SQUARE INTEGRABLE HARMONIC FORMS AND REPRESENTATION THEORY

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1. Introduction

An interesting class of irreducible unitary representations of semisimple Lie groups consists of the representations associated to elliptic coadjoint orbits. An important open problem is to give a construction of these representations in terms of the geometry of the orbits. For example one may construct some Hilbert space of sections of a bundle over the orbit; the Hilbert space inner product being a G–invariant L_2 inner product. Consider an elliptic coadjoint orbit $G \cdot \chi = G/L$. As explained in Section 2, G/L has an invariant complex structure and is biholomorphic to an open orbit in a complex flag manifold. Associated to this orbit is the representation of G naturally occurring on the sheaf cohomology space $H^s(G/L, \mathcal{O}(\mathcal{L}_\chi))$ (under some negativity condition on the line bundle $\mathcal{L}_\chi$). It is a (difficult) known result that this gives a continuous representation of G on a Frechet space. In fact, $H^s(G/L, \mathcal{O}(\mathcal{L}_\chi))$ is a maximal globalization in the sense of [18]. See [25]. It follows that the unitary globalization lies inside the cohomology space. Thus, it is reasonable to look for "L_2" representatives of cohomology classes and define an invariant inner product. This space of representatives should give a Hilbert space with an invariant inner product on these representatives. A somewhat simple example occurs when G is a compact group. The orbit then carries a positive definite hermitian metric (defining a notion of L_2 and harmonic) and the Hodge theorem provides a space of L_2–harmonic forms, one from each cohomology class. These L_2–harmonic spaces are realizations of the unitary representations of the compact group.

As we are interested in arbitrary semisimple Lie groups, often the only invariant hermitian metric available is indefinite. This indefinite metric can be used to define a global invariant hermitian form on $\mathcal{L}_\chi$–valued type $(0, s)$ differential forms. Of course the integral defining this global form must converge. In a very general setting we show in Theorem 3.4 how to choose represenatives for each K–finite cohomology class for which the integral defining the global form converges. In case G/L is an indefinite Kähler symmetric space (ie., L is the fixed point group of an involution) we get the following stronger result. Following the definition in [14], using an auxilary metric on G/L which is positive definite but not G–invariant, we define a Hilbert space of square integrable $\mathcal{L}_\chi$–valued type $(0, s)$ differential forms containing representatives for all K–finite cohomology classes. On this L_2–space the integral defining the global invariant form is convergent and semidefinite. However, we are not able to show it is nonzero in general. If we also assume $\mathrm{rank}_{\mathbf{R}}(G) = \mathrm{rank}_{\mathbf{R}}(L)$ then we show this global invariant form is nonzero. In this case, passing to a quotient, we obtain a continuous representation of G on a Hilbert space $H_2^{(0,s)}$. This Hilbert space representation is infinitesimally equivalent to the cohomology representation. Furthermore, the global invariant form is positive definite on $H_2^{(0,s)}$.

The main tool for picking out cohomology classes is the intertwining operator studied in [2] and [1]. The intertwining operator maps a principal series representation into the space of closed forms. It is given by a very explicit integral formula. A sketch of the construction and some key facts are given in Section 2. In Section 3 we show that the integral defining the global G–invariant hermitian form converges. This is acomplished by showing that the global invariant form is a multiple of the standard form on the principal series defined in terms of intertwining operators. Square integrability in the case when G/L is indefinite

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Kähler symmetric is proved in Section 4. Harish–Chandra type estimates are used. In Section 5 a continuous Hilbert space representation is constructed.

Other 'L_2 cohomology' constructions of representations are given in [16], [17] and [14]. Schmid's work gives a realization of the discrete series representations. An important difference from our work is that L is compact, thus the orbit has a positive definite G–invariant metric. Parts of our work (including the construction of the intertwining map in [1]) depend on Schmid's results. However, the case for which we get our strongest results, $\operatorname{rank}_{\mathbf{R}}(G) = \operatorname{rank}_{\mathbf{R}}(L)$, can be viewed as the opposite extreme from the case considered by Schmid (and does not rely on the results in [16] and [17]). Rawnsley, Schmid and Wolf realize certain singular highest weight representations as L_2–harmonic spaces in the setting of an indefinite metric. They assume G/L is indefinite Kähler symmetric satisfying a holomorphic condition. The framework for indefinite quantization, in particular the definition of the L_2–harmonic space studied here, was formulated in [14]. There is a small overlap of our results with [14], however our methods are very different than theirs. The results in [26] are related to the results here. In particular the square integrability here is proved in a more elementary way and in a much more general setting.

2. Preliminaries

As mentioned in the introduction, our main tool is an intertwining operator $\mathcal{S}$ from a principal series representation into the space of forms of type $(0, s)$. We begin this section by recalling some facts about representations in cohomology. Since we will need detailed information about $\mathcal{S}$, we will also state the main results of [2] and [1]. We will then prove several lemmas that will be needed in Sections 3, 4 and 5.

Let $G_{\mathbf{C}}$ be a complex semisimple Lie group and G a real form of $G_{\mathbf{C}}$. Fix a Cartan involution θ and let $K \subset G$ be the fixed point group of θ, a maximal compact subgroup of G. We denote the Lie algebras of $G_{\mathbf{C}}, G, K$, etc. by $\mathfrak{g}, \mathfrak{g}_0, \mathfrak{k}_0$, etc. The Cartan decomposition of $\mathfrak{g}$ is written as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. For a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ we use the common notation of $\Delta(\mathfrak{g}, \mathfrak{h})$ for the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Similar notation is used for the other subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$. We let Z be an arbitrary complex flag manifold for $G_{\mathbf{C}}$ and $D \subset Z$ a measurable open G-orbit. The structure of G-orbits on Z is studied in detail in [23]. In particular, [23] contains the definition of a measurable open orbit as well as the following facts. A base point $z_0 \in D$ may be chosen so that:

(a) $D = G \cdot z_0$ and $Q = \operatorname{stab}_{G_{\mathbf{C}}}(z_0)$ is a θ-stable parabolic subgroup of $G_{\mathbf{C}}$.
(b) $Q \cap \overline{Q} = L = \operatorname{stab}_G(z_0)$ is connected and contains a fundamental Cartan subgroup H. Write $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}'_0$ with $\mathfrak{t}_0 \subset \mathfrak{k}_0, \mathfrak{a}'_0 \subset \mathfrak{p}_0$.
(c) L is the centralizer of some torus in $\mathfrak{t}_0$, hence there is some $\lambda_0 \in i\mathfrak{t}_0^* \subset i\mathfrak{h}_0^* \subset i\mathfrak{g}_0^*$ so that the set of $\mathfrak{h}$-roots of $\mathfrak{q}$ is $\Delta(\mathfrak{q}, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \langle \lambda_0, \alpha \rangle \geq 0\}$. Then $\mathfrak{q}$ is given by $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ with

$$\Delta(\mathfrak{l}, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \langle \lambda_0, \alpha \rangle = 0\}$$
$$\Delta(\mathfrak{u}, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \langle \lambda_0, \alpha \rangle > 0\}.$$

(d) D is an open complex submanifold of Z with anti-holomorphic tangent space at z_0 identified with $\mathfrak{u}$.

It follows that the measurable open G-orbits are elliptic coadjoint orbits. The converse also holds. We assume that $z_0, \mathfrak{q}, L$, etc. are chosen as in (a)-(d). A positive system of roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$ containing $\Delta(\mathfrak{u}, \mathfrak{h})$ is chosen. Half the sum of the positive roots is denoted by ρ.

The parameters for our representations in cohomology are given by $\lambda \in \mathfrak{t}_0^*$ and $\nu \in i{\mathfrak{a}'_0}^*$ so that $\lambda + \nu \in \mathfrak{h}^*$ is the weight of a one–dimensional representation of L, which we denote by $\mathbf{C}_{\lambda,\nu}$. Let $\mathbf{C}_\chi$ be the one–dimensional representation $\mathbf{C}_{\lambda,\nu} \otimes \wedge^{\mathrm{top}}\mathfrak{u}$ of L. The weight of this representation is denoted by χ. Hence, $\chi = \lambda + \nu + 2\rho(\mathfrak{u})$ where $2\rho(\mathfrak{u}) = \sum_{\alpha \in \Delta(\mathfrak{u},\mathfrak{h})} \alpha$. The corresponding homogeneous line bundle is denoted by $\mathcal{L}_\chi \to G/L$. The sheaf cohomology spaces $H^s(G/L, \mathcal{O}(\mathcal{L}_\chi))$ can be calculated by the standard C^∞–Dolbeault complex:

$$A^\bullet(G/L,\mathcal{L}_\chi)=\{C^\infty(G)\otimes \mathbf{C}_\chi\otimes \wedge^\bullet\mathfrak{u}^*\}^L. \tag{2.1}$$

Thus, $A^p(G/L,\mathcal{L}_\chi)=\{f: G\to \mathbf{C}_\chi\otimes\wedge^p\mathfrak{u}^* |\, f$ is smooth and $f(gl)=l^{-1}\cdot f(g),$ for all $l\in L\}$, the smooth $\mathcal{L}_\chi$–valued $(0,p)$–forms.

