which $X = \sum_w a_w\, w \cdot X'$, can be computed explicitly.

We remark that the multiplicity polynomials for all representations in any cell which contains a discrete series representation are computed by the algorithm.

A byproduct of §3 is that we determine exactly which nilpotent orbits occur as associated varieties of discrete series representations. It is shown in [9] that for a general semisimple Lie group the associated variety of a discrete series representation satisfies a compactness condition; it follows from §3 that the converse not true. This is discussed in §7.

Remark. This article depends heavily on [4]. The construction of certain 'generic' nilpotent elements is given in §2 of [4]; the details of this construction are used here. The reader is assumed to be familiar with the notation and content of §2 of [4]. Other than the statement of the main result, other sections of [4] are not used in this article.

1. Coherent continuation and the multiplicity polynomials

Suppose $\{X_\lambda\}_{\lambda \in \Lambda}$ is a coherent family with $X = X_\rho$ some irreducible Harish-Chandra module (not necessarily in the discrete series). Then the associated variety of the representations $X_\lambda, \lambda \in \Lambda^+$, coincide. Let us assume that $AV(X_\lambda) = \overline{\mathcal{O}}$, the closure of a single K-orbit in $\mathcal{N}_\theta$. This assumption simplifies somewhat our discussion below. It is a fact that for the groups (2) under consideration the associated varieties of irreducible representations with regular integral infinitesimal character are in fact the closure of a single orbit in $\mathcal{N}_\theta$ ([11]), so there is no loss of generality for our purposes. We therefore have that $AC(X_\lambda) = m_X(\lambda)\,\overline{\mathcal{O}}$. The multiplicity $m_X(\lambda)$ extends to a W-harmonic polynomial on $\mathfrak{h}^*$. We refer to $m_X(\lambda)$ as the multiplicity polynomial. In the notation $m_X(\lambda)$ we omit reference to $\overline{\mathcal{O}}$, since the associated variety has just the one irreducible component $(\overline{\mathcal{O}})$. Note that X determines the coherent family. In the introduction and in Sections 4 and 5, where the K-orbit $\mathcal{Q} \subset \mathfrak{B}$ determines a coherent family of discrete series representations, we use the notation $m_{\mathcal{Q}}(\lambda)$ for the multiplicity polynomial.

Consider a Harish-Chandra cell $\mathcal{C}$. All representations in $\mathcal{C}$ have the same associated variety. The fact we need is that the map defined by sending any $X \in \mathcal{C}$ to its multiplicity polynomial $m_X(\lambda)$ (and extended linearly) from the cell representation

$V_{\mathcal{C}}$ to $P(\mathfrak{h}^*)$ is W-equivariant with W acting by coherent continuation on $V_{\mathcal{C}}$ and as usual on polynomials. This fact is well-known to the experts. We need some details on how this map arises.

The following is a (very brief) sketch of a proof of this fact. Begin by writing $\mathcal{O} = K\cdot f$, $f \in \mathcal{N}_\theta$. The homology space $H_{\mathrm{top}}(\mathfrak{B}^f)$ (with rational coefficients) affords a representation of W ([14]). This is referred to as the Springer representation. Let $d = \dim_C(\mathfrak{B}^f)$, so the top degree homology is in degree $2d$. The inclusion $\mathfrak{B}^f \hookrightarrow \mathfrak{B}$ induces a W-homomorphism $H_{\mathrm{top}}(\mathfrak{B}^f) \to H_{2d}(\mathfrak{B})$, as explained in [13, §3]. Following this map by the W-isomorphism ([5]) of $H_{2d}(\mathfrak{B})$ with the W-harmonic polynomials homogeneous of degree $2d$, we get a W-homomorphism

$$H_{\mathrm{top}}(\mathfrak{B}^f) \to P_{2d}(\mathfrak{h}^*). \tag{1.1}$$

A formula for this map is given in ([13, §3, 6]) as follows. The fundamental classes of the components of $\mathfrak{B}^f$ form a $\mathbf{Q}$-basis of $H_{\mathrm{top}}(\mathfrak{B}^f)$. The map (1.1) is defined by

$$[C] \mapsto \int_C e^{\omega_\lambda}$$

where ω_λ is the first Chern class of the homogeneous line bundle for $\lambda \in \mathfrak{h}^*$ and $[C]$ is the fundamental class of an irreducible component C of $\mathfrak{B}^f$.

The characteristic cycle of a representation is a formal integral combination of closures of conormal bundles to K-orbits $\mathcal{Q}$ in $\mathfrak{B}$. Letting $Z(K)$ be the conormal variety for the K-action on $\mathfrak{B}$, that is, the union of the conormal bundles to K-orbits in $\mathfrak{B}$, the characteristic cycle of a Harish-Chandra module lies in the top degree Borel-Moore homology $H_{top}(Z(K))$. We therefore have a characteristic cycle map defined on the Grothendieck group of Harish-Chandra modules of infinitesimal character ρ:

$$CC : \mathcal{HC}_\rho \to H_{top}(Z(K)). \tag{1.2}$$

This map is W-equivariant ([16]) for the coherent continuation representation on $\mathcal{HC}_\rho$ and an appropriate action of W on $H_{top}(Z(K))$, as described in [16].

Now consider cells of Harish-Chandra modules (of infinitesimal character ρ). The cells partition the set of irreducible Harish-Chandra modules into subsets. One property is that any two representations in the same cell have the same associated variety. Each cell $\mathcal{C}$ defines a W-representation $V_{\mathcal{C}}$ as a subquotient of the coherent continuation representation which is spanned by the irreducibles in $\mathcal{C}$. See for example [1] and [2, Appendix] for definitions. On the other hand, $H_{top}(Z(K))$ has a subquotient (as W-representation):

$$\sum_{\mathcal{Q},\mu(T_{\mathcal{Q}})\subset\overline{\mathcal{O}}} \mathbf{Q}\cdot[\overline{T^*_{\mathcal{Q}}\mathfrak{B}}] \Big/ \sum_{\mathcal{Q},\mu(T^*_{\mathcal{Q}}\mathfrak{B})\subsetneq\overline{\mathcal{O}}} \mathbf{Q}\cdot[\overline{T^*_{\mathcal{Q}}\mathfrak{B}}], \tag{1.3}$$

where $AV(X) = \overline{\mathcal{O}}, X \in \mathcal{C}$. The characteristic cycle map is a well-defined W-homomorphism

$$V_{\mathcal{C}} \to \sum_{\mathcal{Q}, \mu(T_{\mathcal{Q}}) \subset \overline{\mathcal{O}}} \mathbf{Q} \cdot [\overline{T^*_{\mathcal{Q}}\mathfrak{B}}] / \sum_{\mathcal{Q}, \mu(T^*_{\mathcal{Q}}\mathfrak{B}) \subsetneq \overline{\mathcal{O}}} \mathbf{Q} \cdot [\overline{T^*_{\mathcal{Q}}\mathfrak{B}}]. \tag{1.4}$$

