

EXAMPLES OF LEADING TERM CYCLES OF HARISH-CHANDRA MODULES

L. BARCHINI AND R. ZIERAU

ABSTRACT. Many examples are given of irreducible Harish-Chandra modules for Type C_n having reducible leading term cycles. Examples are also given of irreducibles in category \mathcal{O} having reducible associated varieties in type C_n .

INTRODUCTION

The characteristic cycle is an important invariant of a Harish-Chandra module X . It is defined in terms of the localization of X . The leading term cycle is the portion of the characteristic cycle that contributes to the associated variety. In principle the leading term cycle is easier to calculate than the characteristic cycle, but there is no known method to compute either in any generality. Until recently it was not known whether or not the leading term cycle of an irreducible Harish-Chandra module of $SU(p, q)$ is necessarily irreducible; an example given in [15] for category \mathcal{O} implies that there are in fact examples of reducible leading term cycles for $SU(p, q)$. It was suspected that leading term cycles of irreducible Harish-Chandra modules for $Sp(p, q)$ are always irreducible. In this article we show there are many Harish-Chandra modules of indefinite symplectic groups having reducible leading term cycles. An example of an irreducible Harish-Chandra module for $Sp(4, 4)$ having reducible leading term cycle is provided in [2]. Here we give a systematic way to find many examples.

The argument is based on a method given in [13, §4] to transfer information about leading term cycles from one real form to another. Briefly, we construct examples of reducible leading term cycles for $Sp(2n, \mathbf{R})$, then make conclusions about reducible leading term cycles for indefinite symplectic groups. A

2010 *Mathematics Subject Classification.* 22E46.

Key words and phrases. Harish-Chandra module, symplectic group, characteristic cycle.
Research supported by an NSA grant.

byproduct is that many irreducibles in category \mathcal{O} in type C are located that have reducible associated varieties. The Harish-Chandra modules we work with for $Sp(2n, \mathbf{R})$ have highest weights. This puts us in the intersection of a category of (\mathfrak{g}, K) -modules and a category of (\mathfrak{g}, B) -modules; the appendix gives a geometric lemma, special to this situation, relating Schubert varieties and K -orbits in the flag variety.

1. PRELIMINARY STATEMENTS

1.1. Highest weight Harish-Chandra modules occur for a connected simple Lie group G_0 exactly when G_0 is of hermitian type (i.e., when the center of a maximal compact subgroup K_0 is one-dimensional). Write the complexified Cartan decomposition as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} contained in \mathfrak{k} . Let K denote the complexification of K_0 . As K -representation, \mathfrak{p} decomposes into a direct sum of two irreducible subrepresentations. It is convenient to choose one of these subrepresentations and call it \mathfrak{p}_+ (and call the other \mathfrak{p}_-). Let $\Delta(\mathfrak{p}_+)$ be the set of \mathfrak{h} -weights in \mathfrak{p}_+ . We fix a positive system of roots $\Delta_c^+ \subset \Delta^+(\mathfrak{h}, \mathfrak{k})$ and set

$$\Delta^+ := \Delta_c^+ \cup \Delta(\mathfrak{p}_+), \quad (1.1)$$

a positive system of roots in $\Delta(\mathfrak{h}, \mathfrak{g})$.

Let \mathfrak{b} be the Borel subalgebra of \mathfrak{g} determined by Δ^+ ; $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, with $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{(\alpha)}$. Then each highest weight Harish-Chandra module has a highest weight with respect to either Δ^+ or $\Delta_c^+ \cup \Delta(\mathfrak{p}_-)$.

The highest weight modules with respect to Δ^+ are of the form $L(\lambda)$, the irreducible quotient of $M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbf{C}\lambda$, where $\lambda \in \mathfrak{h}^*$. Those of infinitesimal character ρ may be written as

$$L_w := L(-w\rho - \rho), w \in W(\mathfrak{h}, \mathfrak{g}).$$

The highest weight module $L(\lambda)$ is a Harish-Chandra module if and only if λ is Δ_c^+ -dominant and analytically integral. Therefore, the highest weight Harish-Chandra modules of infinitesimal character ρ are precisely the the L_w for which $-w\rho$ is Δ_c^+ -dominant. The number of these is $\#(W/W(\mathfrak{h}, \mathfrak{k}))$.

We let \mathcal{N} denote the nilpotent cone in \mathfrak{g} and set $\mathcal{N}_\theta := \mathcal{N} \cap \mathfrak{p}$. The definition of associated variety immediately implies that each irreducible highest weight Harish-Chandra module has associated variety contained in $\mathfrak{p}_+ \subset \mathcal{N}_\theta$. See, for

example, [10]. There are $r + 1$ K -orbits in \mathfrak{p}_+ , where $r = \text{rank}_{\mathbf{R}}(G_0)$. We may label them so that

$$\{0\} = \overline{\mathcal{O}}_0 \subset \overline{\mathcal{O}}_1 \subset \cdots \overline{\mathcal{O}}_r = \mathfrak{p}_+.$$

It follows that the associated variety of any highest weight Harish-Chandra module is the closure of a single orbit \mathcal{O}_k .

1.2. The group K acts on \mathfrak{B} with a finite number of orbits; we write $K \backslash \mathfrak{B}$ for the set of K -orbits on \mathfrak{B} . The conormal variety for this action of K is

$$T_K^* \mathfrak{B} := \bigcup_{\mathcal{Q} \in K \backslash \mathfrak{B}} \overline{T_{\mathcal{Q}}^* \mathfrak{B}}.$$

Suppose that X is a Harish-Chandra module with corresponding \mathcal{D} -module \mathcal{M}_X . Write the characteristic cycle of \mathcal{M}_X as

$$CC(X) = \sum n_{\mathcal{Q}}(X) [\overline{T_{\mathcal{Q}}^* \mathfrak{B}}],$$

where the sum is over various K -orbits in \mathfrak{B} . See [3, Section 1].

