POSITIVITY OF ZETA DISTRIBUTIONS AND SMALL UNITARY REPRESENTATIONS

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Abstract. This paper studies the positivity of certain zeta distributions associated to simple noneuclidean Jordan algebras. The distributions are calculated in the cases where they are positive. The main technique revolves around an explicit form of the corresponding functional equation. Using an identity relating these zeta distributions to the standard intertwining operators for the associated conformal groups, explicit families of singular unitary representations are then constructed.

1. Introduction

The purpose of this article is to give a construction of families of singular unitary representations of certain simple Lie groups. We consider Lie groups which are related to conformal groups of noneuclidean Jordan algebras. See Table 1 for a precise list of the groups under consideration. A key property of such groups is that each has a parabolic subgroup $P = LN$ for which the adjoint action of L on $\mathfrak{n} = Lie(N)$ has a finite number of orbits. In particular, there is an open orbit which is in fact dense and has complement defined by a polynomial equation $P(X) = \nabla^2(X) = 0$. It is well-known that ∇^s defines a family of tempered distributions (meromorphic in $s \in \mathbf{C}$) known as zeta distributions. This family may be regularized to give a family of distributions which is complex analytic in s. It turns out that this family of distributions is intimately related to intertwining operators between certain degenerate principal series representations (for $P = LN$) and to the unitarity of certain subrepresentations. We show how these distributions play a role in giving unitary realizations of these representations.

To be slightly more precise, let R_s be the regularized distribution corresponding to ∇^{-s}; see (6.1). Then our first main theorem is that R_s is a positive distribution if and only if $s \in (-\infty, e+1) \cup \{\frac{m}{n} - qd : q = 0, 1, \ldots, n-1\}$, where d, e, m and n are certain integers which depend on G. Theorem 5.12 states that for the discrete points $s = \frac{m}{n} - qd, q = 0, 1, \ldots, n-1$, R_s is a quasi-invariant measure on an L-orbit $\mathcal{O}_q$ in $\mathfrak{n}$. Under our conditions on G there are $n+1$ L-orbits $\mathcal{O}_q$ which we may write as $\{0\} = \mathcal{O}_0 \subset \mathcal{O}_1 \subset \cdots \subset \mathcal{O}_n$; thus each orbit corresponds to a distribution. Next we consider certain smooth degenerate principal series representations, which we denote by $I(s)$. We will use the realization of $I(s)$ as certain smooth functions on $\bar{\mathfrak{n}}$ in the usual way. By general principles there is a complex analytic family of G-intertwining operators

$$A_s : I(s) \to I(-s).$$

We show that for Schwartz functions f on $\bar{\mathfrak{n}}$

$$(A_s f)(Y) = R_s((\tau_Y f)\widehat{\ }) \tag{1.1}$$

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where τ_Y is translation by Y in $\overline{\mathfrak{n}}$ and $\widehat{\ }$ denotes Fourier transform. When $s \in \mathbf{R}$ the intertwining operator defines a G-invariant hermitian form (often indefinite) on the image of A_s by

$$\langle A_s f_1, A_s f_2 \rangle = \int_{\overline{\mathfrak{n}}} f_1(Y) \overline{A_s f_2(Y)} \, dY \tag{1.2}$$

for functions f_1 and f_2 in $I(s)$. For the following results we exclude $G = SO(p, q)$ (in case 4. on Table 1) as the results about unitarity are somewhat different in this case. Assume for the moment that $s = \frac{m}{n} - qd$, $q = 0, 1, \ldots, n-1$, so that R_s is a quasi-invariant measure on $\mathcal{O}_q$. In view of (1.1), formula (1.2) for the hermitian form becomes

$$\langle A_s f_1, A_s f_2 \rangle = \int_{\mathcal{O}_q} \widehat{f_1} \overline{\widehat{f_2}} d\nu_q$$

for Schwartz functions f_1 and f_2 on $\overline{\mathfrak{n}}$. Lemma 8.10, Theorem 8.11 and Corollary 8.12 state that the completion $\mathcal{H}_s$ of $Im(A_s)$ with respect to this inner product is an irreducible unitary G-representation. For this we show that the Fourier transform provides a factoring of A_s through $L^2(\mathcal{O}_q)$. We show that this gives a unitary equivalence between $\mathcal{H}_s$ and $L^2(\mathcal{O}_q)$ as P-representations. Therefore the natural (irreducible) action of P, via the Fourier transform, on $L^2(\mathcal{O}_q)$ extends to an irreducible unitary representation of G. A similar result holds for the continuous parameter $s \in [0, e+1)$.

Our main technique for studying the zeta distributions is a functional equation. Functional equations in this type of setting have a long history. See for example [25], [22], [3] and [15]. The version we use is contained in [15]. The distributions R_s are related to classical Riesz distributions and generalizations. For instance, when G is the conformal group of a *euclidean* Jordan algebra (i.e., G is of tube type) then families of Riesz distributions are associated to the open convex L-orbits in $\mathfrak{n}$. Many of the results of this article, such as positivity of the distributions for certain parameters and the unitary realization, hold in this setting. See for example [7], [9] and [19]. We remark that the main tool used in the case of convex L-orbits is the Laplace transform. In the nonconvex setting of this article the functional equation becomes the main tool.

Our method is, to some degree, inspired by the treatment of $SL(2, \mathbf{R})$ in [8] and the treatment of holomorphic representations in [19]. The representations considered in this article have been studied previously. Our Corollary 8.12 is obtained in [6] by different methods. The unitarizability of the representations is contained in [20]. There is some overlap with techniques in [1] and the recent article [12]. We thank I. Muller for several conversations and for making part of her manuscript [17] available to us.

2. Preliminaries

Each simple noneuclidean Jordan algebra occurs as the abelian nilradical of a maximal parabolic subalgebra of a reductive Lie algebra $\mathfrak{g}$. There is a reductive Lie group G, with Lie algebra $\mathfrak{g}$, having the following properties.

2.1. *G contains a parabolic subgroup $P = LN$ (a Levi decomposition) such that*

(1) *P and its opposite parabolic are G-conjugate, and*
(2) *N is abelian.*
(3) *The symmetric space corresponding to G is not of tube type.*

For a given simple noneuclidean Jordan algebra the $\mathfrak{g}$ and G as above are not quite unique. The choices for the groups G which we work with are given in Table 1 in Appendix A.

Therefore we make the assumption that G is a group listed on Table 1. Then G has a Cartan involution θ so that θ sends P to the opposite parabolic. We let K be the fixed point group of θ, a maximal compact subgroup of G. As is customary we write the Lie algebra of G (resp. K) as $\mathfrak{g}$ (resp. $\mathfrak{k}$). The Cartan involution determines a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$.

Following [13] there is an abelian subalgebra $\mathfrak{b}$ of $\mathfrak{l} \cap \mathfrak{s}$ with the following properties.

(1) There are commuting copies of $\mathfrak{sl}(2, \mathbf{R})$ in $\mathfrak{g}$ spanned by $\{F_j, H_j, E_j\}$, a standard basis in the sense that

$$\theta(E_j) = -F_j \text{ and } \theta(H_j) = -H_j,$$
$$[E_j, F_j] = H_j, [H_j, E_j] = 2E_j \text{ and } [H_j, F_j] = -2F_j$$

with $E_j \in \mathfrak{n}, F_j \in \overline{\mathfrak{n}}$ and $\mathfrak{b} = \sum_{j=1}^n \mathbf{R} H_j$.

(2) For $\epsilon_k(\sum_{j=1}^n a_j H_j) \equiv a_k$, the $\mathfrak{b}$-roots in $\mathfrak{g}, \mathfrak{l}$ and $\mathfrak{n}$ are

$$\Sigma(\mathfrak{g}, \mathfrak{b}) = \{\pm(\epsilon_j - \epsilon_k) : 1 \leq j < k \leq n\} \cup \{\pm(\epsilon_j + \epsilon_k) : 1 \leq j, k \leq n\},$$
$$\Sigma(\mathfrak{l}, \mathfrak{b}) = \{\pm(\epsilon_j - \epsilon_k) : 1 \leq j < k \leq n\} \text{ and}$$
$$\Sigma(\mathfrak{n}, \mathfrak{b}) = \{\epsilon_j + \epsilon_k : 1 \leq j, k \leq n\}.$$

For each G the roots in $\mathfrak{n}$ have just two multiplicities, defining integers[1] d and e:

$$\begin{aligned} &\text{each short root has multiplicity } 2d \text{ and} \\ &\text{each long root has multiplicity } e+1. \end{aligned} \tag{2.2}$$

Define

$$\Sigma^+(\mathfrak{g}, \mathfrak{b}) = \{\epsilon_j - \epsilon_k : 1 \leq j < k \leq n\} \cup \Sigma(\mathfrak{n}, \mathfrak{b}). \tag{2.3}$$

Definition 2.4. Taking $n = \dim(\mathfrak{b})$ as above, we make the following definitions.

(1) The *rank* of $\mathfrak{n}$ is n.
(2) $\Lambda_0 \equiv \sum_{j=1}^n \epsilon_j$.
(3) χ_q is the positive character of L with differential $2qd\Lambda_0$ for $q = 0, 1, 2, \ldots n$.