Furthermore, (1.3) is isomorphic to $H_{\text{top}}(\mathfrak{B}^f)^{A_K(f)}$ as W-representation, where $A_K(f)$ is the component group of the centralizer in K of f. The isomorphism is given as follows. It is a fact that $\mathfrak{B}^f \cap T^*_{\mathcal{Q}}\mathfrak{B}$ is the union of the components in an $A_K(f)$-orbit in the irreducible components of $\mathfrak{B}^f$. If $\mathfrak{B}^f \cap T^*_{\mathcal{Q}}\mathfrak{B} = \cup_i C_i$, then the isomorphism of (1.3) with $H_{\text{top}}(\mathfrak{B}^f)^{A_K(f)}$ is defined by

$$[\overline{T^*_{\mathcal{Q}}\mathfrak{B}}] \mapsto \sum_i [C_i]. \tag{1.5}$$

Combining the maps (1.4) and (1.5) gives a W-homomorphism

$$\varphi : V_{\mathcal{C}} \to H_{\text{top}}(\mathfrak{B}^f). \tag{1.6}$$

Finally, following φ by the map of (1.1) gives a W-equivariant map $V_{\mathcal{C}} \to P(\mathfrak{h}^*)$ with the following formula:

$$X \mapsto \sum_{\mathcal{Q}, \mu(\overline{T^*_{\mathcal{Q}}\mathfrak{B}}) = \overline{\mathcal{O}}} n_{\mathcal{Q}} \int_{\mathfrak{B}^f \cap \overline{T^*_{\mathcal{Q}}\mathfrak{B}}} e^{\omega_\lambda},$$

when $CC(X) = \sum_{\mathcal{Q}} n_{\mathcal{Q}} [\overline{T^*_{\mathcal{Q}}\mathfrak{B}}]$. The expression on the right is the multiplicity polynomial $m_X(\lambda)$ in the associated cycles for $\{X_\lambda\}$ by [7, Cor. 2.5.6].

We conclude from this discussion the following.

Proposition 1.7. *For $G_{\mathbf{R}} = Sp(p,q)$ or $SO^*(2n)$, if $\mathcal{C}$ is a Harish-Chandra cell with associated variety $\overline{\mathcal{O}}$, then the homomorphism*

$$\begin{aligned} V_{\mathcal{C}} &\to P_{2d}(\mathfrak{h}^*) \\ X &\mapsto m_X(\lambda) \end{aligned}$$

is W-equivariant.

2. Action of $A_G(f)$

In this section we consider the action of $A_G(f)$, the component group of the centralizer of f in G, on the set of irreducible components of $\mathfrak{B}^f$. First we give a general criterion for the class $[C] \in H_{\text{top}}(\mathfrak{B}^f)$ of a component of $\mathfrak{B}^f$ to generate $H_{\text{top}}(\mathfrak{B}^f)$ as $\mathbf{Q}[W]$-module. When (G, K) is one of our pairs (1) we see that this criterion is satisfied for the components associated to closed K-orbits in $\mathfrak{B}$. By applying several known results we are then able to conclude that for the Harish-Chandra module X_π of a discrete series representation, $V_{\mathcal{C}} = \mathbf{Q}[W] \cdot X_\pi$, when $X_\pi \in \mathcal{C}$.

For Prop. 2.2 below we let G be any semisimple algebraic group and let $f \in \mathcal{N} \subset \mathfrak{g}$. We consider the Springer representation of W on $H_{\text{top}}(\mathfrak{B}^f)$ ([14]). The action of $A_G(f)$ on $H_{\text{top}}(\mathfrak{B}^f)$, defined by the action on components of $\mathfrak{B}^f$, commutes with the action of W. There is a decomposition as W-representations:

$$H_{\text{top}}(\mathfrak{B}^f) = \bigoplus_{\sigma} H(\sigma), \tag{2.1}$$

where σ ranges over a subset of the irreducible representations of $A_G(f)$ and $H(\sigma)$ is the σ-isotypic subspace of $H_{\text{top}}(\mathfrak{B}^f)$. It is a fact that each $H(\sigma)$ is irreducible as W-representation ([14]).

Define $\overline{A}_G(f)$ to be the quotient of $A_G(f)$ by the common stabilizer of the components of $\mathfrak{B}^f$. Then each σ occurring in (2.1) may be viewed as a representation of $\overline{A}_G(f)$.

Proposition 2.2. *Suppose the stabilizer in $\overline{A}_G(f)$ of a component C of $\mathfrak{B}^f$ is trivial, then $\mathbf{Q}[W] \cdot [C] = H_{\text{top}}(\mathfrak{B}^f)$.*

Proof. Write $[C] = \sum h_\sigma$ according to the decomposition (2.1). Apply $z \in \overline{A}_G(f)$ to get

$$[z \cdot C] = \sum_{\sigma} z \cdot h_\sigma. \tag{2.3}$$

Let σ' be an irreducible representation of $\overline{A}_G(f)$ and write $\chi_{\sigma'}$ for its character. Multiple both sides of (2.3) by $\frac{\dim(\sigma')}{\#(\overline{A}_G(f))}\chi_{\sigma'}(z)$ and sum over $z \in \overline{A}_G(f)$:

$$\frac{\dim(\sigma')}{\#(\overline{A}_G(f))} \sum_z \chi_{\sigma'}(z)\,[z \cdot C] = \sum_\sigma \Big(\frac{\dim(\sigma')}{\#(\overline{A}_G(f))} \sum_z \chi_{\sigma'}(z)\, z \cdot h_\sigma \Big) = h_{\sigma'}.$$

The last equality holds because $P_{\sigma'} = \frac{\dim(\sigma')}{\#(\overline{A}_G(f))} \sum_z \chi_{\sigma'}(z)\, z$ is the projection onto the σ'-isotypic subspace. The left-hand side is nonzero since $\{[z \cdot C] : z \in \overline{A}_G(f)\}$ is independent (by the hypothesis) and $\chi_{\sigma'}(z) \neq 0$ for some $z \in \overline{A}_G(f)$. We conclude that in the expression for $[C]$, $h_\sigma \neq 0$ for *all* $\sigma \in \overline{A}_G(f)\widehat{}$.

Therefore,

$$\mathbf{Q}[W] \cdot [C] \supset \mathbf{Q}[W] \cdot P_\sigma([C]) = H(\sigma).$$

The last equality holds because $P_\sigma([C]) = h_\sigma \neq 0$ and $H(\sigma)$ is an irreducible W-representation, as noted earlier. Therefore, $\mathbf{Q}[W]\cdot[C]$ contains all isotypic subspaces, so is $H_{\text{top}}(\mathfrak{B}^f)$. □

Now return to the pairs of types C and D of (1). Fix a closed orbit $\mathcal{Q} = K \cdot \mathfrak{b}$ in $\mathfrak{B}$ and let $\mu(T^*_{\mathcal{Q}}\mathfrak{B}) = \overline{\mathcal{O}}$, $\mathcal{O} = K \cdot f$, $f \in \mathcal{N}_\theta$. It follows from [4] that for type C (resp., type D) there are at most two rows in the tableau parametrizing $K \cdot f$ having any given even (resp., odd) length. (See §6 for a discussion of this fact.) To simplify the discussion of centralizers slightly, for type D we replace G by $O(2n)$. As our goal is to verify the hypothesis in Prop. 2.2, working in $O(2n)$ suffices.