The moment map $\mu : T^* \mathfrak{B} \rightarrow \mathcal{N}$ relates the characteristic cycle to the associated variety as follows. One easily sees that $\mu(\overline{T_{\mathcal{Q}}^* \mathfrak{B}}) \subset \mathcal{N}_{\theta}$ and $\mu^{-1}(\mathcal{N}_{\theta}) = T_K^* \mathfrak{B}$. By [3]

$$AV(X) = \bigcup_{\mathcal{Q}: n_{\mathcal{Q}}(X) \neq 0} \mu(\overline{T_{\mathcal{Q}}^* \mathfrak{B}}). \quad (1.2)$$

Each $\mu(\overline{T_{\mathcal{Q}}^* \mathfrak{B}})$ is the closure of some K -orbit \mathcal{O} in \mathcal{N}_{θ} . The ‘Springer fiber’ $\mu^{-1}(f)$ is an equidimensional subvariety of $T_K^* \mathfrak{B}$. Its irreducible components are partitioned by the irreducible components of $T_K^* \mathfrak{B}$, i.e., by the conormal bundle closures $\overline{T_{\mathcal{Q}}^* \mathfrak{B}}$. So each component of $\mu^{-1}(f)$ lies inside a unique $\overline{T_{\mathcal{Q}}^* \mathfrak{B}}$. It is a fact that the set of irreducible components of $\mu^{-1}(f)$ that lie in a given conormal bundle is either empty or exactly one orbit under the component group $A_K(f) := Z_K(f)/Z_K(f)_e$. Therefore, for a given $f \in \mathcal{N}_{\theta}$,

$$\#\{\mathcal{Q} : \mu(\overline{T_{\mathcal{Q}}^* \mathfrak{B}}) = \overline{K \cdot f}\} = \#\{A_K(f)\text{-orbits in } \mu^{-1}(f)\}. \quad (1.3)$$

The notion of Harish-Chandra cell is defined in [1]. A cell spans a representation $V_{\mathcal{C}}$ of W , which is defined in terms of coherent continuation. It follows easily from the definition that if a cell \mathcal{C} contains one highest weight Harish-Chandra module, then all representations in \mathcal{C} are highest weight Harish-Chandra modules. It also follows easily from the definitions that any two representations in a Harish-Chandra cell have the same associated variety.

Suppose \mathcal{C} is a cell and $\mathcal{O} \subset \mathcal{N}_\theta$ is a K -orbit for which $\overline{\mathcal{O}}$ occurs in the associated variety for \mathcal{C} . Write $\mathcal{O}^{\mathcal{C}}$ for the G -saturation of \mathcal{O} . Then the representation $\pi(\mathcal{O}^{\mathcal{C}})$ associated to $\mathcal{O}^{\mathcal{C}}$ by the Springer correspondence occurs in $V_{\mathcal{C}}$. Through the projection $T_{\mathcal{Q}}^*\mathfrak{B} \rightarrow \mathfrak{B}$, the Springer fiber $\mu^{-1}(f)$ is identified with $\mathfrak{B}^f := \{\mathfrak{b} \in \mathfrak{B} : f \in \mathfrak{b}\}$ and we have ([12])

$$\pi(\mathcal{O}^{\mathcal{C}}) := (H^{\text{top}}(\mathfrak{B}^f))^{A_G(f)},$$

with

$$A_G(f) := Z_G(f)/Z_G(f)_e.$$

Thus, the dimension of $\pi(\mathcal{O}^{\mathcal{C}})$ is the number of $A_G(f)$ -orbits in $\text{Irr}(\mathfrak{B}^f) := \{\text{irreducible components in } \mathfrak{B}^f\}$ (or in $\mu^{-1}(f)$). It follows that if $A_G(f)$ and $A_K(f)$ have the same orbits in $\text{Irr}(\mathfrak{B}^f)$, then for $\mathcal{O} = K \cdot f$

$$\#\{\mathcal{Q} : \mu(\overline{T_{\mathcal{Q}}^*\mathfrak{B}}) = \overline{\mathcal{O}}\} = \dim(\pi(\mathcal{O}^{\mathcal{C}})).$$

We now make an observation that will be important in Section 2. This counting argument has been used, for example, in [13]. Suppose that \mathcal{C} is a Harish-Chandra cell of highest weight Harish-Chandra modules with associated variety equal to $\overline{\mathcal{O}} = \overline{K \cdot f}$.

Lemma 1.4. *If the $A_G(f)$ and $A_K(f)$ orbits in $\text{Irr}(\mathfrak{B}^f)$ coincide and $\#(\mathcal{C}) \geq \dim(\pi(\mathcal{O}^{\mathcal{C}}))$, then there are representations in \mathcal{C} with reducible characteristic cycle.*

Proof. No two representations in a cell of highest weight Harish-Chandra modules have the same support. This is Cor. A.6 of the Appendix. Thus for some representation in \mathcal{C} , the conormal bundle of the support has moment map image of strictly smaller dimension than \mathcal{O} . In this case the conormal bundle of support occurs in the characteristic cycle along with *another* conormal bundle closure which has moment image exactly $\overline{\mathcal{O}}$. \square

In terms of the *leading term cycle*

$$LTC(X) := \sum_{\mu(T_{\mathcal{Q}}^*\mathfrak{B}) = \overline{\mathcal{O}}} n_{\mathcal{Q}}(X) [\overline{T_{\mathcal{Q}}^*\mathfrak{B}}],$$

the lemma tells us that if the size of the cell is strictly greater than $\dim(\pi(\mathcal{O}^{\mathcal{C}}))$, then there are Harish-Chandra modules $X_{\mathcal{Q}} \in \mathcal{C}$ ($\text{supp}(X_{\mathcal{Q}}) = \overline{\mathcal{Q}}$) for which $\overline{T_{\mathcal{Q}}^*\mathfrak{B}}$ does not occur in $LTC(X_{\mathcal{Q}})$.

1.3. Cohomological induction. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra of \mathfrak{g} and write $\bar{\mathfrak{q}} = \mathfrak{l} + \bar{\mathfrak{u}}$ for the opposite parabolic. We consider cohomologically induced modules $\mathcal{R}_{\bar{\mathfrak{q}}}(Z)$ where Z is in the good range. The good range means that if Λ_Z is the infinitesimal character of Z , then $\langle \Lambda_Z + \rho(\bar{\mathfrak{u}}), \beta \rangle > 0, \beta \in \Delta(\bar{\mathfrak{u}})$. This condition guarantees, for example, that $\mathcal{R}_{\bar{\mathfrak{q}}}(Z)$ is irreducible and nonzero. (See [8, Th. 8.2].)