We will often denote χ_1 by χ. The following lemma is easily proved.

Lemma 2.5. *Set* $m \equiv \dim(\mathfrak{n})$ *and* $\rho(\mathfrak{n}) \equiv \dfrac{1}{2} \displaystyle\sum_{\alpha \in \Sigma(\mathfrak{n}, \mathfrak{b})} \alpha$. *Then*

(1) $m = n(d(n-1) + (e+1))$,
(2) $\rho(\mathfrak{n}) = \frac{m}{n} \Lambda_0$,
(3) $|\det(Ad(\ell)|_{\mathfrak{n}})| = \chi(\ell)^{\frac{m}{dn}}$.

The integers n, m, d and e are listed on Table 1 for each group.

The orbit structure of L acting on $\mathfrak{n}$ will play an important role. If we set

$$X_q \equiv E_1 + \cdots + E_q, \ q = 1, 2, \ldots, n \ \text{ and } X_0 \equiv 0$$

then by [16] and [11] the L-orbits in $\mathfrak{n}$ are precisely

$$\mathcal{O}_q = L(X_q), \ q = 0, 1, 2, \ldots, n. \tag{2.6}$$

[1] In the case of $SO(p, q)$, Case 4 on Tables 1 and 2, d is a half-integer. When $n = 1$, d is zero.

We write $\mathcal{O}_q = L/S_q$, S_q the stabilizer of X_q. As $\mathrm{ad}(X_n) : \mathfrak{l} \to \mathfrak{n}$ is onto, $\mathcal{O}_n$ is open in $\mathfrak{n}$; it is also dense. The orbit $\mathcal{O}_n$ is a semisimple symmetric space. Consider

$$\tau = \mathrm{Ad}\big(\prod_{j=1}^{n} \exp(\frac{\pi}{2}(E_j - F_j))\big). \tag{2.7}$$

Then $\tau(\mathfrak{l}) \subset \mathfrak{l}, \tau(\mathfrak{n}) \subset \overline{\mathfrak{n}}$ and $\tau(\overline{\mathfrak{n}}) \subset \mathfrak{n}$. Furthermore, $\tau|_{\mathfrak{l}}$ is an involution and $\mathfrak{s}_n$, the Lie algebra of S_n, is the subalgebra of $\mathfrak{l}$ fixed by τ. Therefore, L/S_n is a semisimple symmetric space of rank n. We will describe the other orbits and stabilizers in some detail in Section 3.

There is a diffeomorphism of $\overline{\mathfrak{n}} \times L \times \mathfrak{n}$ onto a dense open set in G given by $(Y, \ell, X) \to \overline{n}_Y \ell n_X$, where $\overline{n}_Y = \exp(Y)$ and $n_X = \exp(X)$. Therefore, on a dense open subset of G, there is a decomposition $g = \overline{n}_Y \ell n_X$. Furthermore, $L = MA$ where $A = \exp(\mathfrak{a}), \mathfrak{a} \equiv \bigcap_{j<k} ker(\epsilon_j - \epsilon_k)$. In particular, the L part of the decomposition has a component in A. We define $a(g) \in A$ by

$$g \in \overline{N} M a(g) N. \tag{2.8}$$

By 2.1 there is a $w \in K$ so that $\mathrm{Ad}(w)\mathfrak{n} = \overline{\mathfrak{n}}$. In particular we may define functions on dense open subsets of $\overline{\mathfrak{n}}$ and $\mathfrak{n}$ by

$$\begin{aligned} &\overline{\nabla}(Y) \equiv \mathrm{e}^{\Lambda_0(\log(a(w\overline{n}_Y)))},\ Y \in \overline{\mathfrak{n}} \quad \text{and} \\ &\nabla(X) \equiv \overline{\nabla}(\theta(X)),\ X \in \mathfrak{n}. \end{aligned} \tag{2.9}$$

Lemma 2.10. $\overline{\nabla}(Y)$ *(respectively* $\nabla(X)$*) extends to a well-defined function on* $\overline{\mathfrak{n}}$ *(respectively* $\mathfrak{n}$*) and the following hold.*

(1) $\overline{\nabla}(Y)^2$ *(respectively* $\nabla(X)^2$*) is a homogeneous polynomial on* $\overline{\mathfrak{n}}$ *(respectively* $\mathfrak{n}$*) of degree* $2n$*.*
(2) $\overline{\nabla}(\ell \cdot Y) = |\det(Ad(\ell)|_{\overline{\mathfrak{n}}})|^{\frac{n}{m}} \overline{\nabla}(Y) = \chi(\ell)^{-\frac{1}{d}} \overline{\nabla}(Y)$ *for* $\ell \in L$ *and* $Y \in \overline{\mathfrak{n}}$*, and*
$\nabla(\ell \cdot X) = |\det(Ad(\ell)|_{\mathfrak{n}})|^{\frac{n}{m}} \nabla(X) = \chi(\ell)^{\frac{1}{d}} \nabla(X)$ *for* $\ell \in L$ *and* $X \in \mathfrak{n}$*.*
(3) *Both* $\overline{\nabla}$ *and* ∇ *are invariant under* $L \cap K$*.*

Proof. Consider a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{b}$. Choose a positive system of $\mathfrak{h}$-roots $\Sigma^+(\mathfrak{g}, \mathfrak{h})$ with the property that the positive $\mathfrak{h}$-roots restrict exactly to the roots in $\Sigma^+(\mathfrak{g}, \mathfrak{b})$. Let $\tilde{\Lambda}_0$ be the extension of Λ_0 to $\mathfrak{h}$ (by 0 on $\mathfrak{b}^\perp$). Then $2\tilde{\Lambda}_0$ is dominant and analytically integral (since each $2\epsilon_j$ is a root). In particular there is a finite dimensional representation U_1 of G with highest weight $2\tilde{\Lambda}_0$. Fix an inner product $\langle\ ,\ \rangle$ so that $\langle gu, v\rangle = \langle u, \theta(g^{-1})v\rangle$, for $u, v \in U_1$ and $g \in G$. Then for a highest weight vector u_+,

$$\mathrm{e}^{2\Lambda_0(\log(a(g))} = \langle gu_+, u_+\rangle,\ g \in \overline{N}LN.$$

In particular

$$\overline{\nabla}(Y)^2 = \mathrm{e}^{2\Lambda_0(\log(a(w\overline{n}_Y)))} = \langle w\overline{n}_Y u_+, u_+\rangle$$

is a polynomial in Y. It follows that $\overline{\nabla}(Y)$ is defined on all of $\overline{\mathfrak{n}}$.

As representation of L, $U_1^{\mathfrak{n}}$ (the $\mathfrak{n}$-invariants in U_1) is 1-dimensional and the L-action is by the character $\chi^{\frac{1}{d}}$ (which has differential $2\Lambda_0$ on $\mathfrak{b}$). Also, $\mathrm{Ad}(w)|_{\mathfrak{b}} = -1$ and $\chi(\theta(\ell)) = \chi(\ell^{-1})$. Now

$$\begin{aligned}\overline{\nabla}(\ell\cdot Y)^2 &= \langle w\ell\overline{n}_Y\ell^{-1}u_+, u_+\rangle \\ &= \langle w\ell w^{-1}w\overline{n}_Y\ell^{-1}u_+, u_+\rangle \\ &= \chi(\ell^{-1})^{\frac{1}{d}}\langle w\overline{n}_Y u_+, \theta(w\ell w^{-1})^{-1}u_+\rangle \\ &= \chi(\ell)^{-\frac{2}{d}}\langle w\overline{n}_Y u_+, u_+\rangle \\ &= \chi(\ell)^{-\frac{2}{d}}\overline{\nabla}(Y)^2 \\ &= \left(|\det(\mathrm{Ad}(\ell)|_{\overline{\mathfrak{n}}})|^{\frac{n}{m}}\overline{\nabla}(Y)\right)^2.\end{aligned}$$

The corresponding statements for ∇ follow. □

Remark 2.11. The functions ∇ are closely related to the determinant functions associated to the Jordan algebras; see Table 2 in Appendix A.

3. Integral formulas for the orbits

As in Section 2, write the L-orbits in $\mathfrak{n}$ as

$$\mathcal{O}_q = L(X_q) \cong L/S_q,\ q = 0, 1, 2, \dots, n.$$

The orbit $\mathcal{O}_n$ is open and dense in $\mathfrak{n}$, and is a semisimple symmetric space. In particular $\mathcal{O}_n$ has a unique (up to scalar multiple) invariant measure, which we denote by ν_n. The semisimple symmetric space L/S_n has a decomposition in terms of $K\cap L$ and $B \equiv \exp(\mathfrak{b})$. We note that $\mathfrak{b}$ is a Cartan subspace in $\mathfrak{l}$ perpendicular to both $\mathfrak{l}\cap\mathfrak{k}$ and $\mathfrak{s}_n$. Therefore, the Mostow decomposition is

$$L = (K\cap L)B^+S_n, \tag{3.1}$$

with $B^+ = \exp(\mathfrak{b}^+)$, $\mathfrak{b}^+ = \{H\in\mathfrak{b} : \alpha(H)\geq 0,\ \text{for all } \alpha\in\Sigma^+(\mathfrak{l},\mathfrak{b})\}$. The invariant measure on $\mathcal{O}_n$ is of the form

$$\int_{\mathcal{O}_n} F(X)\,dX = \int_{K\cap L}\int_{\mathfrak{b}^+} F(kb\cdot X_n)\delta(b)\,dbdk. \tag{3.2}$$

Since $\mathcal{O}_n$ is open in $\mathfrak{n}$ we may express this measure in terms of Lebesgue measure dX on $\mathfrak{n}$. The formula is

$$\int_{\mathcal{O}_n} F(X)d\nu_n(X) = \int_{\mathfrak{n}} F(X)\nabla(X)^{-\frac{m}{n}}dX. \tag{3.3}$$

This is easily verified since the action of L on $\mathfrak{n}$ is linear.