In both cases the component group is $A_G(f) = (\mathbf{Z}_2)^k$, where k is the number of pairs of rows of even (resp., odd) length in the tableau of f. See [8, Thm. 6.1.3]. In order to compute the $A_G(f)$-action on components we need to determine representatives for $z \in A_G(f)$ explicitly.

First consider the cases where the tableau of f has just one pair of rows, which have even length for type C and odd length for type D. Form an $\mathfrak{sl}(2)$ triple $\mathrm{span}_{\mathbf{C}}\{f, h, e\}$ and use the notation $K_{p,q}$ and $I_{n,n}$ as in [4, §1.2 ans 1.3]. Since we are in the two-row case $\mathbf{C}^n = V \oplus V'$, a direct sum of two n-dimensional irreducible $\mathfrak{sl}(2)$-representations. As described in [4, §1.3], there is a basis $\{v_i\}$ of V so that $v_i = f^{i-1}v_1$. Similarly there is a basis $\{v_i'\}$ of V'. The signs of the signed tableau correspond to the eigenvalues of $K_{p,q}$ (resp. $I_{n,n}$) of the eigenvectors v_i and v_i'. One easily checks that a Levi factor of the centralizer of f in G is given by matrices

$$\begin{pmatrix} \alpha I_n & \beta I_n \\ \gamma I_n & \delta I_n \end{pmatrix}, \text{ with } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in O(2),$$

with respect to the ordered basis $v_1, \dots, v_n, v_1', \dots, v_n'$. Note that (because of the parity of n) the rows of the signed tableau begin with different signs, so v_i and v_i' are in in different eigenspaces of $K_{p,q}$ or $I_{n,n}$. Consider

$$z = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

an element of the centralizer representing the nontrivial element of $A_G(f) \simeq O(2)/SO(2) \simeq \mathbf{Z}_2$. We have $z(v_i) = v_i'$, $i = 1, \dots, n$.

Lemma 2.4. *$z \notin K$.*

Proof. Note that $z(v_i) = v_i'$, $i = 1, \dots, n$. If z were in K, then z would commute with $K_{p,q}$ (resp., $I_{n,n}$), so preserve the eigenspaces of $K_{p,q}$ (resp., $I_{n,n}$). As noted above, v_i and v_i' are in in different eigenspaces. □

Note that if C is the component of $\mathfrak{B}^f$ corresponding to a closed orbit $\mathcal{Q}$, then $C \subset K \cdot \mathfrak{b}\,(= \mathcal{Q})$. We check that $z \cdot C \neq C$ by checking that $z \cdot \mathfrak{b} \notin K \cdot \mathfrak{b}$. Otherwise, $z \cdot \mathfrak{b} = k \cdot \mathfrak{b}$, for some $k \in K$. Therefore, there is $b \in K \cap B$ so that $b^{-1}k^{-1}z \cdot \mathfrak{h} = \mathfrak{h}$, by [12, Lemma 5.3]. But $b^{-1}k^{-1}z \cdot \mathfrak{b} = \mathfrak{b}$, so $b^{-1}k^{-1}z$ represents the identity in the Weyl group. Therefore, $b^{-1}k^{-1}z$ is in the Cartan subgroup H. Since $H \subset K$, this says that $z \in K$, contradicting the lemma.

Now suppose $\mathcal{O} = K \cdot f$ is dense in the associated variety of a discrete series representation, i.e., $\mu(T^*_{\mathcal{Q}}\mathfrak{B}) = \overline{\mathcal{O}}$, with $\mathcal{Q}$ closed in $\mathfrak{B}$. Then $A_G(f) \simeq (\mathbf{Z}_2)^k$, and each nontrivial element z is represented by a product of commuting order two elements described above. In particular $z \notin K$. The argument that $z \cdot C \neq C$, for the component C corresponding to $\mathcal{Q}$ is identical to the argument given for the two row case.

We conclude that the stabilizer in $A_G(f)$ of any component C corresponding to a closed K-orbit $\mathcal{Q} \subset \mathfrak{B}$ is trivial. Applying the criterion in Prop. 2.2 we get the following.

Proposition 2.5. *Suppose $\mathcal{Q} \subset \mathfrak{B}$ is a closed K-orbit, $\mu(T^*_{\mathcal{Q}}\mathfrak{B}) = \overline{K \cdot f}$ and $C = \mathfrak{B}^f \cap T^*_{\mathcal{Q}}\mathfrak{B}$. Then $\mathbf{Q}[W] \cdot [C] = H_{\mathrm{top}}(\mathfrak{B}^f)$.*

Corollary 2.6. *Suppose X_π is a discrete series representation of $G_{\mathbf{R}} = Sp(p,q)$ or $SO^*(2n)$. Let $\mathcal{C}$ be the Harish-Chandra cell containing X_π. Then under the coherent continuation action we have $\mathbf{Q}[W] \cdot X_\pi = V_{\mathcal{C}}$.*

Proof. Consider the W-homomorphism $\varphi : V_{\mathcal{C}} \to H_{\mathrm{top}}(\mathfrak{B}^f)$ as given in (1.6). We first show that φ is an isomorphism. Since it is shown in [11] that $V_{\mathcal{C}}$ and $H_{\mathrm{top}}(\mathfrak{B}^f)$ are isomorphic, it suffices to show that φ is surjective. The characteristic cycle map (1.2) is surjective ([16, Remark 3]). Since no other cell has associated variety $\mathcal{O}$, the map (1.4) is surjective. But as noted in §1, the quotient (1.3) is isomorphic to $H_{\mathrm{top}}(\mathfrak{B}^f)^{A_K(f)}$. Since $A_K(f)$ is trivial (as follows easily from the discussion preceding Lemma 2.4), we conclude that φ is surjective.

Since $\mathcal{Q}$ has smooth closure, $CC(X_\pi) = 1 \cdot [T^*_{\mathcal{Q}}\mathfrak{B}]$. Therefore, $\varphi(X_\pi) = [C]$, and the proposition follows. $\square$

Remark 2.7. It is not the case that any irreducible representation in a cell generates the $V_{\mathcal{C}}$ as $\mathbf{Q}[W]$-module. An example occurs in $Sp(1,1)$.