It is well-known that the characteristic cycle of $\mathcal{R}_{\bar{\mathfrak{q}}}(Z)$ may be given in terms of the characteristic cycle of Z . For the statement of this we need to associate a K -orbit in \mathfrak{B} to each $K \cap L$ -orbit in \mathfrak{B}_L , the flag variety for \mathfrak{l} . This is done as follows. Let \mathfrak{F} be the generalized flag variety of all parabolic subalgebras of \mathfrak{g} conjugate to \mathfrak{q} . Let $\pi : \mathfrak{B} \rightarrow \mathfrak{F}$ be the natural projection. Then

$$\pi^{-1}(\mathfrak{q}) = \{\mathfrak{b} \in \mathfrak{B} : \mathfrak{b} \subset \mathfrak{q}\} \simeq \mathfrak{B}_L.$$

The bijection is given by $\mathfrak{b}_L \mapsto \mathfrak{b}^L := \mathfrak{b}_L + \mathfrak{u}$, for any $\mathfrak{b}_L \in \mathfrak{B}_L$. For $\mathcal{Q}_L \in K \cap L \backslash \mathfrak{B}_L$ write $\mathcal{Q}_L = (K \cap L) \cdot \mathfrak{b}_L$. Now set $\mathcal{Q} := K \cdot \mathfrak{b}^L$. Thus we have a map $\mathcal{Q}_L \mapsto \mathcal{Q}$ from $K \cap L$ -orbits in \mathfrak{B}_L to K -orbits in \mathfrak{B} . Now we may state the formula for the characteristic cycle of $\mathcal{R}_{\bar{\mathfrak{q}}}(Z)$.

Proposition 1.5. *If $CC(Z) = \sum_i n_i [\overline{T_{\mathcal{Q}_{L,i}}^* \mathfrak{B}_L}]$, then*

$$CC(\mathcal{R}_{\bar{\mathfrak{q}}}(Z)) = \sum_i n_i [\overline{T_{\mathcal{Q}_i}^* \mathfrak{B}}],$$

where $\mathcal{Q}_{L,i} \mapsto \mathcal{Q}_i$ as described above.

By observing that

$$\mu(\overline{T_{\mathcal{Q}}^* \mathfrak{B}}) = \overline{K \cdot (\mathfrak{n}^L \cap \mathfrak{p})} = \overline{K \cdot (\mathfrak{n}^L \cap \mathfrak{p} + \mathfrak{u} \cap \mathfrak{p})} = \overline{K \cdot (\mu(\overline{T_{\mathcal{Q}_L}^* \mathfrak{B}_L}) + \mathfrak{u} \cap \mathfrak{p})},$$

we have the following (well-known) corollary.

Corollary 1.6. $AV(\mathcal{R}_{\bar{\mathfrak{q}}}(Z)) = \overline{K \cdot (AV(Z) + \mathfrak{u} \cap \mathfrak{p})}$.

2. $Sp(2n, \mathbf{R})$

We now consider $G_0 = Sp(2n, \mathbf{R})$. Then K is the complex group $GL(n)$. Then, with the usual meaning of $\epsilon_j \in \mathfrak{h}^*$, the positive system

$$\Delta^+(\mathfrak{h}, \mathfrak{g}) = \{\epsilon_i \pm \epsilon_j : i < j\} \cup \{2\epsilon_i\}$$

is as in (1.1).

2.1. The action of K on $\mathfrak{p}_+ \simeq \text{sym}(n, \mathbf{C})$ is by $k \cdot X = k^t X k$. The orbits are

$$\mathcal{O}_k = \{X \in \text{sym}(n, \mathbf{C}) : \text{rank}(X) = k\}, \text{ for } k = 0, 1, 2, \dots, n.$$

Base points may be chosen so that $f_k := E_{11} + \dots + E_{kk}$ (where the matrix E_{ii} has 1 in the (i, i) position and 0's elsewhere). Each f_k corresponds to the sum of root vectors for $2\epsilon_1, \dots, 2\epsilon_k$. Thus

$$\mathcal{O}_k = K \cdot f_k = K \cdot \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \subset \text{sym}(n, \mathbf{C}).$$

Let $\mathcal{O}_k^{\mathbf{C}}$ be the nilpotent G orbit of f_k .

We collect some information on these orbits. The partition associated to $\mathcal{O}_k^{\mathbf{C}}$ has k rows of length two and $2n - 2k$ rows of length one. The representation $\pi(\mathcal{O}_k^{\mathbf{C}})$ of W associated to $\mathcal{O}_k^{\mathbf{C}}$ by the Springer correspondence is an action of W on $(H_{\text{top}}(\mathfrak{B}^f))^{A_G(f)}$ ([11]). By (for example) [6, §10.1] we have

$$\dim(\pi(\mathcal{O}_k^{\mathbf{C}})) = \binom{n}{\lfloor \frac{k}{2} \rfloor}. \quad (2.1)$$

As $H_{\text{top}}(\mathfrak{B}^f)$ is spanned by the irreducible components of \mathfrak{B}^f , we see that $\dim(\pi(\mathcal{O}_k^{\mathbf{C}}))$ is the number of $A_G(f)$ -orbits in $\text{Irr}(\mathfrak{B}^f)$. The orbits $\mathcal{O}_k^{\mathbf{C}}$ are ‘special’ exactly when k is either even or equal to n . It is known that for the pair $(G, K) = (Sp(2n), GL(n))$ the orbits of $A_G(f)$ and $A_K(f)$ in $\text{Irr}(\mathfrak{B}^f)$ coincide (for any $f \in \mathcal{N}_\theta$). We may conclude from our earlier discussion that

$$\#\{\mathcal{Q} \in K \backslash \mathfrak{B} : \mu(\overline{T_{\mathcal{Q}}^* \mathfrak{B}}) = \overline{\mathcal{O}_k}\} = \binom{n}{\lfloor \frac{k}{2} \rfloor}. \quad (2.2)$$

A complex nilpotent orbit is called *special* if it is the associated variety of a primitive ideal (of infinitesimal character ρ). Thus, the only associated varieties of Harish-Chandra modules are real forms of special orbits.