The following theorem is fundamental to the study of the representations in cohomology. Proofs of this theorem, in various degrees of generality, can be found in [15], [14], [19], [24] and [25]. We state the version which is proved in [25].

Theorem 2.2. (a) *For each p, $H^p(G/L,\mathcal{O}(\mathcal{L}_\chi))$ is a continuous admissible representation. It is the maximal globalization of a $(\mathfrak{g},K)$–module cohomologically induced in the sense of* [27].

(b) *The infinitesimal character of $H^p(G/L,\mathcal{O}(\mathcal{L}_\chi))$ is $\chi+\rho$.*

(c) *If $Re(\langle\chi+\rho,\beta\rangle)>0$ for all $\beta\in\Delta(\mathfrak{u},\mathfrak{h})$, then $H^p(G/L,\mathcal{O}(\mathcal{L}_\chi))=0$ unless $p=s=\dim_{\mathbf{C}}K/K\cap L$, and $H^s(G/L,\mathcal{O}(\mathcal{L}_\chi))$ is irreducible.*

It is the irreducible representation $H^s(G/L,\mathcal{O}(\mathcal{L}_\chi))$ which is of interest to us.

The intertwining map $\mathcal{S}$ of [2] and [1] maps a principal series representation $I(W)$ into $A^s(G/L,\mathcal{L}_\chi)$. The parameters for the principal series are given carefully in [1], Section 5 and we will only briefly review the construction here. In [2], Section 2 and [1], Section 2, a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{l}_0\cap\mathfrak{p}_0$ is given in terms Cayley transforms with respect to a set of strongly orthogonal roots in $\Delta(\mathfrak{g},\mathfrak{h})$. This set of strongly orthogonal roots determines a positive system of restricted roots $\sum^+(\mathfrak{g},\mathfrak{a})$ and thus a parabolic subgroup $P=MAN$ of G. The parabolic subgroup P has the following properties:

(a) $A\subset L, MA=$ centralizer of A in $G, M\cap L$ is compact and $\sum(\mathfrak{n},\mathfrak{a})=\sum^+(\mathfrak{g},\mathfrak{a})$.

(b) P is a cuspidal parabolic and

$$P\cap L=(M\cap L)A(N\cap L) \text{ is a minimal parabolic subgroup of } L. \tag{2.3}$$

(c) $L=K\cap L\cdot N\cap L\cdot A$ is an Iwasawa decomposition of L.

A discrete series representation δ_M of M is specified in [1], Proposition 4.1. Letting ρ_G (ρ_L respectively) be half the sum of the roots in $\sum^+(\mathfrak{g},\mathfrak{a})$ $\left(\sum^+(\mathfrak{l},\mathfrak{a}) \text{ respectively}\right)$, we denote by W the space of the representation

$$\delta_M\otimes e^{\rho_L+\nu}\otimes 1 \tag{2.4}$$

of $P=MAN$. Then we denote by $I(W)$ the smooth induced representation $\{C^\infty(G)\otimes W\otimes\mathbf{C}_{\rho_G}\}^P$.

The main results of [2] and [1] are contained in the following theorem.

Theorem 2.5. *If $\langle\chi-2\rho(\mathfrak{u}\cap\mathfrak{k}),\alpha\rangle>0$, for $\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})$, then there is a G–intertwining operator $\mathcal{S}$: $I(W)\to A^s(G/L,\mathcal{L}_\chi)$ so that:*

(a) *The image of $\mathcal{S}$ consists of closed forms.*

(b) *The map $\mathcal{S}$, followed by the natural map to cohomology, is nonzero.*

(c) *If $\langle\mathrm{Re}(\chi+\rho),\beta\rangle>0$, for $\beta\in\Delta(\mathfrak{u},\mathfrak{h})$, then each K–finite cohomology class is represented by a closed form in the image of $\mathcal{S}$.*

We will need the following explicit formula for $\mathcal{S}$:

$$\mathcal{S}f(x)=\int_{K\cap L}\pi(l)T\left(f(xl)\right)\,dl \tag{2.6}$$

This requires some explanation. The map $T: W\to\mathbf{C}_\chi\otimes\wedge^s\mathfrak{u}^*$ is an $M\cap L$-homomorphism which decomposes as $T=\Phi\cdot ev\cdot t$ as follows.

The first map t is simply an identification of W, as M–representation, with a space $\mathcal{H}_2^{s_1}(M/M\cap L,\hat{\mathbf{C}}_{(\mu,\chi)})$ of square integrable harmonic forms on $M/M\cap L$. This L_2–harmonic space consists of $(0,s_1)$-forms on $M/M\cap L$ with $s_1=\dim(\mathfrak{m}\cap\mathfrak{u}\cap\mathfrak{k})$. The bundle parameter is $\hat{\mathbf{C}}_{(\mu,\chi)}$ and is related to $\mathbf{C}_{\chi|M\cap L}$ as in

Proposition 4.1 of [1]. The construction of this L_2–harmonic space is essentially given in [17]. See [1], Appendix 2 for the adaptation to the situation here.

Remark 2.7. Type $(0, s_1)$–forms on $M/M \cap L$ are C^∞-functions $M \to \hat{\mathbf{C}}_{(\mu,\chi)} \otimes \wedge^{s_1}(\mathfrak{m} \cap \mathfrak{u})^*$ satisfying a right translation property with respect to $M \cap L$. These forms are square integrable with respect to a positive metric on $M/M \cap L$ obtained from the Killing form B on $\mathfrak{g}$ as follows. Define $\langle \xi, \eta \rangle_{\text{pos}} = -B(\xi, \theta(\eta))$ for $\xi, \eta \in \mathfrak{m}$, with θ the Cartan involution. This defines an invariant metric on $M/M \cap L$ since $M \cap L$ is compact. This metric coincides with the restriction of the metric $\langle\, , \,\rangle_{\text{pos}}$ on G/L which is defined in (4.3).

The next map in the decomposition of T is evaluation at the identity e. This gives some element of $\hat{\mathbf{C}}_{(\mu,\chi)} \otimes \wedge^{s_1}(\mathfrak{m} \cap \mathfrak{u})^*$. Then Φ is the natural map $\hat{\mathbf{C}}_{(\mu,\chi)} \otimes \wedge^{s_1}(\mathfrak{m} \cap \mathfrak{u})^* \to \mathbf{C}_\chi \otimes \wedge^s \mathfrak{u}$ given as follows. The Cayley transform $c : \mathfrak{h} \to \mathfrak{b} \oplus \mathfrak{a}$ is the product of the Cayley transforms with respect to a set $\{\alpha_j\}$ of strongly orthogonal roots in $\Delta(\mathfrak{g}, \mathfrak{h})$. It is shown in [1], Proposition 6.1 that $\{\gamma \in \Delta(\mathfrak{u}, \mathfrak{h}) : c(\gamma)|_{\mathfrak{a}} < 0\}$ has precisely $s - s_1$ elements. Enumerating this set as $\{\gamma_1, \cdots, \gamma_{s-s_1}\}$ and letting ω^γ be the element of $\mathfrak{u}^*$ dual to a root vector for $c(\gamma)$ define

$$\omega_{s-s_1} = \omega^{\gamma_1} \wedge \cdots \wedge \omega^{\gamma_{s-s_1}}. \tag{2.8}$$

Then there is an isomorphism (as $L \cap M$ representations)

$$\hat{\mathbf{C}}_{(\mu,\chi)} \simeq \mathbf{C}_\chi \otimes \omega_{s-s_1}.$$

The map Φ is the corresponding map

$$\begin{aligned} \hat{\mathbf{C}}_{(\mu,\chi)} \otimes \wedge^{s_1}(\mathfrak{u} \cap \mathfrak{m})^* &\to \mathbf{C}_\chi \otimes \wedge^s \mathfrak{u}^*, \\ 1 \otimes v &\mapsto 1 \otimes (v \wedge \omega_{s-s_1}). \end{aligned} \tag{2.9}$$

A very important property of T is that there is an L-subrepresentation (π, V_π) of $\mathbf{C}_\chi \otimes \wedge^s \mathfrak{u}^*$ so that

$$\text{the image of } T \text{ lies in } V_\pi^{\mathfrak{n} \cap \mathfrak{l}}, \tag{2.10}$$

the highest restricted weight space of V_π. Furthermore, $\mathfrak{a}$ acts on $V_\pi^{n \cap \mathfrak{l}}$ by the weight $\rho_G - \rho_L + \nu$.