3. Certain components

The coherent families of discrete series representations of $G_{\mathbf{R}}$ are in one-to-one correspondence with the set of positive systems $\Delta^+ \subset \Delta(\mathfrak{h}, \mathfrak{g})$ which contain a fixed positive system of compact roots, that is the set of closed K-orbits in $\mathfrak{B}$. The closed orbit is $\mathcal{Q} = K \cdot \mathfrak{b}$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$, $\mathfrak{n}^- = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{(-\alpha)}$. We need to use a construction that is given in [4, §2]. The positive system Δ^+ (equivalently, the closed orbit $\mathcal{Q}$) determines what is called an array consisting of two rows of dots, each row separated into blocks. By a simple combinatorial procedure, one obtains a nilpotent $f \in \mathfrak{n}^- \cap \mathfrak{p}$, which is represented in the array by connecting certain dots, which we refer to as forming 'strings through the array'. We refer to [4, §2] for the details of this procedure. The strings tell us explicitly which root vectors occur in f. The nilpotent element f that is obtained is generic in $\mathfrak{n}^- \cap \mathfrak{p}$ in the sense that $K \cdot f$ meets $\mathfrak{n}^- \cap \mathfrak{p}$ in a dense set. It follows that the discrete series representations in the coherent family corresponding to Δ^+ has associated variety equal to $\overline{K \cdot f}$, where f is obtained from the array corresponding to Δ^+.

Consider one of the pairs (G, K) of type C or D as in (1) and the corresponding real forms $G_{\mathbf{R}}$. We prove that each discrete series representation of $G_{\mathbf{R}}$ (of infinitesimal character ρ) is in a cell that contains a discrete series representation with corresponding array having a particularly nice form (see (P1) and (P2) below). As mentioned in §1, for the real forms under consideration distinct cells have distinct associated varieties. Therefore, each cell containing a discrete series representation contains a discrete series representation having a nice array. When the array is nice, as we shall see in §5, the multiplicity polynomial can be computed.

An array will be called *nice* if the following two properties hold.

(P1) The block sizes increase then decrease from left to right.
(P2) At each stage the longest string has length $2(\ell+1)-\delta$, with $\delta = 1$ or 2 for type C and $\delta = 0$ or 1 for type D (where $\ell+1$ is the number of blocks in the array).

The following observation will be useful. If N is the length of the first string (i.e., the number of boxes in the first row of the signed tableau), then $N \leq 2(\ell+1)-\delta$, where

$$\delta = \begin{cases} 2, & \text{if } N \text{ is even and} (G,K) \text{ is of type } C, \\ 1, & \text{if } N \text{ is odd,} \\ 0, & \text{if } N \text{ is even and } (G,K) \text{ is of type } D. \end{cases} \tag{3.1}$$

Therefore, if (P2) holds, then the array is minimal in the sense that there is no array with fewer blocks having the same first string length.

Proposition 3.2. *Given a discrete series representation π of $G_{\mathbf{R}}$, there is another discrete series representation that (a) has the same associated variety as π and (b) corresponds to a nice array.*

Proof. We use induction on n, the rank of G, to prove that for any array there is a nice array with the same associated variety. (Here we are using a slight abuse of notation by referring to the associated variety of the discrete series representation as the associated variety of the array.)

<u>Case 1.</u> The first string already has length $N = 2(\ell+1)-\delta$ (with δ as in (3.1)). It follows that the first string not only passes through each block at least once, it passes through each block twice except for the first and last when $\delta = 2$ in type C, the last if $\delta = 1$ in type C, and the first if $\delta = 1$ in type D. Delete the first string from the array to obtain a smaller array. By induction this smaller array has the same associated variety as an array satisfying (P1) and (P2); we refer to this as the *nice smaller array*. Our goal is to fit this nice smaller array into the part of the array passed through by the first string so as to form a new array for which (P1) and (P2) hold and the associated variety is unchanged.

The part of the original array (i.e., the set of dots) passed through by the first string looks like

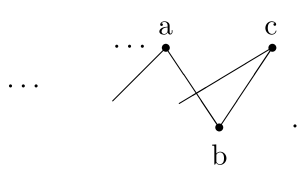
$$\begin{array}{cccccccccc} \bullet\ (\bullet) & & \bullet\ \bullet & & & & \bullet\ \bullet & \\ & & & & \cdots & & & \\ & \bullet\ \bullet & & \bullet\ \bullet & & & & \bullet\ (\bullet) \end{array} \tag{3.3}$$

where the $(\bullet)$'s may or may not occur (and the array may begin/end either up or down). Suppose the second string starts in the i_0^{th} block of the original array (where we number the blocks from left to right in the array as in [4]). Since the nice smaller array has no more blocks than the smaller array, we may fit the nice smaller array into (3.3) by combining its i^{th} block with the $(i+i_0-1)^{\text{th}}$ block of (3.3). This new array clearly satisfies (P1) and (P2); it has the same associated variety because the first two rows of the signed tableau are the same and, on omitting the first string in each, the smaller arrays have the same associated variety.

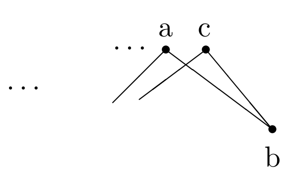
Case 2. The first string has length $N < 2(\ell+1)-\delta$. We show that there is an array with the same associated variety that has fewer blocks. Therefore we can eventually arrive at an array for which $N = 2(\ell+1)-\delta$, so that Case 1 applies. This will complete the proof.

Claim A. If the last two blocks are singletons and we are in type C or the last block is a singleton in type D, then there is an array with the same associated variety that has one less block. This is easy to see in each case as follows.

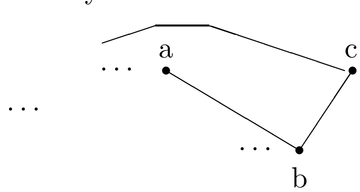
For type C the array ends with



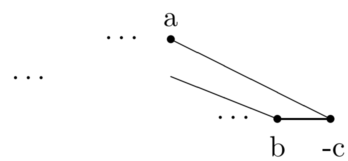
Moving the dot labelled by c to the second to the last block in the upper row gives



In type D the array ends with



Moving the dot labelled with c to the last block in the lower row gives



In each case the new array has one less block and has the same associated variety (since it has the same first row of the signed tableau and has the same smaller array).

Claim B. The array has the same associated variety as one for which the last two blocks are singletons in type C and the last block is a singleton in type D.

Observe that at least one block other than the last (resp., the first) in type C (resp., type D) is a singleton, otherwise $N = 2(\ell+1) - \delta$, Let B_j be the singleton block farthest to the right, with $j \neq \ell + 1$ for type C.

Let's consider type C first. If $j = \ell$ and the $B_{\ell+1}$ is a singleton, then we are done. If $j = \ell$ and $B_{\ell+1}$ is not a singleton, then form a new array by moving all dots except one from $B_{\ell+1}$ to $B_{\ell-1}$. This new array has the same associated variety and satisfies the condition of Claim B. If $j < \ell$, then B_{j+1} is not a singleton. Again move all dots except one from B_{j+1} to B_{j-1} to obtain an array with the same associated variety and the singleton one block farther to the right. After repeating a finite number of times we get to the case $j = \ell$.