Now consider only even n . Since $\mathcal{O}_{n-1}^{\mathbf{C}}$ is not special, there are no irreducible Harish-Chandra modules of infinitesimal character ρ with associated variety $\overline{\mathcal{O}_{n-1}}$. Therefore the irreducible highest weight Harish-Chandra modules having supports \mathcal{Q} with

$$\mu(\overline{T_{\mathcal{Q}}^* \mathfrak{B}}) = \overline{\mathcal{O}_{n-1}} \text{ or } \overline{\mathcal{O}_n}$$

all have associated variety equal to $\overline{\mathcal{O}_n} = \mathfrak{p}_+$. In particular, there are $\binom{n}{\frac{n}{2}-1}$ irreducible highest weight Harish-Chandra modules (of infinitesimal character ρ) for which the closure of the conormal bundle of the support is *not* in the

leading term cycle. Let X_Q be such an irreducible highest weight Harish-Chandra module (with Q its support). We may conclude from Lemma 1.4 that there is a Q_0 so that

$$CC(X_Q) = \overline{[T_Q^* \mathfrak{B}]} + m_{Q_0} \overline{[T_{Q_0}^* \mathfrak{B}]} + \dots, \quad (2.3)$$

with $m_{Q_0} \neq 0$ and $\mu(\overline{[T_{Q_0}^* \mathfrak{B}]}) = \mathfrak{p}_+$.

2.2. For any $m = 0, 1, 2, \dots, n-1$, a θ -stable parabolic subalgebra \mathfrak{q} is defined by

$$H_n := (n - m, \dots, 2, 1, 0, \dots, 0) = \sum_{j=1}^{n-m} (n - m - j + 1) \epsilon_j \in \mathfrak{h}^*.$$

The Levi subalgebra is $\mathfrak{l} \simeq \mathbf{C}^{n-m} \oplus \mathfrak{sp}(2m)$. Note that $\Delta(\mathfrak{u}) \subset \Delta^+(\mathfrak{h}, \mathfrak{g})$ and it follows that

$$\mathfrak{u} \cap \mathfrak{p} = \mathfrak{u} \cap \mathfrak{p}_+. \quad (2.4)$$

Consider an irreducible Harish-Chandra module X for $L = Sp(2m, \mathbf{R})$ as described in (2.3), that is, the conormal bundle of support does not contribute to the leading term cycle. Since we are now working in the group L , we will use a subscript L for the orbits in \mathfrak{B}_L and set $Z_{Q_L} := \mathbf{C} \otimes X$. The support is some $\overline{Q}_L \subset \mathfrak{B}_L$ and $CC(Z_{Q_L}) = \overline{[T_{Q_L}^* \mathfrak{B}_L]} + m_{Q_{L,0}} \overline{[T_{Q_{L,0}}^* \mathfrak{B}_L]} + \dots$. Then the infinitesimal character of Z_{Q_L} is $\rho(\mathfrak{l})$, so Z_{Q_L} is in the good range. By §2.2

$$CC(\mathcal{R}_{\overline{\mathfrak{q}}}(Z_{Q_L})) = \overline{[T_Q^* \mathfrak{B}]} + m_0 \overline{[T_{Q_0}^* \mathfrak{B}]} + \dots,$$

for some K -orbits Q, Q_0 in \mathfrak{B} , and

$$\begin{aligned} AV(\mathcal{R}_{\overline{\mathfrak{q}}}(Z_{Q_L})) &= \overline{K \cdot (AV(Z_{Q_L}) + \mathfrak{u} \cap \mathfrak{p})} \\ &= \overline{K \cdot (\mathfrak{p}_+ \cap \mathfrak{l} + \mathfrak{u} \cap \mathfrak{p})} \\ &= \mathfrak{p}_+, \text{ by (2.4).} \end{aligned}$$

Lemma 2.5. *Both $\overline{[T_Q^* \mathfrak{B}]}$ and $\overline{[T_{Q_0}^* \mathfrak{B}]}$ occur in $LTC(\mathcal{R}_{\overline{\mathfrak{q}}}(Z_{Q_L}))$.*

Proof. We check that $\mu(\overline{[T_Q^* \mathfrak{B}]}) = \mu(\overline{[T_{Q_0}^* \mathfrak{B}]}) = \mathfrak{p}_+$. As noted in §2, under the correspondence $Q_L \rightarrow Q$

$$\mu(\overline{[T_Q^* \mathfrak{B}]}) = \overline{K \cdot (\mu(\overline{[T_{Q_L}^* \mathfrak{B}]} + \mathfrak{u} \cap \mathfrak{p})},$$

and similarly for $Q_{L,0} \rightarrow Q_0$. Thus,

$$\mu(\overline{[T_{Q_0}^* \mathfrak{B}]}) = \overline{K \cdot (\mathfrak{p}_+ \cap \mathfrak{l} + \mathfrak{u} \cap \mathfrak{p})},$$

By (2.4). For \mathcal{O}_L we have

$$\mu(\overline{T_{\mathbb{Q}}^* \mathfrak{b}}) = \overline{K \cdot (\overline{\mathcal{O}_{n-1} + \mathfrak{u} \cap \mathfrak{p}_+})}.$$

So we need to check that $\mathcal{O}_{n-1} + \mathfrak{u} \cap \mathfrak{p}_+$ contains matrices of rank n (on identifying \mathfrak{p}_+ with $\text{sym}(n, \mathbf{C})$). In terms of root vectors such a matrix is

$$\sum_{i=n-m+1}^{N-1} X_{2\epsilon_i} + \left(\sum_{j=2}^{n-m} X_{2\epsilon_j} + (X_{\epsilon_1 + \epsilon_N}) \right) \in f_{n-1} + \mathfrak{u} \cap \mathfrak{p}_+. \quad \square$$

Proposition 2.6. *Fix m even and $m < n$. There are $\binom{m}{\frac{m}{2}-1}$ highest weight Harish-Chandra modules Z of $Sp(2m, \mathbf{R})$ for which $\mathcal{R}_{\overline{\mathfrak{q}}}(Z)$ has associated variety \mathfrak{p}_+ and has reducible leading term cycles.*

2.3. This section contains a refined version of the previous section, giving a larger number of examples of Harish-Chandra modules with reducible leading term cycle, however we rely on a deeper result. It is proved in [9, Thm. 7] that each Harish-Chandra cell for $Sp(2n, \mathbf{R})$ is ‘Lusztig’, that is, it is isomorphic to a corresponding left cell representation. The dimensions of these may be calculated from [5, Ch. 11]. This is an easy calculation for the orbits $\mathcal{O}_{2j}^{\mathbf{C}}$ and one gets the following.