The other orbits however do not have invariant measures. Instead, they have quasi-invariant measures ν_q, $q = 1, 2, \dots, n-1$. This means that for each q there is a character $\tilde{\chi} : L \to \mathbf{R}^\times$ so that

$$\int_{\mathcal{O}_q} F(\ell\cdot X)d\nu_q(X) = \tilde{\chi}(\ell)^{-1}\int_{\mathcal{O}_q} F(X)d\nu_q(X). \tag{3.4}$$

In general a quasi-invariant measure ν on a homogeneous space L/S corresponding to a character $\tilde{\chi}$ is determined by

$$\int_{L/S} F_f(\ell S)d\nu(\ell S) = \int_L \tilde{\chi}(\ell)f(\ell)d\ell \tag{3.5}$$

where $d\ell$ is a left invariant Haar measure on L and

$$F_f(\ell S) = \int_S f(\ell s)ds,$$

with ds equal to left invariant Haar measure on S. Letting Ξ denote the modular function for a Lie group, $\tilde{\chi}$-quasi-invariant measures exist if and only if $\tilde{\chi}$ is an extension of $\Xi_S\Xi_L^{-1}|_S$ to a character of L. The modular function for a Lie group is $\Xi(\cdot) = |\det(\mathrm{Ad}(\cdot))|^{-1}$. See [14, Section 33] for details on quasi-invariant measures. In our situation L is a reductive group, so its modular function is 1. We need to compute the modular function of S_q.

We first describe the stabilizers $S_q = \mathrm{Stab}_L(X_q)$ in some detail. Let $\mathfrak{n}_q$ be the $+2$-eigenspace of $\sum_{j=1}^q H_j$ in $\mathfrak{n}$. Then $\mathfrak{n}_q$ and $\overline{\mathfrak{n}}_q \equiv \theta(\mathfrak{n}_q)$ generate a semisimple Lie subalgebra $\mathfrak{g}_q$ for which the corresponding subgroup G_q satisfies 2.1; we let L_qN_q be the corresponding parabolic subgroup. Let $\mathfrak{b}_q \equiv \mathrm{span}_{\mathbf{R}}\{H_1,\dots,H_q\}$. Then $\Sigma(\mathfrak{l}_q,\mathfrak{b}_q) = \{\pm(\epsilon_j-\epsilon_k) : 1\leq j<k\leq q\}$. For each q there is another subalgebra $\mathfrak{n}'_{n-q}$ defined as the $+2$-eigenspace of $\sum_{j=q+1}^n H_j$. The corresponding subalgebras are denoted by $\mathfrak{b}'_{n-q}, \mathfrak{l}'_{n-q}$ and $\overline{\mathfrak{n}}'_{n-q}$. Then $\Sigma(\mathfrak{l}'_{n-q},\mathfrak{b}'_{n-q}) = \{\pm(\epsilon_j-\epsilon_k) : q+1\leq j<k\leq n\}$. The integers d and e for $\mathfrak{n}_q$ and $\mathfrak{n}'_{n-q}$ are the same as for $\mathfrak{n}$, unless $q=1$ (resp. $q=n-1$) in which case $d=0$ for $\mathfrak{n}_q$ (resp. $\mathfrak{n}'_{n-q}$). We set $m_q\equiv\dim(\mathfrak{n}_q)$ and $m'_{n-q}=\dim(\mathfrak{n}'_{n-q})$. Note that $\mathfrak{l}_q$ and $\mathfrak{l}'_{n-q}$ commute. The following is straightforward to check.

Lemma 3.6. *Let $S_q = Stab_L(X_q)$ be the stabilizer in L of X_q. Then the following statements hold.*

1. *$Q_q \equiv \{\ell\in L : \ell\cdot\mathfrak{n}_q\subset\mathfrak{n}_q\}$ is a parabolic subgroup of L.*
2. *$S_q\subset Q_q$.*
3. *Setting $\mathcal{O}_q(q)\equiv L_q(X_q)$, the open orbit of L_q in $\mathfrak{n}_q$, $Q_q/S_q\cong\mathcal{O}_q(q)$. The corresponding Mostow decomposition is $L_q=(K\cap L_q)B_q(S_q\cap L_q)$.*
4. *Define $N_q^L\equiv\exp(\mathfrak{n}_q^L)$, where $\Sigma(\mathfrak{n}_q^L,\mathfrak{b}_q)=\{\epsilon_i-\epsilon_j : 1\leq i\leq q<j\leq n\}$. Then $Q_q=L_qL'_{n-q}N_q^L$ (with N_q^L the nilradical) and $S_q=(L_q\cap S_q)L'_{n-q}N_q^L$.*

Corollary 3.7. *The modular function Ξ_{S_q} for S_q has differential $2dq\sum_{j=q+1}^n\epsilon_j$. Therefore Ξ_{S_q} extends to the character χ_q on L; this is the character for the quasi-invariant measure on $\mathcal{O}_q$.*

For $\mathfrak{n}_q$ (resp. $\mathfrak{n}'_{n-q}$) the functions defined in (2.9) are denoted by $\overline{\nabla}_q,\nabla_q$ (resp. $\overline{\nabla}'_{n-q},\nabla'_{n-q}$).

The standard integration formula in terms of the Mostow decomposition $L_q=(K\cap L_q)B_q(S_q\cap L_q)$ (see (3.2)) is

$$\int_{\mathcal{O}_q(q)} F(X)d\nu_q^q(X) = \int_{K\cap L_q}\int_{B_q^+} F(kb\cdot E_q)\delta_q(b)\,dbdk. \tag{3.8}$$

The exact form of δ_q, which is not needed here, is given in [23, Section 8.1]. An invariant measure ν_q^q on $\mathcal{O}_q(q)$ may also be given in terms of Lebesgue measure on $\mathfrak{n}_q$ (as in (3.3)) by

$$\int_{\mathcal{O}_q(q)} F(X)d\nu_q^q(X) = \int_{\mathcal{O}_q(q)} F(X)\nabla_q(X)^{-\frac{m_q}{q}}\,dX.$$

In the decomposition $Q_q=L_qL'_{n-q}N_q^L$ we may write $L_q=M_qA_q$ and $L'_{n-q}=M'_{n-q}A'_{n-q}$ so that the Langlands decomposition of Q_q is $M_qM'_{n-q}A_qA'_{n-q}N_q^L$.

Lemma 3.9. *For $q=1,2,\dots,n-1$ the following formula gives a χ_q-quasi invariant measure on $\mathcal{O}_q$.*

$$\int_{\mathcal{O}_q} F(X)\,d\nu_q(X) = \int_{K\cap L}\int_{B_q^+} F(kb\cdot E_q)\chi_n(b)\delta_q(b)\,dbdk.$$

Proof. The standard integration formula for a group in terms of a parabolic subgroup gives us

$$\int_L \chi_q(\ell) f(\ell)\, d\ell$$
$$= \int_{K\cap L}\int_{M_q M'_{n-q} A_q A'_{n-q}}\int_{N_q^L} \chi_q(aa') f(kmm'aa') \mathrm{e}^{2\rho(\mathfrak{n}_q^L)}(aa')\, dndada'dmdm'dk$$

Since $2\rho(\mathfrak{n}_q^L) = 2d((n-q)\sum_1^q \epsilon_j - q\sum_{q+1}^n \epsilon_j)$ we have $\chi_q(aa')\; e^{2\rho(\mathfrak{n}_q^L)}(aa') = e^{2dn\Lambda_{0,q}}(a)$, for $\Lambda_{0,q} = \sum_{j=1}^q \epsilon_j$. Thus,

$$\begin{aligned}
&\int_L \chi_q(\ell) f(\ell)\, d\ell\\
&= \int_{K\cap L}\int_{M_q A_q}\int_{M'_{n-q}A'_{n-q}}\int_{N_q^L} f(kmam'a'n) \mathrm{e}^{2dn\Lambda_{0,q}}(a) dndada'dmdm'dk\\
&= \int_{K\cap L}\int_{M'_{n-q}A'_{n-q}}\int_{M_qA_q/M_qA_q\cap S_q}\int_{M_qA_q\cap S_q}\int_{N_q^L} f(kmam'a'n) \mathrm{e}^{2dn\Lambda_{0,q}}(a)\, dndada'dmdm'dk\\
&= \int_{L\cap K}\int_{M'_{n-q}A'_{n-q}}\int_{S_q} f(kmas) \mathrm{e}^{2dn\Lambda_{0,q}} dsdmdadk\\
&= \int_{K\cap L}\int_{K\cap L_q}\int_{B_q^+}\int_{S_q} f(kk_1bs) \mathrm{e}^{2dn\Lambda_{0,q}}(b)\delta_q(b)\, dsdbdk_1dk\\
&= \int_{K\cap L}\int_{B_q^+} F_f(kb)\chi_q(b)^{\frac{n}{q}}\delta_q(b)\, dbdk.
\end{aligned}$$

The lemma follows from (3.5). □

Inserting the integration formula (3.8) for the dense orbit $\mathcal{O}_q(q) \subset \mathfrak{n}_q$ we obtain the following useful formula. We remark that normalizations of the Lebesgue measures on $\mathfrak{n}, \overline{\mathfrak{n}}$ and subspaces, and on $\mathcal{O}_q$ have not yet been given. We give normalizations of the Lebesgue measures just before Prop. 3.13 and the normalization of ν_q will be given in (5.3).