For type D, if $j = \ell + 1$ we are done by Claim A. Otherwise, all dots but one can be moved from B_{j+1} to B_{j-1} as above without changing the associated variety. Repeating we eventually get to the case for which $j = \ell + 1$.

The proof is now complete. □

Note that the proof given above gives an *algorithm* for finding a nice array having the same associated variety as a given array.

4. The structure of the Springer fibers

We now consider only discrete series representations corresponding to nice arrays (as defined by (P1) and (P2)). The associated cycles of these discrete series representations will be calculated. A theorem of J.-T. Chang ([6]) states that the if $\{X_\lambda\}$ is the coherent family of discrete series representations associated to a closed K-orbit $K \cdot \mathfrak{b}$, $\mathfrak{b} = \mathfrak{h} + \bar{\mathfrak{n}}$, and f is generic in $\bar{\mathfrak{n}} \cap \mathfrak{p}$, then

$$AC(X_\lambda) = m_{\mathfrak{Q}}(\lambda) \cdot \overline{K \cdot f}.$$

The multiplicity is given by $m_{\mathfrak{Q}}(\lambda) = \dim(H^0(\mathfrak{B}^f \cap T^*_{\mathfrak{Q}}\mathfrak{B}, \mathcal{O}(\lambda + \rho - 2\rho_c)))$. Proposition 4.12 describes the structure of $\mathfrak{B}^f \cap T^*_{\mathfrak{Q}}\mathfrak{B}$ in a convenient way. This, along with the Borel-Weil Theorem, will allow us to compute the multiplicity polynomials $m_{\mathcal{Q}}(\lambda)$.

The content of much of this section is contained in [4]. However, in [4] the results are for arbitrary closed K-orbit $\mathcal{Q} \subset \mathfrak{B}$, which is considerably more involved. We consider it worthwhile to present here the much simpler direct arguments for the case of nice arrays.

Consider type D first. Suppose that the discrete series corresponds to a nice array and let b_j be the number of dots in the j^{th} block, for $j = 1, 2, \dots, \ell + 1$. Let L be the Levi subgroup of K corresponding to the simple compact roots for the positive system from which the array is constructed. Then ([4, formula 2.10])

$$L \simeq GL(b_1) \times GL(b_2) \times \cdots \times GL(b_{\ell+1}),$$

and $L \cdot \mathfrak{b}$ is an irreducible closed subvariety of $\mathfrak{B}^f \cap T^*_{\mathcal{Q}}\mathfrak{B}$. Since the component group $A_K(f)$ is trivial, we may conclude $L \cdot \mathfrak{b} = \mathfrak{B}^f \cap T^*_{\mathcal{Q}}\mathfrak{B}$ once we show

$$\dim(L \cdot \mathfrak{b}) = \dim(\mathfrak{B}^f). \tag{4.1}$$

This will be accomplished by induction on the number of rows in the signed tableau. If there are just two rows, then

$$L \simeq \begin{cases} GL(1) \times GL(2)^{\ell}, & \text{if } \delta = 1 \, (\text{i.e., if } n \text{ is odd}) \\ GL(2)^{\ell+1}, & \text{if } \delta = 0 \, (\text{i.e., if } n \text{ is even}) \end{cases}$$

Therefore $\dim(L \cdot \mathfrak{b}) = \ell + 1 - \delta$.

$$\begin{aligned} \dim(\mathfrak{B}^f) &= \frac{1}{2}(\dim(Z_G(f) - \operatorname{rank}(G)), \quad \text{by, e.g., [10, Section 6.7]}, \\ &= \frac{1}{4}(\sum c_i - \#(\text{odd rows}) - 2n), \quad \text{by, e.g., [8, Section 6.1]} \end{aligned} \tag{4.2}$$

where c_i is the number of boxes in the i^{th} column of the signed tableau for f (so $c_i = 2$ in the present case). Therefore,

$$\begin{aligned} \dim(\mathfrak{B}^f) &= \frac{1}{4}(4n - 2\delta - 2n) \\ &= \frac{1}{4}(4(\ell+1) - 4\delta) \\ &= \ell + 1 - \delta. \end{aligned}$$

Now assume there are more than two rows in the tableau of f and (4.1) holds whenever the tableau has fewer rows. In the construction of f, f_0 is constructed from the first string, then a generic element f' for a smaller rank pair (G_2, K_2) of the same type is constructed, and $f = f_0 + f'$. The tableau of f begins with two rows of length N $(= 2(\ell+1) - \delta)$, which are followed by the rows of the tableau for f'. The inductive hypothesis gives

$$\dim(\mathfrak{B}_2^{f'}) = \dim(GL(b_1 - 2 + \delta) \times \cdots \times GL(b_{\ell+1} - 2) \cdot \mathfrak{b}),$$

equivalently

$$\frac{1}{4}\Big(\sum(c_i-2)^2-\#(\text{odd rows}-2\delta)-2(n-N)\Big)=\sum_{j=1}^{\ell+1}\frac{(b_j-2)(b_j-3)}{2}-\delta.$$

(Note that if $\delta=1$, then $b_1=1$, explaining the occurrence of δ in the above formula.) We compute:

$$\begin{aligned}
\dim(\mathfrak{B}^f)&=\frac{1}{4}\big(\sum c_i^2-\#(\text{odd rows})-2n\big)\\
&=\frac{1}{4}\big(\sum(c_i-2)^2+8n-4N-(\#(\text{odd rows})-2\delta)-2\delta-2(n-N)-2N\big)\\
&=\sum_j\frac{(b_j-2)(b_j-3)}{2}-\delta+\frac{1}{4}\big(8n-6N-2\delta\big)\\
&=\sum_j\frac{b_j(b_j-1)}{2}-2\sum_j b_j+\sum_j 3-\delta+\frac{1}{4}\big(8n-12(\ell+1)+4\delta\big)\\
&=\dim(L\cdot\mathfrak{b})-2n+3(\ell+1)-\delta+\frac{1}{4}\big(8n-12(\ell+1)+4\delta\big)\\
&=\dim(L\cdot\mathfrak{b}).
\end{aligned}$$

This proves (4.1).

We now turn to type C. Consider the parabolic subgroups $Q_{2i}=L_{2i}\overline{U}_{2i}$ of K_{2i} defined by the simple compact groups in the array for (G_{2i},K_{2i}); see [4, §2.4]. Then

$$L_0\simeq GL(b_1)\times\cdots\times GL(b_{\ell+1}),$$

and similarly for $L_2,\dots,L_{2m}$. We note that an easy induction ([4, Rem. 2.11]) gives

$$L_{2m}\cdots L_2L_0\cdot\mathfrak{b}\subset\mathfrak{B}^f\cap T^*_{\mathfrak{Q}}\mathfrak{B}. \tag{4.3}$$

We show that $L_{2m}\cdots L_2L_0\cdot\mathfrak{b}$ is closed and has the same dimension as $\mathfrak{B}^f\cap T^*_{\mathfrak{Q}}\mathfrak{B}$.