The following lemma will be crucial.

Lemma 2.11. *Assume ρ_L is dominant regular for $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Then there is a Weyl group element $w_o \in W(L, A)$ with the property that $w_o \Sigma^+(\mathfrak{g}, \mathfrak{a}) = -\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Letting $p \in L \cap K$ be any representative of w_o, $Ad(p)\omega_{s-s_1} = C\theta(\omega_{s-s_1})$, for some constant C.*

Proof. Since $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0 \cap \mathfrak{l}_0$ (by 2.3), there is a Weyl group element w_o sending $\Sigma^+(\mathfrak{l}, \mathfrak{a})$ to $-\Sigma^+(\mathfrak{l}, \mathfrak{a})$. Since ρ_L is regular dominant and $w_o \rho_L = -\rho_L$, w_o has the required property. Thus $\text{Ad}(p)$ and θ both preserve $\Delta(\mathfrak{u}, \mathfrak{h})$ and send $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ to $-\Sigma^+(\mathfrak{g}, \mathfrak{a})$, therefore $\text{Ad}(p)\{c(\gamma_1), \ldots, c(\gamma_{s-s_1})\} = \theta(\{c(\gamma_1), \ldots, c(\gamma_{s-s_1})\})$. The lemma now follows. $\square$

We end this section by setting some more notation and recalling several standard integration formulas. There is a decomposition of G given by

$$G = K \exp(\mathfrak{m}_0 \cap \mathfrak{p}_0) N A \tag{2.12}$$

$$g = \kappa(g)\mu(g)n(g)\mathrm{e}^{H(g)}. \tag{2.13}$$

The expression for g is unique. See [5], Lemma 11 for this decomposition and [5], Lemma 14 the following integration formula. For $\varphi \in C_0(G/A)$ the invariant measure on G/A satisfies

$$\int_{G/A} \varphi(g)\, dg = \int_K \int_{M_0} \int_N \varphi(kmn)\, du\, dm\, dk \tag{2.14}$$

for some normalization of Haar measures.

By (2.3), (2.13) is the Iwasawa decomposition. of $g \in L$. Thus, by [8], Lemma 44 we have the following integration formula.

$$\int_{K\cap L}\psi(l)\ dl=\int_{\overline{N}\cap L}\mathrm{e}^{-2\rho_L(H(\overline{n}_L))}\psi\left(\kappa(\overline{n}_L)\right)\ d\overline{n}_L, \tag{2.15}$$

for ψ any continuous function on $K\cap L$ which is right invariant under $M\cap L$.

3. The Invariant form

The Killing form defines a G–invariant possibly indefinite hermitian metric on G/L. This in turn defines a global G–invariant hermitian form on a space of $(0,s)$–forms. In this section we will determine a space of $(0,s)$–forms for which the integral defining this global hermitian form converges.

Let B be the Killing form of $\mathfrak{g}$ and set $\langle X,Y\rangle_{\text{inv}}\equiv -B\left(X,\tau(Y)\right)$, where τ is conjugation of $\mathfrak{g}$ over $\mathfrak{g}_0$. As $\langle\,,\,\rangle_{\text{inv}}$ is invariant, a G–invariant hermitian metric on G/L is defined in the usual way. We are interested in the cases for which G/L noncompact, therefore the metric is indefinite. Let $\#\eta$ be the Hodge-Kodaira orthocomplementation followed by contraction in $\mathcal{L}_\chi$. Then the global hermitian form, also denoted by $\langle\,,\,\rangle_{\text{inv}}$, is defined by

$$\langle\omega,\eta\rangle_{\text{inv}}=\int_{G/L}\omega\wedge\#\eta,\qquad \omega,\eta\in A^{(0,s)}(G/L,\mathcal{L}_\chi), \tag{3.1}$$

provided this integral is finite. Since G/L has an invariant measure, (3.1) may be expressed as

$$\langle\omega,\eta\rangle_{\text{inv}}=\int_{G/L}\langle\omega(g),\eta(g)\rangle_{\text{inv}}\ dg \tag{3.2}$$

where $\langle\,,\,\rangle_{\text{inv}}$ has been extended to a hermitian form on $\mathbf{C}_\chi\otimes\wedge^s\mathfrak{u}^*$.

Theorem 3.4 below will give a formula for $\langle\mathcal{S}f_1,\mathcal{S}f_2\rangle_{\text{inv}}$ in terms of standard intertwining operators for principal series representations. In particular, the integral defining the global invariant form is finite in K–finite vectors.

We use the notation of (2.3) and (2.4) for parabolics and principal series parameters. The standard intertwining operator for the Weyl group element w_o in (2.11), is denoted by

$$D(w_o)A_P(\delta_M\otimes\nu):\mathrm{Ind}_P^G(\delta_M,\nu)\to\mathrm{Ind}_P^G(\delta_M\otimes w_o\cdot\nu).$$

For details on these intertwining operators see [11], [10] and [22]. The key properties we need are:

(a) $A_P(\delta_M\otimes\nu)f(x)=\displaystyle\int_{\overline{N}}f(xp\overline{n})\ d\overline{n}$ with integral converging for f K-finite and $\mathrm{Re}(\nu)$ dominant regular,

(b) In the case that $\mathrm{Re}(\nu)$ is regular dominant, the unique maximal subrepresentation of $\mathrm{Ind}_P^G(\delta_M,\nu)_{K\text{–finite}}$ is the kernel of $A_P(\delta_M\otimes\nu)$.

We will now specify $D(\omega_o)$ explicitly. Recall from (2.11) that w_o is represented by $p\in K\cap L$. The operator $D(w_o)$ is an intertwining operator from δ_M^p to δ_M, where $\delta_M(m)=\delta_M(pmp^{-1})$. Recall that the representation δ_M is in the discrete series and acts on the harmonic space $\mathcal{H}_2^{s_1}(M/M\cap L,\hat{\mathbf{C}}_{(\mu,\chi)})$. Define

$$(D(w_0)(\omega))(m)=p^{-1}\cdot\omega(pmp^{-1}),\ \text{for any }(0,s_1)\text{–form }\omega\text{ on }M/M\cap L. \tag{3.3}$$

Note that since $p\in K\cap L$ and p normalizes A, p preserves $\mathfrak{m}\cap\mathfrak{u}$ and $\wedge^{s_1}(\mathfrak{m}\cap\mathfrak{u})\subset\wedge^s\mathfrak{u}$. The action of p on $M/M\cap L$ defined by $p\cdot mM\cap L=pmp^{-1}M\cap L$ is a well defined holomorphic diffeomorphism preserving the metric. Thus the Laplacian is preserved by the action of p and the corresponding harmonic space is preserved by $D(w_o)$. Now it is easy to check that $D(w_o)$ intertwines δ_M^p and δ_M.

Theorem 3.4. *Suppose ρ_L is regular. Let f_1 and f_2 be K–finite vectors in $I(W)$. Then the integral (3.2) defining $\langle\mathcal{S}f_1,\mathcal{S}f_2\rangle_{\text{inv}}$ converges and*

$$\langle\mathcal{S}f_1,\mathcal{S}f_2\rangle_{\text{inv}}=C\int_K\langle f_1(k),D(w_o)A_P(\delta_M\otimes\rho_L+\nu)f_2(k)\rangle_M\ dk, \tag{3.5}$$

for some constant C.