Corollary 3.10. *For $q = 1, 2, \ldots, n-1$*

$$\int_{\mathcal{O}_q} F(X)\, d\nu_q(X) = \int_{K\cap L}\int_{\mathfrak{n}_q} F(k\cdot Y)\nabla_q(Y)^{dn-\frac{m_q}{q}}\, dY dk.$$

Proof. Note that $\chi_n(kb) = \nabla_q(kb\cdot X_q)^{dn}$. We therefore have

$$\begin{aligned}
&\int_{\mathcal{O}_q} F(X) d\nu_q(X)\\
&= \int_{K\cap L}\int_{B_q^+} F(kb\cdot X_q)\chi_n(b)\delta_q(b) dbdk\\
&= \int_{K\cap L}\int_{\mathcal{O}_q(q)} F(k\cdot Y)\nabla_q(Y)^{nd}\nabla_q(Y)^{-\frac{m_q}{q}}\; dY dk
\end{aligned}$$

Since $\mathcal{O}_q(q)$ is dense in $\mathfrak{n}_q$ we may integrate over $\mathfrak{n}_q$. □

Corollary 3.11. *Let $\overline{\mathfrak{n}}_q^L = \theta(\mathfrak{n}_q^L)$. Then*

$$\int_{\mathcal{O}_q} F(X)\, d\nu_q(X) = C\int_{\overline{\mathfrak{n}}_q^L}\int_{\mathfrak{n}_q} F(\exp(u)\cdot X)\nabla_q(X)^{nd-\frac{m_q}{q}} dX du.$$

Proof. We convert the integral over $K\cap L$ in the preceding corollary into an integral over $\overline{\mathfrak{n}}_q^L$ using [10, Eq. 5.25]. We write the 'Iwasawa' decomposition of $\exp(u), u \in \overline{\mathfrak{n}}_q^L$ with respect to the parabolic Q_q as

$\exp(u) = \kappa(\exp(u)) m_u \exp(H(\exp(u)) n_u \in K M_q A_q N_q$. Then

$$\begin{aligned}
&\int_{\mathcal{O}_q} F(X)\, d\nu_q(X) \\
&\quad = \int_{\overline{\mathfrak{n}}_q^L} \int_{\mathfrak{n}_q} F(\kappa(\exp(u)\cdot X) \nabla_q(X)^{nd-\frac{m_q}{q}} \mathrm{e}^{-2\rho(\overline{\mathfrak{n}}_q^L)(H(\exp(u)))} dX du \\
&\quad = \int_{\overline{\mathfrak{n}}_q^L} \int_{\mathfrak{n}_q} F(\exp(u) \exp(H(\exp(u)))^{-1} m_u^{-1} n_u^{-1} \cdot X) \nabla_q(X)^{nd-\frac{m_q}{q}} \mathrm{e}^{-2\rho(\overline{\mathfrak{n}}_q^L)(H(\exp(u)))} dX du
\end{aligned}$$

Note that $\exp(H(\exp(u)))^{-1} m_u^{-1} n_u^{-1}$ stabilizes $\mathfrak{n}_q$. Therefore, by Lemma 2.10 the result of the change of variables $X \mapsto \exp(H(\exp(u)))^{-1} m_u^{-1} n_u^{-1} \cdot X$ on $\mathfrak{n}_q$ is

$$\begin{aligned}
\int_{\overline{\mathfrak{n}}_q^L} \int_{\mathfrak{n}_q} F(\exp(u)\cdot X) \nabla_q(X)^{nd-\frac{m_q}{q}} \big(\det(\mathrm{Ad}(\mathrm{e}^{H(\exp(u))}|_{\mathfrak{n}_q})^{\frac{q}{m_q}}\big)^{nd-\frac{m_q}{q}} \\
\mathrm{e}^{-2\rho(\overline{\mathfrak{n}}_q^L)(H(\exp(u)))} \det(\mathrm{Ad}(\mathrm{e}^{H(\exp(u))}|_{\mathfrak{n}_q}) dX du.
\end{aligned}$$

Claim: The terms involving $H(\exp(u))$ cancel out. This follows from

$$\begin{aligned}
&\big(\det(\mathrm{Ad}(\mathrm{e}^{H(\exp(u))}|_{\mathfrak{n}_q})^{\frac{q}{m_q}}\big)^{nd-\frac{m_q}{q}} \mathrm{e}^{-2\rho(\overline{\mathfrak{n}}_q^L)(H(\exp(u)))} \det(\mathrm{Ad}(\mathrm{e}^{H(\exp(u))}|_{\mathfrak{n}_q}) \\
&\quad = \mathrm{e}^{2qd\Lambda_0(H(\exp(u)))} = \chi_q(\exp(H(\exp(u)))) = 1,
\end{aligned}$$

since $H(\exp(u))$ is in the semisimple part of $\mathfrak{l}$ (as $\exp(u)$ is).

□

We will need another integration formula. This one relates Lebesgue measure on $\mathfrak{n}$ to Lebesgue measures on $\overline{\mathfrak{n}}_q^L$, $\mathfrak{n}_q$ and $\mathfrak{n}'_{n-q}$. We take care in normalizing these measures as we wish to obtain exact formulas later.

Let $B_{\mathfrak{g}}$, or simply B, denote the Killing form of $\mathfrak{g}$. Then B gives a nondegenerate pairing between $\mathfrak{n}$ and $\overline{\mathfrak{n}}$. Recall that m is the dimension of $\mathfrak{n}$. Set

$$\langle\ ,\ \rangle_{\mathfrak{g}} \equiv \frac{n}{4m} B_{\mathfrak{g}},$$

giving a nondegenerate pairing between $\mathfrak{n}$ and $\overline{\mathfrak{n}}$.

Fact : The restriction of $\langle\ ,\ \rangle_{\mathfrak{g}}$ to $\mathfrak{g}_q \times \mathfrak{g}_q$ is $\langle\ ,\ \rangle_{\mathfrak{g}_q}$. To see this note that, since $\mathfrak{g}_q$ is simple and $\langle\ ,\ \rangle_{\mathfrak{g}}$ is $\mathfrak{g}_q$-invariant, there is a nonzero constant c so that $\langle\ ,\ \rangle_{\mathfrak{g}} = c\langle\ ,\ \rangle_{\mathfrak{g}_q}$ on $\mathfrak{g}_q \times \mathfrak{g}_q$. To see that $c = 1$ compute

$$\langle H_1, H_1\rangle_{\mathfrak{g}} = \mathrm{trace}(ad(H_1)^2) = \frac{n}{4m} 8(d(n-1)+(e+1)) = 2,$$

(since $m = n(d(n-1)+(e+1))$) and

$$\langle H_1, H_1\rangle_{\mathfrak{g}_q} = \mathrm{trace}(ad(H_1)^2|_{\mathfrak{g}_q}) = \frac{q}{4m_q} 8(d(q-1)+(e+1)) = 2.$$

We will use $\langle\ ,\ \rangle$ to denote $\langle\ ,\ \rangle_{\mathfrak{g}}$ for any $\mathfrak{g}$. The significance of the above fact is that when we pass from $\mathfrak{n}$ to $\mathfrak{n}_q$ in the induction arguments below the pairing $\langle\ ,\ \rangle$ is unchanged.

As the form $-B(\,\cdot\,,\theta(\,\cdot\,))$ is positive definite on $\mathfrak{g}$, $\langle ,\rangle$ defines a inner products on $\mathfrak{n}$. Denote this inner product by $\langle X_1, X_2\rangle_\theta = -\langle X_1, \theta(X_2)\rangle$. The corresponding norms

$$||X||^2 = \langle X, X\rangle_\theta = -\langle X, \theta(X)\rangle, X \in \mathfrak{n}, \quad \text{and} \quad ||Y||^2 = \langle Y, Y\rangle_\theta = -\langle Y, \theta(Y)\rangle, Y \in \overline{\mathfrak{n}}.$$

determine Lebesgue measures dX on $\mathfrak{n}$ and dY on $\overline{\mathfrak{n}}$ in a standard way.