Let

$$d_j:=\#\big(\Delta(\mathfrak{n}\cap\mathfrak{l}_{2j})\setminus\Delta(\mathfrak{n}\cap\mathfrak{l}_{2j+2})\big). \tag{4.4}$$

Since $\Delta(\mathfrak{n}\cap\mathfrak{l}_{2j})\setminus\Delta(\mathfrak{n}\cap\mathfrak{l}_{2j+2})$ are pairwise disjoint, we may conclude from [15, 8.2.1] that

$$\sum_{j=0}^{m}d_j\le\dim(L_{2m}\cdots L_2L_0\cdot\mathfrak{b}). \tag{4.5}$$

We show that

$$\sum_0^m d_j=\dim(\mathfrak{B}^f) \tag{4.6}$$

by induction on the number of rows in the signed tableau of f. It suffices to show that

$$d_0=\dim(\mathfrak{B}^f)-\dim(\mathfrak{B}_2^{f'}), \tag{4.7}$$

where $f = f_0 + f'$ as above and $\mathfrak{B}_2$ is the flag variety for $G_2 = Sp(2(n-N))$.

$$\begin{aligned}
\dim(\mathfrak{B}^f) &- \dim(\mathfrak{B}_2^{f'}) \\
&= \frac{1}{4}\big(\sum c_i^2 + \#(\text{odd rows}) - 2n\big) \\
&\quad - \frac{1}{4}\big(\sum (c_i-2)^2 + (\#(\text{odd rows}) - 2(2-\delta) - 2(n-N)\big) \\
&= \frac{1}{4}\big(8n - 6N + (4-2\delta)\big) \\
&= 2n - 3(\ell+1) + \delta + 1.
\end{aligned}$$

Now we compute d_0. Note first that

$$\begin{aligned}
&L_0 \simeq GL(b_1) \times GL(b_2) \times \cdots GL(b_\ell) \times Sp(2b_{\ell+1}) \\
&L_0 \cap L_2 \simeq GL(b_1 - 3 + \delta) \times GL(b_2 - 2) \times \cdots \times GL(b_\ell - 2) \times Sp(2(b_{\ell+1} - 1)).
\end{aligned} \tag{4.8}$$

Therefore,

$$\begin{aligned}
d_0 &= \frac{1}{2}\dim(L_0/L_0 \cap L_2) \\
&= (3-\delta)\big(\frac{2b_1 + \delta - 4}{2}\big) + \sum_{j=2}^{\ell} \frac{2(2b_j - 3)}{2} + 2b_{\ell+1} - 1 \\
&= \sum_{j=0}^{\ell+1}(2b_j - 3) + \delta + 1 + \frac{(1-\delta)(2b_1 + \delta - 4)}{2} \\
&= 2n - 3(\ell+1) + \delta + 1,
\end{aligned}$$

the last equality holding since $\delta = 2$ implies $b_1 = 1$, so $(1-\delta)(2b_1 + \delta - 4) = 0$. We conclude that (4.6) holds.

Now we check that $L_{2m} \cdots L_2 L_0 \cdot \mathfrak{b}$ is closed. Note that since $\overline{U}_{2j+2} \subset \overline{U}_{2j}$, we have

$$Q_{2j+2} L_{2j} \subset L_{2j+2} Q_{2j} \tag{4.9}$$

An easy induction argument (on j) shows that

$$Q_{2j} \cdots Q_2 Q_0 \cdot \mathfrak{b} = L_{2j} \cdot L_2 L_0 \cdot \mathfrak{b}, \tag{4.10}$$

for each $j = 0, 1, \ldots, m$. Now $L_{2j+2} \cap Q_{2j}$ is a parabolic subalgebra of L_{2j+2} which stabilizes $L_{2j} \cdots L_2 L_0 \cdot \mathfrak{b}$. It follows that if $L_{2j} \cdots L_2 L_0 \cdot \mathfrak{b}$ is closed, then so is $L_{2j+2} \cdots L_2 L_0 \cdot \mathfrak{b}$ (e.g., [10, Section 0.15]). We conclude that (4.3) is closed, so

$$L_{2j} \cdots L_2 L_0 \cdot \mathfrak{b} = \mathfrak{B}^f \cap T^*_{\mathfrak{Q}}\mathfrak{B}. \tag{4.11}$$

Each L_{2j} is a product of $GL(k)$'s and a symplectic group (4.8); we let S_j denote this symplectic group. One easily sees that the $GL(k)$'s are contained in L_0 and

$$L_{2m} \cdots L_2 L_0 \cdot \mathfrak{b} = S_m \cdots S_1 L \cdot \mathfrak{b}.$$

Summarizing the above calculations we have the following.

Proposition 4.12. *For any nice array and corresponding generic element f*

$$\mathfrak{B}^f \cap T^*_{\mathcal{Q}}\mathfrak{B} = \begin{cases} S_m \cdots S_1 L \cdot \mathfrak{b}, & \text{in type } C \\ L \cdot \mathfrak{b}, & \text{in type } D. \end{cases}$$

5. Associated cycles

We have seen that each cell containing a discrete series representation of $G_{\mathbf{R}}$ contains a discrete series representation corresponding to a closed K-orbit $\mathcal{Q}$ having a nice array. To compute the multiplicity polynomials for all discrete series representations (in fact all representations in a cell containing a discrete series representation) it therefore suffices to compute the multiplicity polynomials for those discrete series representations corresponding to nice arrays (cf. §1). We therefore fix a closed K-orbit $\mathcal{Q}$ in $\mathfrak{B}$ having a nice array, and compute the multiplicity polynomial $m_{\mathcal{Q}}(\lambda)$.

By (3),

$$m_{\mathcal{Q}}(\lambda) = \dim(H^0(\mathfrak{B}^f \cap T^*_{\mathcal{Q}}\mathfrak{B}, \mathcal{O}(\tau))),$$

with $\tau = \lambda + \rho - 2\rho_c$. In type D this is easy to compute by the Borel-Weil Theorem and the Weyl Dimension Formula.

$$m_{\mathcal{Q}}(\lambda) = \dim(H^0(\mathfrak{B}_L, \mathcal{O}(\tau))) = \prod_{\alpha \in \Delta(\mathfrak{l})} \frac{\langle \tau + \rho_{\mathfrak{l}}, \alpha\rangle}{\langle \rho_{\mathfrak{l}}, \alpha\rangle}.$$

Since $\mathfrak{l}$ is the Levi subalgebra of parabolic subalgebras of both $\mathfrak{g}$ and $\mathfrak{k}$ defined be a set of simple roots,

$$\rho - 2\rho_c + \rho_{\mathfrak{l}} = (\rho - \rho_{\mathfrak{l}}) - 2(\rho_c - \rho_{\mathfrak{l}})$$

is orthogonal to $\Delta(\mathfrak{l})$. Therefore,

$$m_{\mathcal{Q}}(\lambda) = \prod_{\alpha \in \Delta(\mathfrak{l})} \frac{\langle \lambda, \alpha\rangle}{\langle \rho_{\mathfrak{l}}, \alpha\rangle}.$$