Proof. Let f_1 and f_2 be K–finite vectors in $I(W) = \operatorname{Ind}_P^G(\delta_M, \rho_L{+}\nu)$. The definition of $\mathcal{S}f_1$ gives

$$\begin{aligned}\langle \mathcal{S}f_1, \mathcal{S}f_2\rangle_{\text{inv}} &= \int_{G/L}\int_{L\cap K} \langle \pi(l)T\left(f(gl)\right), \mathcal{S}f(g)\rangle_{\text{inv}}\, dl\; dg\\ &= \int_{G/L}\int_{L\cap K} \langle T\left(f(gl)\right), \mathcal{S}f(gl)\rangle_{\text{inv}}\, dl\; dg.\end{aligned}$$

Make the change of variables $l \mapsto lp$, use the definition of $\mathcal{S}f_2$ and apply the integration formula (2.15) to obtain:

(3.6)

$$\begin{aligned}\langle \mathcal{S}f_1, \mathcal{S}f_2\rangle_{\text{inv}} &= \int_{G/L}\int_{L\cap K} \langle Tf_1(glp), \mathcal{S}f_2(glp)\rangle_{\text{inv}}\, dl\; dg\\ &= \int_{G/L}\int_{L\cap K} \langle \pi(p)Tf_1(glp), \mathcal{S}f_2(gl)\rangle_{\text{inv}}\, dl\; dg\\ &= \int_{G/L}\int_{L\cap K}\int_{L\cap K} \langle \pi(p)Tf_1(glp), \pi(l')\left(Tf_2(gll')\right)\rangle_{\text{inv}}\, dl' dl\; dg\\ &= \int_{G/L}\int_{L\cap K}\int_{L\cap \overline{N}} e^{-2\rho(H(\overline{n}_L))}\langle \pi(p)T\left(f_1(glp)\right), \pi\left(\kappa(\overline{n}_L)\right)T\left(f_2\left(gl\kappa(\overline{n}_L)\right)\right)\rangle_{\text{inv}}\, d\overline{n}\; dl\; dg.\end{aligned}$$

Using the invariance property of f_2 and (2.10) we get

$$\pi(\overline{n_L})T\left(f_2(gl\overline{n}_L)\right) = e^{-2\rho_L(H(\overline{n}_L))}\pi\left(\kappa(\overline{n}_L)\right)T\left(f_2(gl\kappa(\overline{n}_L))\right). \tag{3.7}$$

The first equality below now follows from (3.6).

$$\begin{aligned}\langle \mathcal{S}f_1, \mathcal{S}f_2\rangle_{\text{inv}} &= \int_{G/L}\int_{L\cap K}\int_{L\cap\overline{N}} \langle \pi(p)T\left(f_1(glp)\right), \pi(\overline{n}_L)T\left(f_2(gl\overline{n}_L)\right)\rangle_{\text{inv}}\, d\overline{n}_L\; dl\; dg\\ &= \int_{G/L}\int_{L\cap K}\int_{L\cap\overline{N}} \langle \pi(p)\pi(p^{-1}\overline{n}_L^{-1}p)T\left(f_1(glp)\right), T\left(f_2(gl\overline{n}_L)\right)\rangle_{\text{inv}}\, d\overline{n}_L\; dl\; dg\\ &\qquad\qquad \text{by right } N\text{–invariance of } f_1\\ &= \int_{G/L}\int_{L\cap K}\int_{L\cap\overline{N}} \langle \pi(p)T\left(f_1(gl\overline{n}_Lp)\right), T\left(f_2(gl\overline{n}_L)\right)\rangle_{\text{inv}}\, d\overline{n}_L\; dl\; dg\\ &\qquad\qquad \text{by (2.10)}\\ &= \int_{G/L}\int_{L/A} \langle \pi(p)T\left(f_1(glp)\right), T\left(f_2(gl)\right)\rangle_{\text{inv}}\, dl\; dg\\ &\qquad\qquad \text{by right } N\text{–invariance of } f_1 \text{ and (2.14)}\\ &= \int_{G/A} \langle \pi(p)T\left(f_1(gp)\right), T\left(f_2(g)\right)\rangle_{\text{inv}}\, dg\\ &= \int_K\int_{M^0}\int_{\overline{N}} \langle \pi(p)T\left(f_1(km\overline{n}p)\right), T\left(f_2(km\overline{n})\right)\rangle_{\text{inv}}\, d\overline{n}_L\; dm\; dk\\ &\qquad\qquad \text{by integration formula (2.14)}\\ &= \int_K\int_{M^0}\int_{\overline{N}} \langle \pi(p)T\left(f_1(kpp^{-1}mp)\right), T\left(f_2(km\overline{n})\right)\rangle_{\text{inv}}\, d\overline{n}\; dm\; dk\\ &\qquad\qquad \text{since } p^{-1}\overline{n}p \in N,\end{aligned}$$

$$= \int_K\int_{M^0}\int_{\overline{N}} \langle \pi(p)T\left(f_1(kp^{-1}mp)\right), T\left(f_2(kp^{-1}\overline{n}m)\right)\rangle_{\text{inv}}\, d\overline{n}\; dm\; dk \tag{3.8}$$

by changes of variables $k \mapsto kp^{-1}$ and $\overline{n} \mapsto m^{-1}\overline{n}m$.

We prove the following lemma before concluding the proof of (3.4). Recall that $T = \Phi \circ ev \circ t$ as in the paragraph containing (2.6).

Lemma 3.9. *There is a nonzero constant C' such that* $\langle \pi(p)T\left(f_1(kp^{-1}mp)\right), T\left(f_2(kp^{-1}\overline{n}m)\right)\rangle_{\mathrm{inv}}$ $= C'\langle p\cdot t\left(f_1(k)\right)(p^{-1}mp), t\left(f_2(kp^{-1}\overline{n})\right)(m)\rangle_{\mathrm{inv},M}$.

Proof. By (2.10), $T\left(f_1(kp^{-1}mp)\right) = u_1\otimes\omega_{s-s_1}$ and $T\left(f_2(kp^{-1}\overline{n}m)\right) = u_2\otimes\omega_{s-s_1}$ where $u_1 = t(f_1(k)(p^{-1}mp)$ and $u_2 = t(f_2(k)(p^{-1}mp)\in \mathbf{C}_\chi\otimes\wedge^{s_1}(\mathfrak{m}\cap\mathfrak{u})^*$ and ω_{s-s_1} as in (2.8). By Lemma 2.11, $p\cdot\omega_{s-s_1} = \theta(\omega_{s-s_1})$. Thus

$$\begin{aligned}\langle \pi(p)u_1\otimes\omega_{s-s_1}, u_2\otimes\omega_{s-s_1}\rangle_{\mathrm{inv}} &= \langle p\cdot u_1\otimes\theta(\omega_{s-s_1}), u_2\otimes\omega_{s-s_1}\rangle_{\mathrm{inv}}\\ &= \langle p\cdot u_1, u_2\rangle_{\mathrm{inv},M}\langle\theta(\omega_{s-s_1}),\omega_{s-s_1}\rangle_{\mathrm{inv}}\\ &= \langle p\cdot t\left(f_1(k)\right)(p^{-1}mp), t\left(f_2(kp^{-1}\overline{n})\right)(m)\rangle_{\mathrm{inv},M}\langle\theta(\omega_{s-s_1}),\omega_{s-s_1}\rangle_{\mathrm{inv}}.\end{aligned}$$

□

Now we continue with the proof of (3.4) by applying Lemma 3.9 to the last expression in (3.8). We let $\langle,\rangle_{\mathrm{pos},M}$ ($\langle,\rangle_{\mathrm{inv},M}$, respectively) denote the positive (invariant, respectively) form restricted to $M/M\cap L$.

$$\begin{aligned}\langle\mathcal{S}f_1,\mathcal{S}f_2\rangle_{\mathrm{inv}} &= C'\int_K\int_{M^0}\int_{\overline{N}}\langle p\cdot t\left(f_1(k)\right)(p^{-1}mp), t\left(f_2(kp^{-1}\overline{n})\right)(m)\rangle_{\mathrm{inv},M}\,d\overline{n}\,dm\,dk\\ &= C'\int_K\int_{M^0}\int_{\overline{N}}\langle p\cdot t\left(f_1(k)\right)(m), t\left(f_2(kp^{-1}\overline{n})\right)(pmp^{-1})\rangle_{\mathrm{inv},M}\,d\overline{n}\,dm\,dk\\ &\qquad\qquad\text{by change of variables } m\to pmp^{-1},\\ &= C'\int_K\int_{M^0}\langle t\left(f_1(k)\right)(m), \int_{\overline{N}}p\cdot^{-1} t\left(f_2(kp^{-1}\overline{n})\right)(pmp^{-1})d\overline{n}\rangle_{\mathrm{inv},M}\,dm\,dk\\ &= C'\int_K\int_{M^0}\langle t\left(f_1(k)\right)(m), D(w_o)\int_{\overline{N}}t\left(f_2(kp^{-1}\overline{n})\right)(m)\rangle_{\mathrm{inv},M}\,dm\,dk\\ &\qquad\qquad\text{by change of variables } m\to pmp^{-1},\end{aligned}$$

$$= C\int_K\langle f_1(k), D(w_o)A_P(\delta_M\otimes\rho_L+\nu)f_2(k)\rangle_{\mathrm{pos},M}\,dk. \tag{3.10}$$

By Lemma 5.2 below the integral defining the global invariant form on the L_2–harmonic space (defining the discrete series) converges. Therefore the global invariant form is a multiple of the global positive form. This justifies (3.10). □

Corollary 3.11. *If* $\mathrm{rank}_{\mathbf{R}}(G) = \mathrm{rank}_{\mathbf{R}}(L)$ *then* $\langle,\rangle_{\mathrm{inv}}$ *is not identically zero on the image of* $\mathcal{S}$.