(a) When n is even, for each $j = 0, 1, \dots, \frac{n}{2}$, there is a cell \mathcal{C}_{2j} having associated variety \mathcal{O}_{2j} of size $\binom{n}{j} + \binom{n}{j-1}$.

(b) When n is odd, for each $j = 0, 1, \dots, \frac{n-1}{2}$, there is a cell \mathcal{C}_{2j} having associated variety \mathcal{O}_{2j} of size $\binom{n}{j} + \binom{n}{j-1}$ and a cell \mathcal{C}_n with associated variety $\overline{\mathcal{O}_n} = \mathfrak{p}_+$ of size $\binom{n}{\frac{n-1}{2}}$.

From this, along with (2.2), we may conclude the following.

(a) When n is even,

$$\mathcal{C}_{2j} = \{X_{\mathbb{Q}} : \mu(\overline{T_{\mathbb{Q}}^* \mathfrak{B}}) = \overline{\mathcal{O}_{2j}} \text{ or } \overline{\mathcal{O}_{2j-1}}\}, j = 0, 1, \dots, \frac{n}{2}.$$

(b) When n is odd,

$$\begin{aligned} \mathcal{C}_{2j} &= \{X_{\mathbb{Q}} : \mu(\overline{T_{\mathbb{Q}}^* \mathfrak{B}}) = \overline{\mathcal{O}_{2j}} \text{ or } \overline{\mathcal{O}_{2j-1}}\}, j = 0, 1, \dots, \frac{n-1}{2} \text{ and} \\ \mathcal{C}_n &= \{X_{\mathbb{Q}} : \mu(\overline{T_{\mathbb{Q}}^* \mathfrak{B}}) = \overline{\mathcal{O}_n}\}. \end{aligned}$$

Proposition 2.7. *With $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ as in §2.2*

$$\overline{K \cdot (\overline{\mathcal{O}}_{2j-1} + \mathfrak{u} \cap \mathfrak{p})} = \overline{K \cdot (\overline{\mathcal{O}}_{2j} + \mathfrak{u} \cap \mathfrak{p})} = \mathfrak{p}_+,$$

when $j \geq m - \lfloor \frac{n-1}{2} \rfloor$.

Proof. Let $0 \leq k \leq m$. We have

$$\begin{aligned} & \overline{K \cdot (\overline{\mathcal{O}}_k + \mathfrak{u} \cap \mathfrak{p})} \\ &= \overline{K \cdot (\overline{\mathcal{O}}_k + \mathfrak{u} \cap \mathfrak{p}_+)}, \text{ since } \mathfrak{u} \cap \mathfrak{p} = \mathfrak{u} \cap \mathfrak{p}_+, \\ &= \overline{K \cdot (\mathcal{O}_k + \mathfrak{u} \cap \mathfrak{p}_+)}, \\ &= \overline{K \cdot (f_k + \mathfrak{u} \cap \mathfrak{p}_+)}, \text{ since } K \cap L \text{ preserves } \mathfrak{u} \cap \mathfrak{p}_+. \end{aligned}$$

Claim: $\overline{K \cdot (f_k + \mathfrak{u} \cap \mathfrak{p})} = \begin{cases} \overline{\mathcal{O}}_{2(n-m)+k}, & k = 0, 1, \dots, 2m - n - 1 \\ \overline{\mathcal{O}}_n, & k \geq 2m - n. \end{cases}$

Case 1. $k \geq 2m - n$ (i.e., $m - k \leq n - m$). Then

$$f_k + \sum_{i=1}^{m-k} X_{\epsilon_i + \epsilon_{n-m+k+i}} + \sum_{i=m-k+1}^{n-m} X_{2\epsilon_i} \in f_k + \mathfrak{u} \cap \mathfrak{p}_+$$

has rank n . Note that in $\text{sym}(n, \mathbf{C})$ this is the matrix

$$\begin{bmatrix} & & & I_{m-k} \\ & & & \\ & & I_{(n-m)-(m-k)} & \\ & & & I_k \\ I_{m-k} & & & \end{bmatrix}.$$

Case 1. $k \leq 2m - n$ (i.e., $m - k \geq n - m$). In this case $f_k + \mathfrak{u} \cap \mathfrak{p}_+$ contains

$$f_k + \sum_{i=1}^{n-m} X_{\epsilon_i + \epsilon_{n-m+k+i}},$$

which has rank $k + 2(n - m)$. In $\text{sym}(n, \mathbf{C})$ this is the matrix

$$\begin{bmatrix} & & I_{n-m} \\ & & \\ & I_k & \\ I_{n-m} & & \\ & & & 0_{m-k} \end{bmatrix}.$$

Since the lower right $m \times m$ block has rank k , the greatest possible rank in $f_k + \mathfrak{u} \cap \mathfrak{p}_+$ is $(n - m) + k + (n - m)$. \square

We may conclude the following.

Proposition 2.8. *Let $j_0 = \max\{0, m - \lfloor \frac{n-1}{2} \rfloor\}$ and $j = j_0, j_0 + 1, \dots, \lfloor \frac{m}{2} \rfloor$. Then there are $\binom{m}{j-1}$ highest weight Harish-Chandra modules Z_{Ω_L} for $(\mathfrak{l}, K \cap L)$ with associated variety $\overline{\mathcal{O}_{2j}}$ for which $X_{\Omega} = \mathcal{R}_{\overline{\mathfrak{q}}}(Z_{\Omega_L})$ has associated variety \mathfrak{p}_+ and has reducible leading term cycle.*

2.4. The previous section tells us how to find highest weight Harish-Chandra modules with reducible leading term cycles. In this section we provide a family of examples very explicitly - we give the highest weights.