The type C case is more involved because $\mathfrak{B}^f \cap T^*_{\mathcal{Q}}\mathfrak{B}$ is not homogeneous. For the computation of $m_{\mathcal{Q}}(\lambda)$ we follow [2, §6]. In particular, let

$$W_{-\tau} = H^0(\mathcal{Q}, \mathcal{O}(\tau)),$$

the irreducible finite dimensional representation with lowest weight $-\tau$. Fix a lowest weight vector $w_{-\tau}$. Then for τ sufficiently dominant,

$$m_{\mathcal{Q}}(\lambda) = \dim(\operatorname{span}_{\mathbf{C}}\{L_{2m} \cdots L_2 L \cdot w_{-\tau}\}).$$

Set $U_{-\tau} = \operatorname{span}_{\mathbf{C}}\{L \cdot w_{-\tau}\}$, an irreducible L-representation of lowest weight $-\tau$. Therefore,

$$m_{\mathcal{Q}}(\lambda) = \dim(\operatorname{span}_{\mathbf{C}}\{L_{2m} \cdots L_2 \cdot U_{-\tau}\}). \tag{5.1}$$

This is computed inductively as follows. Suppose $1 \le j \le m-1$ and we have already computed the decomposition

$$\mathrm{span}_{\mathbf{C}}\{L_{2(j-1)} \cdots L_2 L \cdot U_{-\tau}\} = \sum_i E_{-\tau_i},$$

into irreducible $L_{2(j-1)}$-representations, with $E_{-\tau_i} = \mathrm{span}_{\mathbf{C}}\{L_{2(j-1)} \cdot e_{-\tau_i}\}$ and $e_{-\tau_i}$ a lowest weight vector. Then,

$$\mathrm{span}_{\mathbf{C}}\{L_{2j} \cdots L_2 L \cdot U_{-\tau}\} = \sum_i \mathrm{span}_{\mathbf{C}}\{L_{2j} \cdot E_{-\tau_i}\},$$

Each $\mathrm{span}_{\mathbf{C}}\{L_{2j} \cdot E_{-\tau_i}\}$ can be computed. First decompose

$$E_{-\tau_i}|_{L_{2j} \cap L_{2(j-1)}} = \sum_k F_{-\tau_{ik}}. \tag{5.2}$$

Then,

$$\mathrm{span}_{\mathbf{C}}\{L_{2j} \cdot E_{-\tau_i}\} = \sum_k \mathrm{span}_{\mathbf{C}}\{L_{2j} \cdot F_{-\tau_{ik}}\} = \sum_k \mathrm{span}_{\mathbf{C}}\{L_{2j} \cdot f_{-\tau_{ik}}\},$$

where $f_{-\tau_{ik}}$ is a lowest weight vector for the $L_{2j} \cap L_{2(j-1)}$-representation $F_{-\tau_{ik}}$.

Lemma 5.3. $\mathrm{span}_{\mathbf{C}}\{L_{2j} \cdot f_{-\tau_{ik}}\}$ *is an irreducible representation with lowest weight* $-\tau_{ik}$.

Proof. It suffices to show that $f_{-\tau_{ik}}$ is a lowest weight vector for L_{2j}, i.e., is annihilated by $\mathfrak{n}^- \cap \mathfrak{l}_{2j}$. Since $L_{2(j-1)}$ normalizes $\mathfrak{u}_{2(j-1)}$, $E_{-\tau_i}$ is annihilated by $\mathfrak{u}_{2(j-1)}$. Since $\mathfrak{u}_{2j} \subset \mathfrak{u}_{2(j-1)}$, $E_{-\tau_i}$ is also annihilated by $\mathfrak{u}_{2j}$. As $f_{-\tau_{ik}}$ is a lowest weight vector it is annihilated by $\mathfrak{n}^- \cap \mathfrak{l}_{2j} \cap \mathfrak{l}_{2(j-1)}$. Therefore, $f_{-\tau_{ik}}$ is annihilated by

$$\mathfrak{n}^- \cap \mathfrak{l}_{2j} \subset \mathfrak{n}^- \cap \mathfrak{l}_{2j} \cap \mathfrak{l}_{2(j-1)} + \mathfrak{u}_{2(j-1)}.$$

□

To complete the computation of $m_{\mathcal{Q}}(\lambda)$ we need to explicitly carry out the decomposition (5.2). For this one uses well-known branching rules for restricting finite dimensional representations from $GL(k+2)$ to $GL(k)$ and $Sp(2k)$ to $Sp(2(k-1))$.

6. Associated varieties

In this section we determine exactly which K-orbits in $\mathcal{N}_\theta$ are dense in the associated varieties of discrete series representations for $G_{\mathbf{R}} = Sp(p,q)$ and $SO^*(2n)$. Recall that the K-orbits in $\mathcal{N}_\theta$ are in one-to-one correspondence with signed tableau having $2n$ boxes containing signs that alternate along each row, and for type C (resp., type D)

(a) have $2p$ (resp., n) $+$ signs and $2q$ (resp., n) $-$ signs;
(b) have the number of rows of a given even (resp., odd) length beginning with a $+$ sign coinciding with the number beginning with a $-$ sign;

(c) have an even number of rows of a given odd (resp., even) length that begin with a $+$ sign and also an even number beginning with a $-$ sign.

By Prop. 3.2 any K-orbit that occurs as the associated variety of a discrete series representation occurs as the the associated variety of a discrete series representation corresponding to a nice array. One easily concludes that these tableau in types C (resp., type D) satisfy:

(i) all rows of a given odd (resp., even) length begin with the same sign;
(ii) only one pair of rows of a given even (resp., odd) length can occur.

It is a fact that the K-orbits described by (i)-(ii) are precisely the K-orbits $K \cdot f$ in $\mathcal{N}_\theta$ so that the reductive part of the stabilizer in $\mathfrak{g}$ is contained in $\mathfrak{k}$ (i.e., a Levi subgroup of $Z_K(f)$ is open in a Levi subgroup of $Z_G(f)$). It has recently been proved by B. Harris ([9]) that it is a general fact that the associated variety of any tempered representation (any semisimple Lie group) satisfies this condition. We show the converse fails by showing that there are K-orbits in $\mathcal{N}_\theta$ satisfying (i)-(ii) that are not dense in the associated variety of a discrete series representation. The following examples illustrate the way this can happen.