Proof. The metric defining $\langle,\rangle_{\mathrm{inv},M}$ is the metric on $\wedge^{s_1}(\mathfrak{u}\cap\mathfrak{m})^*$ induced by the Killing form of $\mathfrak{g}$. This coincides with $\langle,\rangle_{\mathrm{pos},M}$ since $\mathfrak{u}\cap\mathfrak{m}\subset\mathfrak{u}\cap\mathfrak{k}$. It follows that the constant C in (3.10) is nonzero. The corollary now follows from [12], Section 3. □

4. The positive definite form

We have seen that G/L has a G-invariant indefinite hermitian metric which defines a global G-invariant hermitian form on the K–finite image of $\mathcal{S}$. In this section we define a Hilbert space of closed forms; the inner product being in terms of a (non G–invariant) positive definite metric on G/L. The important properties of this L_2–harmonic space, which is denoted by $H_2^{(0,s)}$, are that (a) it is G–invariant, (b) it contains the K–finite vectors in the image of $\mathcal{S}$ and (c) the G–invariant form is defined on $H_2^{(0,s)}$.

The 1 metric on G/L we use was introduced in [14]. To define this metric we use the following decomposition of G:

$$G = K\exp(\mathfrak{l}^\perp\cap\mathfrak{p}_0)\exp(\mathfrak{l}\cap\mathfrak{p}_0). \tag{4.1}$$

Here $\mathfrak{l}^\perp$ is the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}$ with respect to the Killing form. Under this decomposition we can write an element of the group uniquely as

$$g = k(g)\exp\left(X(g)\right)\exp\left(Y(g)\right), \text{ with } k(g)\in K, X(g)\in \mathfrak{l}^\perp\cap\mathfrak{p}_0 \text{ and } Y(g)\in\mathfrak{l}\cap\mathfrak{p}_0. \tag{4.2}$$

See [13] for this decomposition. Define $\langle X,Y\rangle_{\rm pos} = \langle X,\theta(Y)\rangle_{\rm inv}$. It is clear that this is a positive definite K–invariant Hermitian form on $\mathfrak{g}$. Define $\langle X,Y\rangle_{{\rm pos},eL} = \langle X,Y\rangle_{\rm pos}$ for $X,Y\in\mathfrak{l}^\perp\cong T_{eL}(G/L)$. Now use the decomposition (4.2) to translate to the other tangent spaces as follows. Let ℓ_x denote the differential of left translation by $x\in G$ on G/L. Write arbitrary tangent vectors at $gL\in G/L$ as $\xi_g = \ell_{k(g)\exp(X(g))}(\xi)$ and $\eta_g = \ell_{k(g)\exp(X(g))}(\eta)$ with $\xi,\eta\in\mathfrak{l}^\perp\cong T_{eL}(G/L)$. Define

$$\langle \xi_g,\eta_g\rangle_{{\rm pos},gL} = \langle\xi,\eta\rangle_{\rm pos}. \tag{4.3}$$

Lemma 4.4. *The expression in (4.3) is independent of the coset representative g of gL and defines a K–invariant metric on G/L.*

Proof. An easy calculation shows that for any $l\in L, k(gl)\exp\left(X(gl)\right) = k(g)\exp\left(X(g)\right)l_1$ for some $l_1\in L\cap K$. So $\ell_{k(gl)\exp(X(gl))}(\xi) = \ell_{k(g)\exp(X(g))}\left(\mathrm{Ad}(l_1)\xi\right)$ and the first part follows from the $L\cap K$–invariance of $\langle\,,\,\rangle_{\rm pos}$ on $\mathfrak{l}^\perp$. For the K–invariance, note that if $g\in K$ then $X(g)=0$. □

As in (3.1) a global inner product is defined on a subspace of $A^s(G/L,\mathcal{L}_\chi)$ by

$$\langle\omega_1,\omega_2\rangle_{\rm pos} = \int_{G/L}\omega_1\wedge\#_p\omega_2. \tag{4.5}$$

Here $\#_p$ denotes the Hodge-Kodaira orthocomplementation with respect to the positive metric on G/L, followed by the hermitian pairing on $\mathcal{L}_\chi$. Letting $\langle\,,\,\rangle_{\rm pos}$ also denote the corresponding inner product on $\wedge^s\mathfrak{u}\otimes\mathbf{C}_\chi$ we have

$$\langle\omega_1,\omega_2\rangle_{\rm pos} = \int_{G/L}\langle\omega_1\left(k(g)\exp(X(g))\right),\omega_2\left(k(g)\exp(X(g))\right)\rangle_{\rm pos}\,dg. \tag{4.6}$$

Definition 4.7. $L_2^{(0,s)}(G/L,\mathcal{L}_\chi)$ is the space of measurable $\mathcal{L}_\chi$–valued $(0,s)$-forms satisfying $||\omega||^2_{\rm pos} = \langle\omega,\omega\rangle_{\rm pos}<\infty$.

Definition 4.8. Define $H_2^{(0,s)}$ to be $\left\{\omega\in L_2^{(0,s)}(G/L,\mathcal{L}_\chi) : \overline{\partial}\omega = 0 \text{ as distributions}\right\}$.

In Section 5 we will see that both $L_2^{(0,s)}(G/L,\mathcal{L}_\chi)$ and $H_2^{(0,s)}$ are invariant under left translation by G. The remainder of this section is devoted to the proof of Theorem 4.9 which establishes the fact that $H_2^{(0,s)}$ is not zero.

Theorem 4.9. *Suppose L is the fixed point group of an involution σ. For χ satisfying the negativity condition of (2.5) and for ρ_L nonsingular, $\mathcal{S}(I(W)_{K\text{–finite}})\subset H_2^{(0,s)}(G/L,\mathcal{L}_\chi)$.*

Note that the assumption on the base point z_0 of Section 2 along with the condition that L is the fixed point group of an involution guarantees that the involution commutes with the Cartan involution θ. Thus, by properly choosing the base point we may assume that the involution commutes with θ. We begin with several lemmas.

Lemma 4.10. *Let $f\in I(W)$, then*

$$||\mathcal{S}f(x)||^2_{\rm pos} = \int_{K\cap L}\int_{\overline{N}\cap L}\langle T\left(f(xk)\right),T\left(f(xk\overline{n}_L)\right)\rangle_{\rm pos}\,d\overline{n}_L\,dk,$$

for any $x\in K\exp(\mathfrak{l}^\perp\cap\mathfrak{p}_0)$.

Proof. By the definition of $\mathcal{S}f$,

$$||\mathcal{S}f(x)||^2_{\rm pos} = \int_{K\cap L} \langle \pi(k)T\left(f(xk)\right), \mathcal{S}f(x)\rangle_{\rm pos}\ dk.$$

Since the positive form is invariant under K and $\mathcal{S}f$ satisfies a transformation property under L, we obtain the first equality below:

$$\begin{aligned} ||\mathcal{S}f(x)||^2_{\rm pos} &= \int_{K\cap L} \langle T\left(f(xk)\right), \mathcal{S}f(xk)\rangle_{\rm pos}\ dk \\ &= \int_{K\cap L}\int_{K\cap L} \langle T\left(f(xk)\right), \pi(k')T\left(f(xkk')\right)\rangle_{\rm pos}\ dk'\ dk \\ &\qquad\qquad\qquad \text{by (2.6)} \\ &= \int_{K\cap L}\int_{\overline{N}\cap L} e^{-2\rho_L(H(\overline{n}_L))}\langle T\left(f(xk)\right), \pi\left(\kappa(\overline{n}_L)\right)T\left(f(xk\kappa(\overline{n}_L)\right)\rangle_{\rm pos}\ d\overline{n}_L\ dk \\ &\qquad\qquad\qquad \text{by the change of variables (2.15).} \end{aligned}$$

By (3.7) the integrand is equal to

$$\begin{aligned} &\langle T\left(f(xk)\right), \pi(\overline{n}_L)T\left(f(xk\overline{n}_L)\right)\rangle_{\rm pos} \\ &= \langle \pi\left(\theta(\overline{n}_L)^{-1}\right)T\left(f(xk)\right), T\left(f(xk\overline{n}_L)\right)\rangle_{\rm pos} \\ &= \langle T\left(f(xk)\right), T\left(f(xk\overline{n}_L)\right)\rangle_{\rm pos}. \end{aligned}$$

The last equality holds since $\theta(\overline{n}_L) \in N\cap L$ and T takes values in $V_\pi^{n_L}$. The lemma follows. ☐

Recall that $\mathfrak{a}_0 \subset \mathfrak{p}_0 \cap \mathfrak{l}_0$ is a maximal abelian subspace. The decomposition $G = K\exp(\mathfrak{m}_0\cap\mathfrak{p}_0)NA$ is therefore stable under σ. In addition to this decomposition we have $G = K\exp(\mathfrak{m}_0\cap\mathfrak{p}_0)\overline{N}A$, also stable under σ. The decomposition of $g\in G$ with respect to this decomposition will be written as

$$g = \overline{\kappa}(g)\overline{\mu}(g)\overline{n}(g)\mathrm{e}^{\overline{H}(g)}. \tag{4.11}$$

An important relationship between the two decompositions is given in the following lemma.