Consider highest weight Harish-Chandra modules L_w for $Sp(2m, \mathbf{R})$, any $m > 0$, with w of the form

$$w_k(\epsilon_i) := \begin{cases} -\epsilon_i, & i = 1, \dots, m - k \\ \epsilon_{2m-i-k+1}, & i = m - k + 1, \dots, m \end{cases}$$

with $k = 0, 1, \dots, m$. Note that $w_k(\rho) = (-m, \dots, -(k+1), 1, 2, \dots, k)$. Since

$$\begin{aligned} \Delta(\mathfrak{n} \cap \mathfrak{n}^{w_k}) &= \{\alpha \in \Delta^+ : w_k^{-1}\alpha > 0\} \\ &= \{\alpha \in \Delta^+ : \langle \alpha, w_k\rho \rangle > 0\} \\ &= \{\epsilon_i + \epsilon_j : m - k + 1 \leq i, j \leq m\}, \end{aligned}$$

we see that the maximum rank of matrices in $\mathfrak{n} \cap \mathfrak{n}^{w_k}$ is k . Letting $B(w_k)$ denote the B -orbit of $w_k \cdot \mathfrak{b}$ in \mathfrak{B} , we therefore have $\mu(\overline{T_{B(w_k)}^* \mathfrak{B}}) = \overline{B \cdot (\mathfrak{n} \cap \mathfrak{n}^{w_k})} = \overline{\mathcal{O}_k}$. When k is odd and $k \neq m$, $\mu(\overline{T_{B(w_k)}^* \mathfrak{B}})$ cannot be not the associated variety of L_{w_k} , so $CC(L_{w_k})$ contains $\overline{T_{B(w_k)}^* \mathfrak{B}}$ and (at least) one other $\overline{T_{B(w'_k)}^* \mathfrak{B}}$, each with nonzero multiplicity.

Remark 2.9. With more effort, one can show that $w'_k = w_{k+1}$. One can further say that $CC(L_{w_k}) = \overline{T_{B(w_k)}^* \mathfrak{B}} + \overline{T_{B(w_{k+1})}^* \mathfrak{B}}$.

Now consider $n > m$ and the θ -stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ defined by H_m as in §2.2. The K -type formula of [14, §6.3] tells us that $\mathcal{R}_{\overline{\mathfrak{q}}}(L_{w_k})$ is a highest weight Harish-Chandra module for $Sp(2n, \mathbf{R})$:

$$\mathcal{R}_{\overline{\mathfrak{q}}}(L_{w_k}) = L_{w^0 w_k \rho - \rho},$$

$w^0 = w_K w_{K \cap L}$, the product of the long elements of the Weyl groups for K and $K \cap L$.

Therefore, the conormal bundle closures (for the flag variety of $Sp(2n)$) $\overline{T_{B(w^0 w_k)}^* \mathfrak{B}}$ and $\overline{T_{B(w^0 w'_k)}^* \mathfrak{B}}$ occur in $CC(L_{w^0 w_k})$. By Prop. 2.7, both occur in the leading term cycle when $n \geq 2m - k, n$ even, and $n \geq 2m - k + 1, n$ odd.

2.5. There is a relationship between leading term cycles of irreducible Harish-Chandra modules and associated varieties of irreducible highest weight modules. This is discussed in detail in [13, Section 4]. We summarize the conclusions that we will use. We may consider arbitrary Harish-Chandra modules for arbitrary reductive algebraic groups for this discussion.

A well-known geometric fact is that there is a one-to-one correspondence

$$\{A_G(f)\text{-orbits in } \text{Irr}(\mathfrak{B}^f)\} \leftrightarrow \{\text{orbital varieties in } \mathfrak{n} \cap (G \cdot f)\}.$$

Let X_Q be an irreducible Harish-Chandra module with support \overline{Q} . Suppose that $AV(X_Q) = \overline{\mathcal{O}_K}$ and $\text{ann}(X_Q) = \text{ann}(L_w)$ for some $w \in W$. We note that if, for $f \in \mathcal{N}_\theta$, the $A_G(f)$ and $A_K(f)$ orbits in $\text{Irr}(\mathfrak{B}^f)$ coincide, then there is a one-to-one correspondence

$$\{Q \in K \backslash \mathfrak{B} : \mu(\overline{T_Q^* \mathfrak{B}}) = \overline{\mathcal{O}_K}\} \leftrightarrow \{\text{orbital varieties in } \mathfrak{n} \cap (\mathcal{O}_K^C)\}. \quad (2.10)$$

The following two statements are contained in [13, Cor. 4.2]

Proposition 2.11. *Suppose the $A_G(f)$ and $A_K(f)$ orbits in $\text{Irr}(\mathfrak{B}^f)$ coincide. If $Q \leftrightarrow \nu$ under the one-to-one correspondence (2.10), then $\overline{T_Q^* \mathfrak{B}}$ occurs in $LTC(X_Q)$ if and only if ν occurs as a component of $AV(L_{w^{-1}})$.*

Note that without the equal orbit hypothesis we might have a reducible leading term cycle with, say, $\overline{T_{Q_1}^* \mathfrak{B}}$ and $\overline{T_{Q_2}^* \mathfrak{B}}$ occurring, for which the corresponding $A_K(f)$ -orbits in $\text{Irr}(\mathfrak{B}^f)$ lie in the same $A_G(f)$ -orbit. In this case $AV(L_{w^{-1}})$ may still be irreducible. However, without the equal orbits hypothesis we have the following.

Proposition 2.12. *Suppose $\text{ann}(X_Q) = \text{ann}(L_w)$. If $AV(L_{w^{-1}})$ is reducible, then $LTC(X_Q)$ must contain at least two conormal bundle closures.*

Prop. 2.11, along with 2.3, gives a family of irreducible highest weight modules $L_{w^{-1}}$ of $\mathfrak{sp}(2n)$ having reducible associated varieties. The w 's can be determined explicitly as follows. Begin with $\sigma \in W_\mathfrak{l}$, $\mathfrak{l} = \mathfrak{sp}(2m)$, so that $\overline{B(\mathfrak{n} \cap \mathfrak{n}^\sigma)} = \overline{\mathcal{O}_{2j-1}}$ with j as in Prop. 2.8. Then $\mathcal{R}_{\overline{\mathfrak{q}}}(L_\sigma)$ is the highest weight Harish-Chandra module L_w , $w = w^0 \sigma$, where w^0 is the product of the long elements of $W_\mathfrak{k}$ and $W_{\mathfrak{k} \cap \mathfrak{l}}$ (as follows easily from the the formula for the lowest K -type of a cohomologically induced representation in the good range ([14])). Thus the annihilator of $\mathcal{R}_{\overline{\mathfrak{q}}}(L_\sigma)$ is $\text{ann}(L_w)$ so Prop. 2.11 tells us that $AV(L_{w^{-1}})$ is reducible.