For type C consider the following signed tableau:

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
+ & - & \cdots & + & - & + \\ \hline
+ & - & \cdots & + & - & + \\ \hline
+ & - & \cdots & + & - & \\ \cline{1-5}
- & + & \cdots & - & + & \\ \cline{1-5}
+ & - & \cdots & + & & \\ \cline{1-4}
+ & - & \cdots & + & & \\ \cline{1-4}
\end{array}
\quad \text{and} \quad
\begin{array}{|c|c|c|c|c|c|}
\hline
- & + & \cdots & - & + & - \\ \hline
- & + & \cdots & - & + & - \\ \hline
+ & - & \cdots & + & - & \\ \cline{1-5}
- & + & \cdots & - & + & \\ \cline{1-5}
- & + & \cdots & - & & \\ \cline{1-4}
- & + & \cdots & - & & \\ \cline{1-4}
\end{array}
\tag{6.1}
$$

(The lengths of the rows are $N, N-1$ and $N-2$, and N is necessarily odd.) These tableau are not dense in the associated variety of a discrete series representation since a corresponding nice array would need to have five dots in the first block. But this would imply that the first four rows in the tableau would start with $+$ signs. This argument also says that any tableau that contains (6.1) as a subtableau is not dense in the associated variety of a discrete series representation. (By *subtableau* we mean obtained from the tableau by deleting certain rows.)

Similarly, in type D the patterns

$$\begin{array}{|c|c|c|c|c|c|}
\hline
+ & - & \cdots & + & - & + \\ \hline
- & + & \cdots & - & + & - \\ \hline
+ & - & \cdots & + & - \\ \cline{1-5}
+ & - & \cdots & + & - \\ \cline{1-5}
+ & - & \cdots & + \\ \cline{1-4}
- & + & \cdots & - \\ \cline{1-4}
\end{array}
\quad \text{and} \quad
\begin{array}{|c|c|c|c|c|c|}
\hline
+ & - & \cdots & + & - & + \\ \hline
- & + & \cdots & - & + & - \\ \hline
- & + & \cdots & - & + \\ \cline{1-5}
- & + & \cdots & - & + \\ \cline{1-5}
+ & - & \cdots & + \\ \cline{1-4}
- & + & \cdots & - \\ \cline{1-4}
\end{array}
\tag{6.2}$$

cannot occur as a subtableau of an orbit dense in the associated variety of a discrete series representation.

Proposition 6.3. *A K-orbit in $\mathcal{N}_\theta$ is dense in the associated variety of a discrete series representation if and only if its signed tableau satisfies (i)-(ii) and the patterns of (6.1) (resp., (6.2)) do not occur as a subtableau for type C (resp., type D).*

Proof. We have already established one direction; a K-orbit dense in the associated variety of a discrete series representation must have signed tableau satisfying (i)-(ii) and the specified patterns not occurring. We now assume that the pair (G, K) is of type C. We proceed by induction on $\text{rank}(G) = n$. Suppose a signed tableau satisfies (i)-(ii) and contains no subtableau as in (6.1). Our goal is to find an array (corresponding to a closed K orbit $\mathcal{Q} \subset \mathfrak{B}$) for which the algorithm computing the generic f results in the signed tableau. The proof roughly begins by writing down part of an array (the first string) which gives the first pair of rows of the array. Then the rest of the signed tableau gives a corresponding smaller array. This smaller array must be fit into the first string in the sense that the resulting array gives (by the algorithm to find the generic element) the signed tableau we want to produce.

There are three possibilities for the first row of the tableau.

1) The row length N is even. The the first pair of rows is therefore

$$\begin{array}{|c|c|c|c|}
\hline
+ & - & \cdots & - \\ \hline
- & + & \cdots & + \\ \hline
\end{array}\ .$$

Consider the smaller tableau obtained by removing the first pair of rows. By induction, this smaller tableau is the associated variety of a nice array (for a rank $n - N$ group). This nice array fits into one of

$$\begin{array}{ccccccc}
\bullet & & \bullet\,\bullet & & & & \\
 & & & & & \cdots & \\
 & \bullet\,\bullet & & & \bullet\,\bullet & &
\end{array}
\quad \text{or} \quad
\begin{array}{ccccccc}
 & \bullet\,\bullet & & & \bullet\,\bullet & & \\
 & & & & & & \cdots \\
\bullet & & & \bullet\,\bullet & & &
\end{array}$$

Since the longest row of the smaller tableau has length strictly less than $N-1$ (by condition (ii)), this is easy to see.

2) The row length N is odd and the first two rows are

$$\begin{array}{|c|c|c|c|} \hline + & - & \cdots & + \\ \hline + & - & \cdots & + \\ \hline \end{array}\ .$$

This case is a little trickier. Again we omit the first pair of rows to obtain a smaller tableau, which is the associated variety of a nice array. There are several subcases to consider. The first is when this smaller tableau has longest row length equal to N; it is immediate that the corresponding nice array fits into

$$\begin{array}{ccccc} \bullet\,\bullet & & \bullet\,\bullet & & \\ & & & & \cdots \\ & \bullet\,\bullet & & \bullet\,\bullet & \end{array} \tag{6.4}$$

to form an array with our tableau as associated variety. When the smaller tableau has length $\leq N-2$, then it is easy to see that the corresponding nice array fits into (6.4).

Now consider the subcase where the smaller tableau has longest pair of rows of length $N-1$ (an even integer). It is not clear how to fit the nice array into (6.4). For example, the nice array might look like

$$\begin{array}{ccccc} & \bullet\,\bullet & & \bullet\,\bullet & \\ & & & & \cdots \\ \bullet & & \bullet\,\bullet & & \end{array}$$

which does not fit in. However, it turns out that there is another nice array with the same associated variety which does fit in. There are several possibilities. First suppose that the third pair of rows has length $\leq N-3$. Then there are (at least) two nice arrays having the smaller tableau as associated variety. This is the statement that if the second pair of rows has at least two fewer boxes than the first, then the smaller array fits into *both*

$$\begin{array}{ccccc} \bullet & & \bullet\,\bullet & & \\ & & & & \cdots \\ & \bullet\,\bullet & & \bullet\,\bullet & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & \bullet\,\bullet & & \bullet\,\bullet & \\ & & & & \cdots \\ \bullet & & \bullet\,\bullet & & \end{array}$$

The first of these fits into (6.4). Second, suppose that the third pair of rows have length $N-2$. If the rows begin with $++, +-, --$, then the smaller nice array fits into (6.4) with no problem, as this smaller nice array is

$$\begin{array}{llll} \bullet & & \bullet\ \bullet\ \cdots & \\ & & & \cdots \\ & \bullet\ \bullet\ \cdots & & \end{array}$$

with the same number of blocks. The rows of the tableau cannot cannot begin with the signs $++, +-, ++$, since we are assuming that our tableau does not contain the pattern (6.1). The third possibility is that there is no third pair of rows. Then the smaller tableau is

$+$	$-$	$\cdots$	$-$
$-$	$+$	$\cdots$	$+$

,

with row length $N-1$, and we may choose the corresponding nice array to be

$$\begin{array}{lllll} \bullet & & \bullet\ \bullet & & \\ & & & \cdots & \\ & \bullet\ \bullet & & \bullet\ \bullet & , \end{array}$$

which fits into (6.4).

3) The case of odd row length with the first two rows

$-$	$+$	$\cdots$	$-$
$-$	$+$	$\cdots$	$-$

is essentially the same as 2) above.

The proof for type D is very similar to the proof for type C. $\square$

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