Lemma 4.12. *If $\sigma\theta(g) = g$ then $\overline{H}(g) = -H(g)$, $\overline{\mu}(g) = \mu(g)$ and $\overline{\kappa}(g) = \sigma\left(\overline{\kappa}(g)\right)$.*

Proof. Since M_0 is preserved by both θ and σ we may write $\mathfrak{m}_0\cap\mathfrak{p}_0 = \mathfrak{m}_0\cap\mathfrak{p}_0\cap\mathfrak{l}^\perp + \mathfrak{m}_0\cap\mathfrak{p}_0\cap\mathfrak{l}_0$. But $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{l}_0\cap\mathfrak{p}_0$, so $\mathfrak{m}_0\cap\mathfrak{p}_0\cap\mathfrak{l}_0 = 0$, i.e., $\mathfrak{m}_0\cap\mathfrak{p}_0 = \mathfrak{m}_0\cap\mathfrak{p}_0\cap\mathfrak{l}^\perp$. Therefore $\sigma\left(\theta\left(\overline{\mu}(g)\right)\right) = \overline{\mu}(g)$. So $g = \sigma\theta(g) = \sigma\theta\left(\overline{\kappa}(g)\right)\sigma\theta\left(\overline{\mu}(g)\right)\sigma\theta\left(\overline{n}(g)\right)e^{\sigma\theta\overline{H}(g)} \in K\exp(\mathfrak{m}_0\cap p_0)NA$ and the lemma follows. ☐

We now begin the proof of Theorem 4.9. Let $x = k\exp(X) \in K\exp(\mathfrak{l}^\perp\cap\mathfrak{p}_0)$. Let $\{\omega_s^i\}$ be an orthonormal basis of $V_\pi^{n_L}$ with respect to $\langle\ ,\ \rangle_{\rm pos}$. Using Lemma 4.10 and the definition of T we write

$$\begin{aligned} ||\mathcal{S}f(x)||^2_{\rm pos} = \sum_{i=1}^m \int_{K\cap L}\int_{\overline{N}\cap L} &\langle \Phi\left(t\left(f(xl)\right)(e)\right), \omega_s^i\rangle_{\rm pos} \\ &\overline{\langle \Phi\left(t\left(f(xl\overline{n}_L)\right)(e)\right), \omega_{s^i}\rangle_{\rm pos}}\ d\overline{n}_L\ dl. \end{aligned} \tag{4.13}$$

The transformation properties of $f\in I(W)$ along with the M-invariance of t give

$$\begin{aligned} t\left(f(xl)\right)(e) &= \mathrm{e}^{-(\rho_G+\rho_L+\nu)H(xl)}t(f\left(\kappa(xl)\right))\left(\mu(xl)\right) \text{ and} \\ t\left(f(xl\overline{n}_L)\right)(e) &= \mathrm{e}^{-(\rho_G+\rho_L+\nu)H(xl\overline{n}_L)}t(f\left(\kappa(xl\overline{n}_L)\right))\left(\mu(xl\overline{n}_L)\right) \end{aligned}$$

substituting into (4.13) given

$$\begin{aligned} ||\mathcal{S}f(x)||^2_{\rm pos} = \sum_{i=1}^m \int_{K\cap L}\int_{\overline{N}\cap L} &\mathrm{e}^{-(\rho_G+\rho_L+\nu)(H(xl)+H(xl\overline{n}_L))}\langle t\left(f\left(\kappa(xl)\right)(\mu(xl)\right), \Phi^*\omega_s^i\rangle\cdot \\ &\langle t\left(f\left(\kappa(xl\overline{n}_L)\right)\right)\left(\mu(xl\overline{n}_L)\right), \Phi^*\omega_s^i\rangle\ d\overline{n}_L\ dl \end{aligned} \tag{4.14}$$

We now let $f \in I(W)$ be a K-finite function. We may choose a basis $\{f_l\}$ of the finite dimensional space span $\{k \cdot f : k \in K\}$ and write $k^{-1} \cdot f = \sum_{l=1}^{d} C_l(k) f_l$ or equivalently $f(k) = \sum_{l=1}^{d} C_l(k) f_l(e)$. Note that the coefficient C_l are continuous functions on K hence are bounded. Now choose an orthonormal basis $\{\varphi_j\}$ of W consisting of $M \cap K$-finite vectors. Since f is K–finite, $f(k)$ is $M \cap K$–finite (by page 141 of [22]). Then for each l we may express

$$f_l(e) = \sum_j \langle f_l(e), \varphi_j \rangle_W \varphi_j$$

with only a finite number of terms being nonzero. For any $g \in G$ it follows that

$$\mu(g)^{-1} t\left(f\left(\kappa(g)\right)\right) = \sum_{l=1}^{M} C_l\left(\kappa(g)\right) \mu(g)^{-1} t\left(f_l(e)\right). \tag{4.15}$$

Now,

$$\begin{aligned}
&\langle t\left(f\left(\kappa(g)\right)\right)\left(\mu(g)\right), \Phi^* \omega_s^i \rangle_{\mathrm{pos}} \\
&= \sum_{l=1}^{m} C_l\left(\kappa(g)\right) \langle \mu(g)^{-1} t\left(f_l(e)\right)(e), \Phi^* \omega_s^i \rangle_{\mathrm{pos}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by (4.15)} \\
&= \sum_{l=1}^{m} \sum_j C_l\left(\kappa(g)\right) \langle \mu(g)^{-1} t(\varphi_j)(e), \Phi^* \omega_s^i \rangle_{\mathrm{pos}} \langle f_l(e), \varphi_j \rangle_W \\
&= \sum_{l=1}^{m} \sum_j C_l\left(\kappa(g)\right) \langle t(\varphi_j)(\mu(g)), \Phi^* \omega_s^i \rangle_{\mathrm{pos}} \langle f_l(e), \varphi_j \rangle_W .
\end{aligned} \tag{4.16}$$

The matrix coefficients in (4.16) are compared with the functions Ξ_0^M defined in [6]. We claim that for every $r > 0$ there exist a constant C_r so that

$$\langle t(\varphi_j)\left(\mu(g)\right), \Phi^* \omega_s^i \rangle_{\mathrm{pos}} \leq C_r \frac{\Xi_0^M\left(\mu(g)\right)}{\left(1 + ||\mu(g)||_{\mathrm{pos}}^2\right)^r}. \tag{4.17}$$

By [7], Lemma 65 it is enough to show that $F(m) = \langle t(\varphi_j)(m), \Phi^* \omega_s^i \rangle_{\mathrm{pos}}$ defines a function F on M so that $F \in L_2(M)$, F is $M \cap K$–finite, and F is $Z(m)$–finite. By Schmid's Theorem $t(\varphi_j)$ is an L_2–harmonic form on $M/M \cap L$. The square integrability of F follows from $|\langle t(\varphi_j)(m), \Phi^* \omega_s^i \rangle_{\mathrm{pos}}| \leq C ||t(\varphi_j)(m)||_{\mathrm{pos}}^2$ (and the compactness of $M \cap L$). The $K \cap M$–finiteness and $Z(\mathfrak{m})$–finiteness of F follow from the same properties of $t(\varphi_j)$.

By (4.14) and (4.17) applied to $g = xl$ and $g = xl\overline{n}_L$, and the finiteness of the sums in (4.16), we have a bound for $||\mathcal{S}f(x)||_{\mathrm{pos}}^2$: for each $r > 0$ there is a constant $C_r > 0$ so that

$$\begin{aligned}
||\mathcal{S}f(x)||_{\mathrm{pos}}^2 \leq C_r \int_{K \cap L} \int_{\overline{N} \cap L} & \mathrm{e}^{-(\rho_G + \rho_L + \nu)(H(xl) + H(xl\overline{n}_L))} \\
& \frac{\Xi_0^M\left(\mu(xl)\right)}{\left(1 + ||\mu(xl)||_{\mathrm{pos}}^2\right)^r} \frac{\Xi_0^M\left(\mu(xl\overline{n}_L)\right)}{\left(1 + ||\mu(xl\overline{n}_L)||_{\mathrm{pos}}^2\right)^r} \, d\overline{n}_L \, dl
\end{aligned} \tag{4.18}$$

for $x \in K \exp(\mathfrak{l}^{\perp} \cap \mathfrak{p}_0)$.