2.6. Reducible leading term cycles for $Sp(n, n)$. Now consider $G'_0 = Sp(p, q)$, $p + q = n$, so $G' = Sp(2n)$ and $K' \simeq Sp(2p) \times Sp(2q)$. Suppose that \mathcal{O} is a K' -orbit in \mathcal{N}'_θ . Then

- (i) there is a Harish-Chandra cell with associated variety $\overline{\mathcal{O}}$ ([9]), and
- (ii) if I is a primitive ideal with $AV(I) = \overline{\mathcal{O}^C}$, then there is a Harish-Chandra module X with $\text{ann}(X) = I$.

There are irreducible representations of $Sp(2n, \mathbf{R})$ with reducible leading term cycles and associated variety $\overline{\mathcal{O}}_n = \mathfrak{p}_+$ (§2.2,2.3). In particular, these representations each have an annihilator I with $AV(I) = \overline{\mathcal{O}^C}_n$. But $\overline{\mathcal{O}^C}_n$ corresponds to the partition $(2, 2, \dots, 2)$. It follows¹ that \mathcal{O}^C_n is \mathcal{O}^C for some K' -orbit \mathcal{O} in \mathcal{N}'_θ with $p = q = \frac{n}{2}$ (and n must be even).

See [2] for an explicit example in type C_4 .

We may now conclude that there are irreducible representations of $Sp(\frac{n}{2}, \frac{n}{2})$ with reducible leading term cycles.

APPENDIX A. GEOMETRIC FACTS

Suppose X is a highest weight Harish-Chandra module. As we know, the associated variety of X considered as a highest weight module, is a union of components of $\mathfrak{n} \cap \mathcal{O}^C$ (where $\overline{\mathcal{O}^C}$ is the associated variety of the annihilator of X). On the other hand, the associated variety of a Harish-Chandra module is a union of closures of K -orbits in $\mathcal{N}_\theta \cap \mathcal{O}^C$. These two seemingly different objects coincide: each may be computed using the good filtration $\{X_n\}$, where X_0 is the K -type generated by the highest weight vector and $X_n = \mathcal{U}_n(\mathfrak{p}^-) \cdot X_0$. This filtration is stable under *both* K and B . Lemma A.2 below shows that the the relevant orbital varieties and nilpotent K orbits are in fact the same.

Similarly, the characteristic cycle of a highest weight module is a union of closures of conormal bundles of B -orbits on \mathfrak{B} and the the characteristic cycle of a Harish-Chandra module is the union of closures of K -orbits on \mathfrak{B} . These also coincide: the localization of X is $\mathcal{X} := \mathcal{D} \otimes_{\mathcal{U}(\mathfrak{g})} X$ and the characteristic cycle is defined by any good filtration, thus independent of whether we consider X

¹The K' -orbits occurring in \mathcal{N}'_θ are those for which (i) there are an even number of even rows and the same number beginning with + as those beginning with a -, and (ii) there are an even number of odd rows and an even number beginning with + and an even number beginning with a -.

a highest weight module or a Harish-Chandra module. Lemma A.3 shows that the relevant Schubert varieties and K -orbit closures coincide. A statement in the same spirit is in [7, Section 1].

Suppose $X = L_w$. As X is a Harish-Chandra module, $-w\rho$ is Δ_c^+ -dominant.

A.1. Orbital varieties. Suppose $y \in W$, then $\nu(y) := \overline{B \cdot (\mathfrak{n} \cap \mathfrak{n}^y)}$, with $\mathfrak{n}^y = \text{Ad}(y)\mathfrak{n}$, is an irreducible component of $\overline{\mathfrak{n} \cap \mathcal{O}^C}$. Each such component has the form $\nu(y)$ for some $y \in W$ and is called an *orbital variety*. One easily sees that

$$\nu(y) \subset \mathfrak{p}_+ \text{ if and only if } -y\rho \text{ is } \Delta_c^+ \text{-dominant.} \quad (\text{A.1})$$

Lemma A.2. (i) *If $\mathcal{O}_K := K \cdot f \subset \mathfrak{p}_+$, then $\overline{\mathcal{O}_K} = \nu(y)$ for some $y \in W$. In this case $-y\rho$ is necessarily Δ_c^+ -dominant.*

(ii) *If $\nu(y) \subset \mathfrak{p}_+$, then $\nu(y) = \overline{\mathcal{O}_K}$ for some K -orbit in \mathfrak{p}_+ .*

Proof. (i) Since $BK \cdot f = KP_+ \cdot f = K \cdot f$ (as P_+ is abelian and $f \in \mathfrak{p}_+$), \mathcal{O}_K is B -stable. Also, $\dim(\mathcal{O}_K) = \frac{1}{2} \dim(\mathcal{O}_K^C)$ is the dimension of any orbital variety for \mathcal{O}^C . It follows that $\overline{\mathcal{O}_K}$ is $\overline{\nu(y)}$ for some $y \in W$. By (A.1), $-y\rho$ is Δ_c^+ -dominant.

(ii) If $\nu(y) \subset \mathfrak{p}_+$, then

$$\begin{aligned} K\nu(y) &= \overline{KB(\mathfrak{n} \cap \mathfrak{n}^y)} \\ &\subset \overline{KB(\mathfrak{n} \cap \mathfrak{n}^y)} \\ &\subset \overline{KP_+(\mathfrak{n} \cap \mathfrak{n}^y)} \\ &= \overline{K(\mathfrak{n} \cap \mathfrak{n}^y)}, \text{ since } \mathfrak{n} \cap \mathfrak{n}^y \subset \mathfrak{p}_+, \\ &\subset \overline{(K \cap B)(K \cap B^-)(\mathfrak{n} \cap \mathfrak{n}^y)}, \\ &\quad \text{since } (K \cap B)(K \cap B^-) \text{ is dense in } K, \\ &\subset \overline{(K \cap B)(K \cap B^-)(\mathfrak{n} \cap \mathfrak{n}^y)}, \\ &\subset \overline{B(\mathfrak{n} \cap \mathfrak{n}^y)} \\ &= \nu(y). \end{aligned}$$

The last inclusion needs justification. It suffices to show that $\alpha \in \Delta_c^+$ and $\beta \in \Delta(\mathfrak{n} \cap \mathfrak{n}^y)$ implies $-\alpha + \beta \in \mathfrak{n} \cap \mathfrak{n}^y$. Since $\beta \in \Delta(\mathfrak{p}_+) \subset \mathfrak{n}$, we need only show $-\alpha + \beta \in \mathfrak{n}^y$, i.e., $y^{-1}(-\alpha + \beta) > 0$. However, $\langle -\alpha + \beta, y\rho \rangle = \langle \alpha, -y\rho \rangle + \langle y^{-1}\beta, \rho \rangle > 0$, since $-y\rho$ is Δ_c^+ -dominant by (A.1) and $y^{-1}\beta > 0$ (since $\beta \in \mathfrak{n}^y$). \square

A.2. Let us set $B(y) = By \cdot \mathfrak{b}$ and $X(y) := \overline{By \cdot \mathfrak{b}}$, the Schubert variety for $y \in W$.