Define

$$\begin{aligned}
\tilde{\mathfrak{n}} &= \sum \mathfrak{g}_{\epsilon_j+\epsilon_k}, \quad 1 \le j \le q < k \le n, \text{ and} \\
\overline{\tilde{\mathfrak{n}}} &= \sum \mathfrak{g}_{-(\epsilon_j+\epsilon_k)}, \quad 1 \le j \le q < k \le n.
\end{aligned}$$

The inner product $\langle\, ,\, \rangle_\theta$ is nondegenerate on each of $\widetilde{\mathfrak{n}}, \widetilde{\overline{\mathfrak{n}}}$ and $\overline{\mathfrak{n}}_q^L$. Give $\widetilde{\mathfrak{n}}, \widetilde{\overline{\mathfrak{n}}}$ and $\overline{\mathfrak{n}}_q^L$ the Lebesgue measures determined by the restriction of $\langle\, ,\, \rangle_\theta$ to each subspace.

Lemma 3.12. *For $X \in \mathfrak{n}_q$ (resp. $Y' \in \overline{\mathfrak{n}}'_{n-q}$) with $\nabla_q(X) \neq 0$ (resp. $\overline{\nabla}_{n-q}(Y') \neq 0$) $ad(X) : \overline{\mathfrak{n}}_q^L \to \widetilde{\mathfrak{n}}$ (resp. $ad(Y') : \overline{\mathfrak{n}}_q^L \to \widetilde{\overline{\mathfrak{n}}}$ is invertible. For $\nabla_q(X) \neq 0$ and $H \in L^1(\widetilde{\mathfrak{n}})$*

$$\nabla_q(X)^{2d(n-q)} \int_{\widetilde{\mathfrak{n}}} H(z)dz = \int_{\overline{\mathfrak{n}}_q^L} H(ad(X)u)du$$

and for $\overline{\nabla}_{n-q}(Y') \neq 0$ and $F \in L^1(\widetilde{\overline{\mathfrak{n}}})$

$$\overline{\nabla}_{n-q}(Y')^{2dq} \int_{\widetilde{\overline{\mathfrak{n}}}} F(z')dz' = \int_{\overline{\mathfrak{n}}_q^L} F(ad(Y')u)du$$

Proof. Suppose $X \in \mathfrak{n}_q$ and $\nabla_q(X) \neq 0$. We may write $X = \ell \cdot X_q, X_q = \sum_{j=1}^q E_j$. Fix orthonormal bases of $\widetilde{\mathfrak{n}}$ and $\overline{\mathfrak{n}}_q^L$. We compute the determinant of $T \equiv \mathrm{ad}(X) : \overline{\mathfrak{n}}_q^L \to \widetilde{\mathfrak{n}}$ with respect to these bases. First we check that $T \equiv \mathrm{ad}(X_q) : \overline{\mathfrak{n}}_q^L \to \widetilde{\mathfrak{n}}$ preserves inner products (so has determinant 1).

Observe that

$$\mathrm{ad}(\theta X_q)\mathrm{ad}(X_q)u = [\theta X_q, [X_q, u]] = [u, [\theta X_q, X_q]] = \sum_{j=1}^q [H_j, u] = -u,$$

therefore, $-\mathrm{ad}(\theta X_q)\mathrm{ad}(X_q)$ is the identity on $\overline{\mathfrak{n}}_q^L$. Now

$$\begin{aligned}\langle \mathrm{ad}(X_q)u, \mathrm{ad}(X_q)v\rangle_\theta &= \langle \mathrm{ad}(X_q)u, \theta(\mathrm{ad}(X_q)v)\rangle \\ &= \langle -\mathrm{ad}(\theta(X_q))\mathrm{ad}(X_q)u, \theta(v)\rangle \\ &= \langle u, v\rangle_\theta\end{aligned}$$

We can now calculate the determinant of T with respect to the orthonormal bases of $\overline{\mathfrak{n}}_q^L$ and $\widetilde{\mathfrak{n}}$.

$$\begin{aligned}\det(T) &= \det(\mathrm{Ad}(\ell)|_{\widetilde{\mathfrak{n}}}) \det(\mathrm{ad}(X) : \overline{\mathfrak{n}}_q^L \to \widetilde{\mathfrak{n}}) \det(\mathrm{Ad}(\ell^{-1}))|_{\overline{\mathfrak{n}}_q^L} \\ &= \chi(\ell)^{n-q} \cdot 1 \cdot \chi(\ell^{-1})^{-(n-q)} \\ &= \chi(\ell)^{2(n-q)} \\ &= \nabla_q(X)^{2d(n-q)}.\end{aligned}$$

Similarly, for $\overline{\nabla}_{n-q}(Y') \neq 0$, $\mathrm{ad}(Y') : \overline{\mathfrak{n}}_q^L \to \widetilde{\overline{\mathfrak{n}}}$ is an isometry and $\det(\mathrm{ad}(Y') : \overline{\mathfrak{n}}_q^L \to \widetilde{\overline{\mathfrak{n}}}) = \overline{\nabla}_{n-q}(Y')^{2dq}$. □

Proposition 3.13. *Let $f \in L^1(\mathfrak{n})$ and $h \in L^1(\overline{\mathfrak{n}})$. Then*

$$\int_{\mathfrak{n}} f(X)dX = \int_{\overline{\mathfrak{n}}_q^L} \int_{\mathfrak{n}_q} \int_{\mathfrak{n}'_{n-q}} f(\exp(u) \cdot (X + X'))\nabla_q(X)^{2d(n-q)} dX dX' du$$

$$\int_{\overline{\mathfrak{n}}} h(Y)dY = \int_{\overline{\mathfrak{n}}_q^L} \int_{\overline{\mathfrak{n}}_q} \int_{\overline{\mathfrak{n}}'_{n-q}} h(\exp(u) \cdot (Y + Y'))\overline{\nabla}_{n-q}(Y')^{2dq} dY dY' du.$$

Proof. We prove only the first formula, the second is proved by essentially the same argument. Since $\overline{Q}_q S_q = \overline{N}_q^L Q_q$ is dense in L and the complement is of measure zero,

$$\overline{N}_q^L L_q L'_{n-q}(X_q + X'_{n-q}) \subset \overline{N}_q^L \cdot (\mathfrak{n}_q + \mathfrak{n}'_{n-q})$$

have full measure in $\mathfrak{n}$. We compute the Jacobian of the transformation

$$\begin{aligned}\phi :\ & \mathfrak{n}_q \times \overline{\mathfrak{n}}_q^L \times \mathfrak{n}'_{n-q} \to \mathfrak{n} \\ & (X, u, X') \mapsto \exp(u) \cdot (X + X')\end{aligned}$$

and see that ϕ is a diffeomorphism on the open subset of $(X,u,X') \in \mathfrak{n}_q \times \overline{\mathfrak{n}}_q^L \times \mathfrak{n}'_{n-q}$ where $\nabla_q(X) \neq 0$.

Note that $\mathfrak{n} = \mathfrak{n}_q + \widetilde{\mathfrak{n}} + \mathfrak{n}'_{n-q}$. The matrix of the differential of ϕ at (X,u,X') with respect to orthonormal bases of $\mathfrak{n}_q, \overline{\mathfrak{n}}_q^L, \mathfrak{n}'_{n-q}$ and $\mathfrak{n}_q, \widetilde{\mathfrak{n}}, \mathfrak{n}'_{n-q}$ has the form

$$\begin{pmatrix} I & 0 & 0 \\ * & T & 0 \\ * & * & I \end{pmatrix}$$

where $T = \mathrm{ad}(X) : \overline{\mathfrak{n}}_q^L \to \widetilde{\mathfrak{n}}$. Therefore the Jacobian is $\det(T) = \nabla_q(X)^{2d(n-q)}$ as computed in Lemma 3.12. $\square$

The following lemma, which is essentially only a restatement of the of what is proved above, will be used in Sections 4 and 5 Let $\pi_q : \mathfrak{n} \to \mathfrak{n}_q$ be the orthogonal projection (with respect to $\langle\, ,\, \rangle_\theta$). Set $\mathcal{U}_q = \{X \in \mathfrak{n} : \nabla_q(X) \neq 0\}$, an open set in $\mathfrak{n}$.

Lemma 3.14. *For $X \in \mathfrak{n}_q$ with $\nabla_q(X) \neq 0$, ad$(X) : \overline{\mathfrak{n}}_q^L \to \widetilde{\mathfrak{n}}$ is invertible. The change of coordinates $\phi : (X,u,X') \mapsto \exp(u)(X+X')$ is a smooth one-to-one map $\mathfrak{n}_q \times \overline{\mathfrak{n}}_q^L \times \mathfrak{n}'_{n-q} \to \mathfrak{n}$ and is a diffeomorphism from $\mathcal{O}_q(q) \times \overline{\mathfrak{n}}_q^L \times \mathfrak{n}'_{n-q}$ onto $\mathcal{U}_q$. The Jacobian is $\nabla_q(X)^{2d(n-q)}$.*

Proof. The Jacobian is computed in the proof of Lemma 3.13. To see that ϕ is a bijection $\mathcal{O}_q(q) \times \overline{\mathfrak{n}}_q^L \times \mathfrak{n}'_{n-q} \to \mathcal{U}_q$ write $\exp(u)(X+X') = X + \mathrm{ad}(u)X + (X' + \frac{1}{2}\mathrm{ad}(u)^2(X)) \in \mathfrak{n}_q + \widetilde{\mathfrak{n}} + \mathfrak{n}'_{n-q}$ and use the fact that ad(X) is invertible when $\nabla_q(X) \neq 0$. $\square$

We end this section with two facts we will need later.