The remainder of the proof follows somewhat standard arguments involving the Iwasawa H–function and changes of variables. However, the assumption that G/L is symmetric is used in a crucial way (in the form of Lemma 4.12) in the equalities of (4.19) and (4.21). Comparing the Iwasawa decomposition for N and $\overline{N}$ one easily sees that

$$H(xl\overline{n}_L) = \overline{H}(xl) + H\left(\overline{\mu}(xl)\overline{n}(xl\overline{n}_L)\right).$$

Thus, the exponential term in the integrand of (4.18) becomes

$$\begin{aligned} \mathrm{e}^{-(\rho_G+\rho_L+\nu)(H(xl)+H(xl\overline{n}_L))} &= \mathrm{e}^{-(\rho_G+\rho_L+\nu)\left(H(xl)+\overline{H}(xl)\right)}\mathrm{e}^{-(\rho_G+\rho_L+\nu)H(\overline{\mu}(xl)\overline{n}(xl\overline{n}_L))} \\ &= \mathrm{e}^{-(\rho_G+\rho_L+\nu)H(\overline{\mu}(xl)\overline{n}(xl\overline{n}_L))}. \end{aligned} \tag{4.19}$$

The last equality follows from Lemma 4.12 since $x = k\exp(X)$ with $X \in \mathfrak{l}^{\perp} \cap \mathfrak{p}_0$ and $H(xl) = H\left(\exp(X)l\right)$ and $\exp(X)l = \sigma\theta\left(\exp(X)l\right)$. Thus, (4.18) becomes

$$\begin{aligned} ||\mathcal{S}f(x)||^2_{\text{pos}} \leq C_r \int_{K\cap L}\int_{\overline{N}\cap L} &\mathrm{e}^{-(\rho_G+\rho_L+\nu)H(\overline{\mu}(xl)\overline{n}(xl\overline{n}_L))} \\ &\frac{\Xi_0^M\left(\mu(xl)\right)}{\left(1+||\mu(xl)||^2_{\text{pos}}\right)^r}\,\frac{\Xi_0^M\left(\mu(xl\overline{n}_L)\right)}{\left(1+||\mu(xl\overline{n}_L)||^2_{\text{pos}}\right)^r}\,d\overline{n}_L\,dl, \end{aligned} \tag{4.20}$$

for $x \in K\exp(\mathfrak{l}^{\perp} \cap \mathfrak{p}_0)$.

We now consider the convergence of the integral (given in (4.6)) over G/L. Replacing x in (4.20) by $k(g)\exp\left(X(g)\right)$ and using

$$\begin{aligned} \overline{\mu}\left(k(g)\exp\left(X(g)\right)l\right) &= \overline{\mu}\left(\exp\left(X(g)\right)l\overline{n}_L\right), \\ \overline{n}\left(k(g)\exp\left(X(g)\right)l\overline{n}_L\right) &= \overline{n}\left(\exp\left(X(g)\right)l\overline{n}_L\right) \text{ and,} \\ \mu\left(\exp\left(X(g)\right)l\right) &= \overline{\mu}\left(\exp\left(X(g)\right)l\right), \text{ by Lemma 4.12} \\ &= \overline{\mu}\left(\exp\left(X(g)\right)l\overline{n}_L\right), \end{aligned} \tag{4.21}$$

we obtain

$$\begin{aligned} ||\mathcal{S}f||^2_{\text{pos}} &\leq C_r \int_{G/L}\int_{K\cap L}\int_{\overline{N}\cap L} \mathrm{e}^{-(\rho_G+\rho_L+\nu)H(\overline{\mu}(\exp(X(g))l\overline{n}_L))\overline{n}(\exp(X(g))l\overline{n}_L)} \\ &\quad \frac{\Xi_0^M\left(\overline{\mu}\left(\exp\left(X(g)\right)l\overline{n}_L\right)\right)}{\left(1+||\overline{\mu}\left(\exp\left(X(g)\right)l\overline{n}_L\right)||^2_{\text{pos}}\right)^r}\frac{\Xi_0^M\left(\mu\left(\exp\left(X(g)\right)l\overline{n}_L\right)\right)}{\left(1+||\mu\left(\exp\left(X(g)\right)l\overline{n}_L\right)||^2_{\text{pos}}\right)^r}\,d\overline{n}_L\,dl\,dg \\ &= C_r \int_{G/L}\int_{L/A} \mathrm{e}^{-(\rho_G+\rho_L+\nu)H(\overline{\mu}(\exp(X(g))l)\overline{n}_L(\exp(X(g))l))} \\ &\quad \frac{\Xi_0^M\left(\overline{\mu}\left(\exp\left(X(g)\right)l\right)\right)}{\left(1+||\overline{\mu}\left(\exp\left(X(g)\right)l\right)||^2_{\text{pos}}\right)^r}\frac{\Xi_0^M\left(\mu\left(\exp\left(X(g)\right)l\right)\right)}{\left(1+||\mu\left(\exp\left(X(g)\right)l\right)||^2_{\text{pos}}\right)^r}\,dl\,dg \end{aligned} \tag{4.22}$$

by integration formula (2.14).

Now use the left L-invariance of dl to replace l by $\exp\left(Y(g)\right)l$ in (4.22). Also observe that

$$\begin{aligned} \mu\left(\exp\left(X(g)\right)\exp\left(Y(g)\right)l\right) &= \mu(gl), \\ \overline{\mu}\left(\exp\left(X(g)\right)\exp\left(Y(g)\right)l\right) &= \overline{\mu}(gl), \text{ and} \\ \overline{n}\left(\exp\left(X(g)\right)\exp\left(Y(g)\right)l\right) &= \overline{n}(gl). \end{aligned}$$

Thus,

$$\begin{aligned} ||\mathcal{S}f||^2_{\text{pos}} &\leq C_r \int_{G/L}\int_{L/A} \mathrm{e}^{-(\rho_G+\rho_L+\nu)H(\overline{\mu}(gl)\overline{n}(gl))}\frac{\Xi_0^M\left(\overline{\mu}(gl)\right)}{\left(1+||\overline{\mu}(gl)||^2_{\text{pos}}\right)^r}\frac{\Xi_0^M\left(\mu(gl)\right)}{\left(1+||\mu(gl)||^2_{\text{pos}}\right)^r}\,dl\,dg \\ &= C_r \int_{G/A} \mathrm{e}^{-(\rho_G+\rho_L+\nu)H(\overline{\mu}(g)\overline{n}(g))}\frac{\Xi_0^M\left(\overline{\mu}(g)\right)}{\left(1+||\overline{\mu}(g)||^2_{\text{pos}}\right)^r}\frac{\Xi_0^M\left(\mu(g)\right)}{\left(1+||\mu(g)||^2_{\text{pos}}\right)^r}\,dg \\ &= C_r \int_K\int_{M_0}\int_{\overline{N}} \mathrm{e}^{-(\rho_G+\rho_L+\nu)H(m_0\overline{n})}\frac{\Xi_0^M(m_0)}{\left(1+||m_0||^2_{\text{pos}}\right)^r}\frac{\Xi_0^M\left(\mu(m_0\overline{n})\right)}{\left(1+||m_0\overline{n}||^2_{\text{pos}}\right)^r}\,d\overline{n}\,dm_0\,dk \\ &\leq C_r \int_{M_0}\left(\int_{\overline{N}} \mathrm{e}^{-(\rho_G+\rho_L+\nu)H(m_0\overline{n})}\cdot\Xi_0^M\left(\mu(m_0\overline{n})\right)d\overline{n}\right)\frac{\Xi_0^M(m_0)}{\left(1+||m_0||^2_{\text{pos}}\right)^r}\,dm_0. \end{aligned} \tag{4.23}$$

The following lemma provides the final step in the proof of Theorem 4.9.

Lemma 4.24. *For $\nu' \in \mathfrak{a}^*$ with ν' dominant regular* $\displaystyle\int_{\overline{N}} e^{-(\nu'+\rho_G)H(m_0\overline{n})}\Xi_0^M\left(\mu(m_0n)\right)\,d\overline{n} = C\Xi_0^M(m_0).$

Proof. We use standard methods (as on page 197 of [10], for example) to show that

$$\int_{\overline{N}} e^{-(\nu'+\rho_G)H(m_0\overline{n})}\Xi_0^M\left(\mu(m_0 n)\right)\ d\overline{n} = \Xi_0^M(m_0)\int_{\overline{N}} e^{-(\nu'+\rho_P)H_P(\overline{n})}\ d\overline{n}$$

where $G = KA_PN_P$ is an Iwasawa decomposition of G with $A \subset A_P$ and $N \subset N_P$, and H_P and ρ_P are the corresponding H–function and ρ–function. The last integral converges by [3] or [9], Corollary 32.1. □

Since $\rho_G + \rho_L + \nu$ is dominant regular the lemma, along with (4.23) gives

$$||\mathcal{S}f||^2_{\text{pos}} \leq C\int_{M_0} \frac{\|\Xi_0^M(m)\|^2}{\left(1+||m||^2_{\text{pos}}\right)^r}\ dm.$$

This integral is finite for r large enough by Theorem 9.3 in [8].