Lemma A.3. (i) *If $\mathcal{Q} \in K \setminus \mathfrak{B}$ is B -stable, then $\overline{\mathcal{Q}} = X(y)$ or some $y \in W$ with $-y\rho$ Δ_c^+ -dominant.*

(ii) *If $X(y)$ is K -stable (so the closure of a K -orbit in \mathfrak{B}), then $-y\rho$ is Δ_c^+ -dominant.*

Proof. (i) A closed B -stable irreducible subvariety of \mathfrak{B} is a Schubert variety, so $\overline{\mathcal{Q}} = X(y)$ for some $y \in W$. We need to show $-y\rho$ is Δ_c^+ -dominant. If not, there is $\alpha \in \Delta_c^+$ so that $\langle y\rho, \alpha \rangle > 0$, i.e., $y^{-1}\alpha > 0$. Therefore, $\ell(y) < \ell(\sigma_\alpha y)$, where σ_α is the simple reflection for α . Since $KBy \cdot \mathfrak{b} \subset X(y)$, $\sigma_\alpha y \cdot \mathfrak{b} \in X(y)$. This implies that $X(\sigma_\alpha y) \subset X(y)$. But this is impossible since $\ell(y) < \ell(\sigma_\alpha y)$.

(ii) Since $X(y)$ is K -stable, it is the closure of a single K -orbit in \mathfrak{B} . The argument above shows that $-y\rho$ is Δ_c^+ -dominant. \square

A.3. We may draw a few conclusions. Suppose $X = L_w$ is a Harish-Chandra module, so $-w\rho$ is Δ_c^+ -dominant. The support of the localization is $X(w)$ ([4, Prop. 6.4]). By Lemma A.3 this is the closure of a K -orbit \mathcal{Q}_w in \mathfrak{B} .

Lemma A.4. *With the above notation $\overline{T_{B(w)}^* \mathfrak{B}} = \overline{T_{\mathcal{Q}_w}^* \mathfrak{B}}$.*

Proof. The varieties $X(w)$ and $\overline{\mathcal{Q}_w}$ contain a smooth dense subvariety in common (namely $(Bw \cdot \mathfrak{b}) \cap \mathcal{Q}_w$). Both conormal bundles appearing in the lemma are the closure of the conormal bundle to the common dense subvariety. \square

Corollary A.5. *The characteristic cycle of X is a combination of conormal bundle closures $\overline{T_{\mathcal{Q}_y}^* \mathfrak{B}}$ with $-y\rho$ Δ_c^+ -dominant.*

Proof. Each conormal bundle occurring is the conormal bundle of both a K -orbit and B -orbit, so Lemma A.3 applies. \square

We also conclude the following fact.

Corollary A.6. *No two highest weight Harish-Chandra modules have the same support.*

REFERENCES

- [1] Barbasch, D., and D. Vogan, *Weyl group representations and nilpotent orbits*, Representation theory of reductive groups (Park City, Utah, 1982) (Boston) (P. C. Trombi, ed.), Progr. Math., vol. 40, Birkhuser, 1983, pp. 21–33.
- [2] Barchini, L., *Two triangularity results and invariants of $(\mathfrak{sp}(p+q, C), Sp(p) \times Sp(q))$ and $(\mathfrak{so}(2n, C), GL(n))$ modules*, submitted.
- [3] Borho, W., and J.-L. Brylinski, *Differential operators on homogeneous spaces. III*, Invent. Math. **80** (1985), 1–68.
- [4] Brylinski, J.-L., and M. Kashiwara, *Kazhdan-lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), 387–410.
- [5] Carter, R., *Finite groups of Lie Type. Conjugacy classes and complex characters*, A Wiley-Interscience, 1985.
- [6] Collingwood, D., and W. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold Co., New York, 1993.
- [7] Collingwood, D., *Orbits and characters associated to highest weight representations*, Proc. Amer. Math. Soc. **114** (1992), no. 4, 1157–1165.
- [8] Knapp, A. W., and D. A. Vogan, *Cohomological Induction and Unitary Representations*, Princeton Mathematical Series, vol. 45, Princeton University Press, 1995.
- [9] McGovern, W., *Cells of Harish-Chandra modules for real classical groups*, Amer. J. Math. **120** (1998), 211–228.
- [10] Nishiyama, K., H. Ochiai, and K. Taniguchi, *Bernstein degree and associated cycles of Harish-Chandra modules - Hermitian case*, Nilpotent Orbits, Associated Cycles and Whittaker Models for Highest Weight Representations (K. Nishiyama, H. Ochiai, K. Taniguchi, H. Yamashita, and S. Kato, eds.), Astérisque, vol. 273, Soc. Math. France, 2001, pp. 13–80.
- [11] Springer, T. A., *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207.
- [12] ———, *A construction of representations of Weyl groups.*, Invent. Math. **44** (1978), 279–293.
- [13] Trapa, P., *Leading-term cycles of Harish-Chandra modules and partial orders on components of the Springer fiber*, Compos. Math. **143** (2007), no. 2, 515–540.
- [14] Vogan, D.A., *Representations of Real Reductive Lie Groups*, Progress in Mathematics, **15**, Birkhäuser, 1981.
- [15] Williamson, G., *A reducible characteristic variety in type A*, arXiv:1405.3479.

OKLAHOMA STATE UNIVERSITY, MATHEMATICS DEPARTMENT, STILLWATER, OK 74078

E-mail address: leticia@math.okstate.edu

OKLAHOMA STATE UNIVERSITY, MATHEMATICS DEPARTMENT, STILLWATER, OK 74078

E-mail address: zierau@math.okstate.edu