Lemma 3.15. *For $\alpha \in \mathbf{C}$*

$$\int_{\overline{\mathfrak{n}}} e^{-\alpha\Lambda_0(H(\overline{n}_Y))} dY$$

is finite for Re$(\alpha) > \frac{m}{n} + d(n-1)$.

Proof. This is a standard argument. See [10, Cor. 7.7]. $\square$

We are also interested in the local integrability of powers of $\nabla(X)$ and $\overline{\nabla}(Y)$. For this we apply the standard integration formula for a semisimple symmetric space (as in equation (3.2)). The invariant measure in the polar coordinates $(K \cap L)B^+$ is of the form $\delta(b)dbdk$. It follows from [23, page 149] that $|\delta(b)| \leq \mathrm{e}^{2\rho(\mathfrak{l})}(b)$.

Lemma 3.16. *For Re$(\alpha) > -(e+1)$ the function $\nabla(X)^\alpha$ (resp. $\overline{\nabla}(Y)^\alpha$) is a locally L_1 function on $\mathfrak{n}$ (resp. $\overline{\mathfrak{n}}$) and defines a tempered distribution.*

Proof. We use the polar coordinates $(K \cap L)B^+$ and formulas (3.2) and (3.3) to check that

$$\int_{\mathfrak{b}_c^+} \nabla(b_t \cdot X_n)^{\mathrm{Re}(\alpha)+\frac{m}{n}} \delta(b_t)\, dt < \infty,$$

where

$$\mathfrak{b}_c^+ = \{t \in \mathbf{R}^n : c > t_1 > t_2 \cdots\}$$

and

$$b_t = \exp(\sum_{j=1}^{n} t_j H_j).$$

By Lemma 2.10 (b)

$$\nabla(b_t \cdot X_n) = \prod_{j=1}^{n} \mathrm{e}^{2t_j}.$$

Now

$$\int_{\mathfrak{b}_c^+} \nabla(b_t \cdot X_n)^{\mathrm{Re}(\alpha)+\frac{m}{n}} \mathrm{e}^{2\rho(\mathfrak{l})}\, dt = \int_{c>t_1>\cdots>t_n} \prod_j \mathrm{e}^{(\mathrm{Re}(\alpha)+\frac{m}{n})2t_j+2d(n-2j+1)t_j} dt_1 \ldots dt_n < \infty$$

for $\mathrm{Re}(\alpha) + \frac{m}{n} + d(n-2j+1) > 0$ for $j = 0, 1, \ldots, n$, i.e., $\mathrm{Re}(\alpha) > -\frac{m}{n} + d(n-1) = -(e+1)$. $\square$

4. Functional equation

The functions $\nabla(X)^s$ and $\overline{\nabla}(Y)^s$ are locally L_1 functions for $\mathrm{Re}(s) > -(e+1)$ by Lemma 3.16. Therefore tempered distributions are defined by the integrals

$$\mathbf{Z}(h,s) = \int_{\mathfrak{n}} h(X)\nabla(X)^s\, dX, \text{ for } h \in \mathcal{S}(\mathfrak{n}) \text{ and } \overline{\mathbf{Z}}(f,s) = \int_{\overline{\mathfrak{n}}} f(X)\overline{\nabla}(X)^s\, dX, \text{ for } f \in \mathcal{S}(\overline{\mathfrak{n}}). \tag{4.1}$$

Here $\mathcal{S}(\mathfrak{n})$ (resp. $\mathcal{S}(\overline{\mathfrak{n}})$) denotes the space of Schwartz functions on $\mathfrak{n}$ (resp. $\overline{\mathfrak{n}}$). Note that in the range $\mathrm{Re}(s) > -(e+1)$ both expressions are complex analytic functions of s. We will see that there is a meromorphic continuation to all of $\mathbf{C}$ and a functional equation relating the two distributions via the Fourier transform.

The fact that there is a meromorphic continuation is well-know ([21], [25] and [22]). The explicit functional equation (Theorem 4.4) below has been studied in various forms. See [15] and [4]. Theorem 4.4 below computes the coefficients which occur in the functional equation and is a special case of Proposition 3 in [15]. When G is a complex group it is contained in [3, Theorem 3.16]. Most of the statements in this section are contained in [15]; we include the details of the proofs since we will need much of the setup and many of the formulas which arise.

By Lemma 2.10 $P(X) \equiv \nabla(X)^2$ and $\overline{P}(Y) \equiv \overline{\nabla}(Y)^2$ are polynomials. They define constant coefficient differential operators characterized by

$$P(\partial_X)\mathrm{e}^{\langle X,Y\rangle} = \overline{P}(Y)\mathrm{e}^{\langle X,Y\rangle} \text{ and } \overline{P}(\partial_Y)\mathrm{e}^{\langle X,Y\rangle} = P(X)\mathrm{e}^{\langle X,Y\rangle}.$$

There is a polynomial $b(s)$ ([2]) so that

$$P(\partial_X)\nabla(X)^s = b(s)\nabla(X)^{s-2} \text{ and } \overline{P}(\partial_Y)\overline{\nabla}(Y)^s = b(s)\overline{\nabla}(Y)^{s-2}.$$

In particular, for $b_k(s) \equiv b(s)b(s-2)b(s-4)\cdots b(s-2(k-1))$,

$$P(\partial_X)^k\nabla(X)^s = b_k(s)\nabla(X)^{s-2k} \text{ and } \overline{P}(\partial_Y)^k\overline{\nabla}(Y)^s = b_k(s)\overline{\nabla}(y)^{s-2k}.$$

It follows that

$$\mathbf{Z}(P(\partial_X)^k h, s) = b_k(s)\mathbf{Z}(h, s-2k) \tag{4.2}$$

and

$$\overline{\mathbf{Z}}(\overline{P}(\partial_Y)^k f, s) = b_k(s)\overline{\mathbf{Z}}(f, s-2k)$$

for $\mathrm{Re}(s) \gg 0$. Since the left hand side is analytic for $\mathrm{Re}(s) > -(e+1)$, $\mathbf{Z}(h,s)$ and $\overline{\mathbf{Z}}(f,s)$ continue to meromorphic functions on $\mathrm{Re}(s) > -(e+1) - 2k$ for any k. Let $\{\alpha_j\}$ be the set of roots of $b(s)$ and set $S = \{\alpha_j - 2l : l \in \mathbf{Z}_+\}$. We may conclude the following lemma.

Lemma 4.3. *For $h \in \mathcal{S}(\mathfrak{n})$ (resp. $f \in \mathcal{S}(\overline{\mathfrak{n}})$) $Z(h,s)$ (resp. $\overline{\mathbf{Z}}(f,s)$) has a meromorphic continuation with S the set of potential poles.*

We now turn to the Fourier transform and functional equation. Recall the definition of the pairing $\langle\, ,\, \rangle$ given in Section 3. Define the Fourier transforms by

$$\widehat{h}(Y) = \int h(X) \mathrm{e}^{-2\pi i \langle Y, X\rangle}\, dX, \text{ for } h \in \mathcal{S}(\mathfrak{n})$$

and

$$\widehat{f}(X) = \int f(Y) \mathrm{e}^{-2\pi i \langle Y, X\rangle}\, dY, \text{ for } f \in \mathcal{S}(\overline{\mathfrak{n}}).$$

The normalization of Lebesgue measure is so that Fourier inversion is

$$\widehat{\widehat{h}}(X) = h(-X) \text{ and } \widehat{\widehat{f}}(Y) = f(-Y).$$

The main result of this section is the following theorem. We let Γ denote the gamma function on $\mathbf{C}$.

Theorem 4.4. ([15]) Let $s \in \mathbf{C}$ and $f \in \mathcal{S}(\overline{\mathfrak{n}})$. As meromorphic functions

$$\frac{\pi^{\frac{ns}{2}}}{\Gamma_n(s)} \mathbf{Z}(\widehat{f}, s - \frac{m}{n}) = \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_n(-s+\frac{m}{n})} \overline{\mathbf{Z}}(f, -s) \tag{4.5}$$

where

$$\Gamma_n(s) \equiv \prod_{j=0}^{n-1} \Gamma(\frac{s - jd}{2}).$$

We begin the proof with a few preliminary propositions. The first step is Weil's integration formula ([27]).