5. The Hilbert Space representation

Recall from Definition 4.8 that our Hilbert space $H_2^{(0,s)}$ is the space of square integrable forms of type $(0,s)$ which are closed in the sense of distributions. The sheaf cohomology $H^s(G/L,\mathcal{O}(\mathcal{L}_\chi))$ can be calculated by using the Dolbeault complex with either smooth or distribution forms. As a result, there is a natural map

$$q : H_2^{(0,s)} \to H^s(G/L,\mathcal{O}(\mathcal{L}_\chi)),\ q(\eta) = [\eta]. \tag{5.1}$$

We assume for the rest of this section that L is the fixed point set of an involution σ. We also assume that the negativity conditions of (2.2) and (2.5) hold. By (2.2) and (4.9) the image of q is dense and contains the K–finite vectors of $H^s(G/L,\mathcal{O}(\mathcal{L}_\chi))$.

Lemma 5.2. *For $\omega_1,\omega_2 \in L_2^{(0,s)}(G/L,\mathcal{L}_\chi)$, $\langle\omega_1,\omega_2\rangle_{\text{inv}} \leq ||\omega_1||^2_{\text{pos}}||\omega_2||^2_{\text{pos}}$. It follows that $\langle\, ,\,\rangle_{\text{inv}}$ is defined on $L_2^{(0,s)}(G/L,\mathcal{L}_\chi)$ and is continuous.*

Proof. It is enough to show that on $\mathfrak{u}$, $|\langle X_1,X_2\rangle_{\text{inv}}| \leq ||X_1||^2_{\text{pos}}||X_2||^2_{\text{pos}}$. This follows from: $|\langle X_1,X_2\rangle_{\text{inv}}| = |\langle X_1,\theta X_2\rangle_{\text{pos}}| \leq ||X_1||^2_{\text{pos}}||\theta X_2||^2_{\text{pos}} = ||X_1||^2_{\text{pos}}||X_2||^2_{\text{pos}}$. □

The following proposition, which is stated in [14], shows that $H_2^{(0,s)}$ is invariant under left translation and this action of G is continuous. We will only outline the proof.

Proposition 5.3. *Left translation by $x \in G$ in $L_2^{(0,s)}(G/L,\mathcal{L}_\chi)$ is a bounded operator and defines a continuous representation of G on $L_2^{(0,s)}(G/L,\mathcal{L}_\chi)$.*

Proof. The crucial fact is given in Lemma 7.3 of [14] which states the following. For the moment let $\|\cdot\|$ denote the operator norm of a linear operator on $\mathbf{C}_\chi \otimes \wedge^s u$ with respect to the positive metric. Then there is a continuous function B on G so that

$$\|\text{Ad}\left(\exp(Y(xk\exp(X))\right)\| \leq B(x) \tag{5.4}$$

for all $k \in K$ and $X \in \mathfrak{l}^\perp \cap \mathfrak{p}_0$. Here Y is as in (4.2).

First we check that ℓ_x ($=$ left translation by $x \in G$) is a bounded operator on $L_2^{(0,s)}(G/L,\mathcal{L}_\chi)$.

$$
\begin{aligned}
||\ell_x\omega||^2_{\rm pos} &= \int_{G/L} ||\omega\left(x^{-1}k(g)\exp(X(g))\right)||^2_{\rm pos}\, dg \\
&= \int_{G/L} ||\omega(x^{-1}k(g)\exp(X(g))\exp(Y(g))\exp(-Y(g))||^2_{\rm pos}\, dg \\
&= \int_{G/L} ||\mathrm{Ad}\,(\exp(Y(g)))\,\omega(x^{-1}g)||^2_{\rm pos}\, dg \\
&= \int_{G/L} ||\mathrm{Ad}\,(\exp(Y(xg)))\,\omega(g)||^2_{\rm pos}\, dg \\
&\qquad\qquad\qquad \text{by the } G\text{–invariance of dg} \\
&= \int_{G/L} ||\mathrm{Ad}\,(\exp(Y(xk(g)\exp(X(g))\exp(Y(g)))\,\omega(g)||^2_{\rm pos}\, dg \\
&= \int_{G/L} ||\mathrm{Ad}\,(\exp(Y(xk(g)\exp(X(g)))))\,\omega\,(k(g)\exp(X(g)))\,||^2_{\rm pos}\, dg \\
&\qquad\qquad\qquad \text{by } K\text{–invariance of the positive norm} \\
&\le B(x)\int_{G/L} ||\omega\,(k(g)\exp(X(g)))\,||^2_{\rm pos}\, dg \\
&= B(x)||\omega||^2_{\rm pos}.
\end{aligned}
$$

For continuity it suffices to show that $x \mapsto ||\ell_x\omega||^2_{\rm pos}$ is continuous and $\|\ell_x\|$ is bounded in some neighborhood of e. The second condition follows from the continuity of B and the above calculation that $\|\ell_x\| \le B(x)$. The first follows from the dominant convergence theorem as follows. In the above calculation of $\|\ell_x\omega\|$ we see that

$$||\ell_x\omega||^2_{\rm pos} = \int_{G/L} ||\mathrm{Ad}\,(\exp(Y(xk(g))))\,\omega\,(k(g)\exp(X(g)))\,||^2_{\rm pos}\, dg$$

and the integrand is bounded by

$$B(x)||\omega\,(k(g)\exp\,(X(g)))\,||^2_{\rm pos}.$$

As B is continuous and $||\omega\,(k(g)\exp\,(X(g)))\,||^2_{\rm pos}$ is L_1, the dominated convergence theorem applies. □

Corollary 5.5. $H_2^{(0,s)}$ *is a Hilbert space and* G *acts continuously .*

Let $\overline{Im(\mathcal{S})}$ denote the closure of $\mathcal{S}\,(I(W)_{K\text{–finite}})$ in the Hilbert space $H_2^{(0,s)}$.

Lemma 5.6. *If* $\eta \in \overline{Im(\mathcal{S})}$ *and* η *represents the zero cohomology class then* $\langle\omega,\eta\rangle_{\rm inv} = 0$*, for every* $\omega \in \overline{Im(\mathcal{S})}$*.*

Proof. If η is K–finite then $\eta = \mathcal{S}(f)$ for some $f \in I(W)_{K\text{–finite}}$. Since η is in the image of $\overline{\partial}$, f lies in the the maximal submodule of $I(W)_{K\text{–finite}}$, which is the kernel of A_P (by [12], Section 3). Thus, by Theorem 3.4, the lemma holds at the K-finite level. By Corollary 5.5, $\overline{Im(\mathcal{S})}$ is an admissible representation of G. It follows that the K–finite vectors in $ker(q)\cap\overline{Im(\mathcal{S})}$ are dense in the null space. Now the continuity of $\langle\,,\,\rangle_{\rm inv}$ gives the lemma. □

Definition 5.7. $\mathcal{H}_2^s \equiv \overline{Im(\mathcal{S})}/ker\{q : \overline{Im(\mathcal{S})} \to H^s(G/L,\mathcal{O}(\mathcal{L}_\chi))\}$.

Theorem 5.8. *If* G/L *is indefinite Kähler symmetric space and* $\mathrm{rank}_{\mathbf{R}}(G) = \mathrm{rank}_{\mathbf{R}}(L)$*, then* $\mathcal{H}_2^s$ *is a Hilbert space, the action of* G *is continuous and* $\langle\,,\,\rangle_{\rm inv}$ *defines a positive definite* G*–invariant inner product on* $\mathcal{H}_2^s$*. The representations* $\mathcal{H}_2^s$ *and* $H^s(G/L,\mathcal{O}(\mathcal{L}_\chi))$ *are infinitesimally equivalent.*

Proof. It follows from (3.11) that $\langle\,,\,\rangle_{\rm inv}$ is not identically zero. It is well defined on $\mathcal{H}_2^s$ by Lemma 5.4 and positive definite by [20], [21] and the choice of parameter χ. □

In the setting of this theorem $H_2^{(0,s)}$ consists of square integrable strongly harmonic forms (in the sense that $\overline{\partial}\omega = 0$ and $\overline{\partial}^*\omega = 0$ as distributions). The square integrability is shown here and the harmonic property is Theorem 9.4 in [2].

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