Recall the subalgebras of $\mathfrak{g}$ defined in Section 3:

$$\begin{aligned}
\mathfrak{n}_q &= \sum \mathfrak{g}_{\epsilon_i+\epsilon_j}, \text{ for } 1 \le i, j \le q \\
\mathfrak{n}'_{n-q} &= \sum \mathfrak{g}_{\epsilon_i+\epsilon_j}, \text{ for } q < i, j \le n \\
\overline{\mathfrak{n}}_q &= \sum \mathfrak{g}_{-\epsilon_i-\epsilon_j} \text{ for } 1 \le i, j \le q \\
\overline{\mathfrak{n}}'_{n-q} &= \sum \mathfrak{g}_{-\epsilon_i-\epsilon_j} \text{ for } q < i, j \le \\
\overline{\mathfrak{n}}^L_q &= \sum \mathfrak{g}_{-\epsilon_i+\epsilon_j}, \text{ for } 1 \le i \le q < j \le n \\
\widetilde{\mathfrak{n}} &= \sum \mathfrak{g}_{\epsilon_i+\epsilon_j}, \text{ for } 1 \le i \le q < j \le n \\
\widetilde{\overline{\mathfrak{n}}} &= \sum \mathfrak{g}_{-\epsilon_i-\epsilon_j} \text{ for } 1 \le i \le q < j \le n.
\end{aligned} \tag{4.6}$$

Observe that for $u \in \overline{\mathfrak{n}}^L_q$

$$\begin{aligned}
&ad(u) : \mathfrak{n}_q \to \widetilde{\mathfrak{n}}, \\
&ad(u) : \widetilde{\mathfrak{n}} \to \mathfrak{n}'_{n-q}, \\
&ad(u) : \mathfrak{n}'_{n-q} \to 0.
\end{aligned} \tag{4.7}$$

Therefore,

$$\mathrm{Ad}(\exp((u)) = I + ad(u) + \frac{1}{2}(ad(u))^2 \tag{4.8}$$

on $\mathfrak{n}_q$ and

$$ad(u)^2 : \mathfrak{n}_q \to \mathfrak{n}'_{n-q}.$$

Define the Fourier transform on $\overline{\mathfrak{n}}^L_q$ by

$$\widehat{\phi}(v) = \int_{\overline{\mathfrak{n}}^L_q} \phi(u) \mathrm{e}^{-2\pi i \langle u, v\rangle_\theta}\, du$$

for $\phi \in \mathcal{S}(\overline{\mathfrak{n}}_L)$. The normalize Lebesgue measure du on $\overline{\mathfrak{n}}^L_q$ given in Section 3 guarantees that $\widehat{\widehat{\phi}}(u) = \phi(-u)$.

Proposition 4.9. ([27], [17]) Suppose that $X \in \mathfrak{n}_q$ with $\nabla_q(X) \neq 0$ and $Y' \in \overline{\mathfrak{n}}'_{n-q}$ with $\overline{\nabla}_{n-q}(Y') \neq 0$. Then for each $\phi \in \mathcal{S}(\overline{\mathfrak{n}}_q^L)$

$$\int_{\overline{\mathfrak{n}}_q^L} \int_{\overline{\mathfrak{n}}_q^L} \phi(v-u) \mathrm{e}^{\pi i \langle Y', ad(u)^2 X\rangle}\, dudv = \nabla_q(X)^{-d(n-q)} \overline{\nabla}_{n-q}(Y')^{-qd} \int_{\overline{\mathfrak{n}}_q^L} \phi(u)\, du.$$

Proof. We follow Sections 13 and 14 in [27]. The first step is to see that

$$q(u) \equiv q_{X,Y'}(u) \equiv \mathrm{e}^{\frac{1}{2} i \langle Y', (ad(u))^2(X)\rangle}$$

is a nondegenerate character of the second degree[2], that is q satisfies

$$q(u+v)q(u)^{-1}q(v)^{-1} = \mathrm{e}^{i\langle u, \rho_{X,Y'}(v)\rangle} \tag{4.10}$$

for some linear isomorphism $\rho_{X,Y'} : \overline{\mathfrak{n}}_q^L \to \theta(\overline{\mathfrak{n}}_q^L)$.

Since $[u,v] = 0$, $ad(u)ad(v) = ad(v)ad(u)$. Therefore

$$\begin{aligned}&\langle Y', \frac{1}{2}(ad(u)ad(v) + ad(v)ad(u))(X)\rangle = \langle Y', ad(u)ad(v)X\rangle = -\langle ad(u)Y', ad(v)X\rangle \\ &= -\langle ad(Y')u, ad(X)v\rangle = \langle u, ad(Y')ad(X)v\rangle.\end{aligned}$$

Therefore, taking $\rho_{X,Y'} = ad(Y')ad(X)$ we see that (4.10) holds. We also need to check that $\rho_{X,Y'}$ is a linear isomorphism. By Lemma 3.12 $ad(X) : \overline{\mathfrak{n}}_q^L \to \widetilde{\mathfrak{n}}$ has determinant $\nabla_q(X)^{2d(n-1)}$. Similarly the determinant of $ad(Y') : \widetilde{\mathfrak{n}} \to \theta(\overline{\mathfrak{n}}_q^L)$ is $\overline{\nabla}_{n-q}(Y')^{2dn}$. Therefore, q is nondegenerate for X and Y' satisfying $\nabla_q(X) \neq 0$ and $\overline{\nabla}_{n-q}(Y') \neq 0$. It also follows that the Jacobian term appearing in [27, Cor. 2] is $\nabla_q(X)^{-d(n-q)}\overline{\nabla}_{n-q}(Y')^{-dq}$.

To complete the proof we compute the 'γ term' in [27, Cor. 2]. A formula for γ is given in [18, Prop. 1-6] in terms of the signature of $\langle\,\cdot\,, \rho_{X,Y'}(\,\cdot\,)\rangle = \langle\,\cdot\,, \theta\rho_{X,Y'}(\,\cdot\,)\rangle_\theta$. The signature of $\rho_{X,Y'}$ is independent of X, Y' (with $\nabla_q(X)\overline{\nabla}_{n-q}(Y') \neq 0$), so we will assume that

$$X = \sum_{j=1}^{q} E_j \quad \text{and} \quad Y' = \sum_{j=q+1}^{n} \theta(E_j).$$

Let $\rho = \rho_{X,Y'}$. Then we claim that the eigenvalues of $\theta\rho = \theta\rho_{X,Y'}$ restricted to $\overline{n}_q^L$ are ± 1, each occurring with the same multiplicity. It follows from this that $\gamma(\rho) = 1$.

To see that the claim holds we show that $\theta\rho = \theta\tau$ (with τ as in (2.7)), then note that the root multiplicities in the 1 and -1 eigenspaces of $\theta\tau$ are all equal (to d), as observed in [6, Section 1.2].

Both $\theta\rho$ and $\theta\tau$ preserve $\mathfrak{a}$-root spaces so it is enough to check that $\rho\theta = \tau\theta$ on root spaces for $-\epsilon_i + \epsilon_k$, $i \leq q < k$. Let Z be in such a root space, then

$$\rho(Z) = [E_k, \theta[E_i, Z]].$$

On the other hand,

$$\tau\theta(Z) = \mathrm{Ad}\big(\exp(\frac{\pi}{2}(E_i + \theta(E_i))\exp(\frac{\pi}{2}(E_k + \theta(E_k))\big)\theta(Z).$$

Now,

$$\mathrm{Ad}\big(\frac{\pi}{2}(E_k + \theta(E_k))\big)\theta(Z) = \cos(\frac{\pi}{2})\theta(Z) + \sin(\frac{\pi}{2})[E_k, \theta(Z)] = [E_k, Z],$$

[2]See [27, pages 145-146] for the definition of a nondegenerate character of the second degree.

so

$$\begin{aligned}\tau\theta(Z) &= \mathrm{Ad}\big(\exp(\frac{\pi}{2}(E_i+\theta(E_i))\big)([E_k,\theta(Z)]) \\ &= \cos(\frac{\pi}{2})[E_k,\theta(Z)] + \sin(\frac{\pi}{2})[\theta(E_i),[E_k,\theta(Z)]] \\ &= [\theta(E_i),[E_k,\theta(Z)]] \\ &= \rho(Z).\end{aligned}$$

This completes the proof. □

For $f \in \mathcal{S}(\overline{\mathfrak{n}})$ define $\overline{T}_f \in \mathbf{C}^\infty(\overline{\mathfrak{n}}_q \times \overline{\mathfrak{n}}'_{n-q})$ by

$$\overline{T}_f(Y,Y') = \int_{\overline{\mathfrak{n}}_q^L} f(\exp(u)(Y+Y'))du$$

and for $h \in \mathcal{S}(\mathfrak{n})$ define $T_h \in \mathbf{C}^\infty(\mathfrak{n}_q \times \mathfrak{n}'_{n-q})$ by

$$T_h(X,X') = \int_{\overline{\mathfrak{n}}_q^L} h(\exp(u)(X+X'))du.$$

The functions T_h and $\overline{T}_f$ are not defined everywhere, for example the integral defining $T_h(0,0)$ is not convergent (when $h(0,0) \neq 0$). The following lemma shows, among other things, that T_h and $\overline{T}_f$ (for Schwartz functions h and f) are defined almost everywhere. Before stating the lemma we make several observations.

As noted in Lemma 3.12 $\mathrm{ad}(X) : \overline{n}_q^L \to \widetilde{\mathfrak{n}}$ is invertible when $X \in \mathfrak{n}_q$ and $\nabla_q(X) \neq 0$, i.e., when $X \in \mathcal{O}_q(q)$. This immediately provides us with an estimate in terms of the operator norm of $\mathrm{ad}(X)$:

$$\frac{1}{||\mathrm{ad}(X)^{-1}||_{\mathrm{op}}}||u|| \leq ||\mathrm{ad}(X)(u)|| \leq ||\mathrm{ad}(X)||_{\mathrm{op}}||u||.$$

Since $||\mathrm{ad}(X)||_{\mathrm{op}}$ and $||\mathrm{ad}(X)^{-1}||_{\mathrm{op}}$ are continuous on $\mathcal{O}_q(q)$ we may conclude that for every compact set $\Omega \subset \mathcal{O}_q(q)$ there are constants $C', C'' > 0$ such that

$$C'||u|| \leq ||\mathrm{ad}(X)(u)|| \leq C''||u||, \text{ for each } X \in \Omega. \tag{4.11}$$

It follows that for each $N \in \mathbf{N}$ there exists a constant C so that

$$(1+||\mathrm{ad}(X)(u)||)^{-N} \leq C(1+||u||)^{-N}, \text{ for each } X \in \Omega. \tag{4.12}$$

Similarly for $Y' \in \overline{\mathfrak{n}}'_{n-q}$ with $\overline{\nabla}'_{n-q}(Y') \neq 0$, $\mathrm{ad}(Y') : \overline{\mathfrak{n}}_q^L \to \overline{\widetilde{\mathfrak{n}}}$ is invertible and given a compact set $\Omega' \subset \{Y' \in \overline{\mathfrak{n}}'_{n-q} : \overline{\nabla}'_{n-q}(Y') \neq 0\}$ and an $N \in \mathbf{N}$ there is a constant C so that

$$(1+||\mathrm{ad}(Y')(u)||)^{-N} \leq C(1+||u||)^{-N}, \text{ for each } Y' \in \Omega'. \tag{4.13}$$

Lemma 4.14. *Let $h \in \mathcal{S}(\mathfrak{n})$ and $f \in \mathcal{S}(\overline{\mathfrak{n}})$. Then the following statements hold.*
(a) The integral defining $T_h(X,X')$ (respectively, $\overline{T}_f(Y,Y')$) is finite when $\nabla_q(X)$ (respectively, $\overline{\nabla}'_{n-q}(Y')$) is nonzero. In particular, T_h and $\overline{T}_f$ are defined almost everywhere.
(b) For $X \in \mathcal{O}_q(q)$, $h(X + ad(u)(X) + X')$ is a Schwartz function in the variables $u \in \overline{\mathfrak{n}}_q^L$ and $X' \in \mathfrak{n}'_{n-q}$.
(c) Let $\Omega \subset \mathcal{O}_q(q)$ be compact and let $N \in \mathbf{N}$. Then there is a constant C so that

$$T_h(X,X') \leq C(1+||X'||)^{-N}, \textit{ for all } X \in \Omega. \tag{4.15}$$

Furthermore, $T_h(X,\cdot\,) \in \mathcal{S}(\mathfrak{n}'_{n-q})$ for $\nabla_q(X) \neq 0$.

Proof. (a) Let $\nabla_q(X) \neq 0$. Then, for any $N \in \mathbf{N}$ there are constants C_1 and C so that

$$\begin{aligned}|h(\exp(u)(X+X'))| &= |h(X+\mathrm{ad}(u)X+(X'+\frac{1}{2}\mathrm{ad}(u)^2(X)))| \\ &\leq C_1(1+||\mathrm{ad}(u)X||)^{-N}, \text{ since } h \text{ is Schwartz in each variable,} \\ &\leq C(1+||u||)^{-N}, \text{ by (4.11).}\end{aligned}$$

Choosing N large enough, this is an L^1 function on $\overline{\mathfrak{n}}_q^L$. The corresponding statement for $\overline{T}_f$ follows from (4.13).

(b) The linear change of coordinates $\mathrm{ad}(X) : \overline{\mathfrak{n}}_q^L \to \widetilde{\mathfrak{n}}$ sends Schwartz functions to Schwartz functions.

(c) Since h is Schwartz in each of the $\widetilde{\mathfrak{n}}, \mathfrak{n}_q$ and $\mathfrak{n}'_{n-q}$ variables, for each $M, N \in \mathbf{N}$ there is a constant C_1 so that

$$|h(\exp(u)(X+X')| \leq C_1(1+||\mathrm{ad}(u)(X)||)^{-M}(1+||X'+\frac{1}{2}\mathrm{ad}(u)^2(X)||)^{-N}.$$

Therefore,

$$|T_h(X,X')| \leq C_1(1+||X'||)^{-N}\int_{\overline{\mathfrak{n}}_q^L}\Big(\frac{1+||X'||}{1+||X'+\frac{1}{2}\mathrm{ad}(u)^2(X)||}\Big)^N(1+||\mathrm{ad}(u)X||)^{-M}\,du.$$

To bound the integrand we use the triangle inequality:

$$\begin{aligned}1+||X'|| &\leq 1+||X'+\frac{1}{2}\mathrm{ad}(u)^2(X)||+||\frac{1}{2}\mathrm{ad}(u)^2(X)|| \\ &\leq (1+||X'+\frac{1}{2}\mathrm{ad}(u)^2(X)||)(1+||\frac{1}{2}\mathrm{ad}(u)^2(X)||).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1+||X'||}{1+||X'+\frac{1}{2}\mathrm{ad}(u)^2(X)||} &\leq 1+\frac{1}{2}||\mathrm{ad}(u)||_{\mathrm{op}}||\mathrm{ad}(u)X|| \\ &\leq (1+\frac{1}{2}||\mathrm{ad}(u)||_{\mathrm{op}})(1+||\mathrm{ad}(u)X||) \\ &\leq C_2(1+||u||)(1+||\mathrm{ad}(u)X||), \text{ by continuity of ad} \\ &\leq C_3(1+||u||)^2, \text{ by (4.11).}\end{aligned}$$

Now choose $M = 3N$, then

$$\begin{aligned}T_h(X,X') &\leq C_3(1+||X'||)^{-N}\int_{\overline{\mathfrak{n}}_{n-q}^L}(1+||u||)^{2N}(1+||u||)^{-M}\,du \\ &\leq C(1+||X'||)^{-N}.\end{aligned}$$

That $T_h(X,\cdot\,)$ is Schwartz follows from the same estimate applied to $\mathfrak{n}'_{n-q}$-derivatives of h.

□

Let $\mathcal{F}_q$ (respectively, $\mathcal{F}'_{n-q}$) denote the Fourier transform in the $\mathfrak{n}_q$-variable (respectively, the $\mathfrak{n}'_{n-q}$-variable).

Proposition 4.16. ([15], [17]) For $h \in \mathcal{S}(\mathfrak{n}), Y \in \overline{\mathfrak{n}}_q$ and $Y' \in \overline{\mathfrak{n}}'_{n-q}$ with $\overline{\nabla}_{n-q}(Y') \neq 0$

$$\overline{T}_{\widehat{h}}(Y,Y') = \overline{\nabla}_{n-q}(Y')^{-qd}\mathcal{F}_q\big(\nabla_q(\,\cdot\,)^{d(n-q)}\mathcal{F}_{n-q}(T_h)\big)(Y,Y').$$

Proof. Let $h \in \mathcal{S}(\mathfrak{n})$ and $\overline{\nabla}'_{n-q}(Y') \neq 0$. Then the following integrals converge by part (a) of Lemma 4.14.

$$\begin{aligned}
\overline{T}_{\widehat{h}}(Y,Y') &= \int_{\overline{\mathfrak{n}}_q^L} \widehat{h}(\exp(u)(Y+Y'))du \\
&= \int_{\overline{\mathfrak{n}}_q^L} \Big(\int_{\mathfrak{n}} h(X) \mathrm{e}^{-2\pi i \langle X, \exp(u)(Y+Y')\rangle} dX \Big) du \\
&= \int_{\overline{\mathfrak{n}}_q^L} \Big(\int_{\mathfrak{n}_q} \int_{\overline{\mathfrak{n}}_q^L} \int_{\mathfrak{n}'_{n-q}} h(\exp(v)(X+X')) \mathrm{e}^{-2\pi i \langle \exp(v)(X+X'), \exp(u)(Y+Y')\rangle} \nabla_q(X)^{2d(n-q)} dX' dv dX \Big) du.
\end{aligned} \tag{4.17}$$

We would like to switch the order of integration of the X and u variables. The orthogonality relations $\widetilde{\mathfrak{n}} \perp (\mathfrak{n}_q + \mathfrak{n}'_{n-q})$, $\widetilde{\overline{\mathfrak{n}}} \perp (\overline{\mathfrak{n}}_q + \overline{\mathfrak{n}}'_{n-q})$, $\mathfrak{n}'_{n-q} \perp \overline{\mathfrak{n}}_q$ and $\overline{\mathfrak{n}}'_{n-q} \perp \mathfrak{n}_q$ in (4.7) and (4.8) give

$$\begin{aligned}
\langle \exp(v)(X+X'), \exp(u)(Y+Y')\rangle = \langle X' + \frac{1}{2} ad(v)^2(X), Y'\rangle + \langle Y, X\rangle + \\
\frac{1}{2}\langle X, ad(u)^2(Y')\rangle + \langle ad(u)(Y'), ad(v)(X)\rangle.
\end{aligned}$$

Moreover, observe that

$$\begin{aligned}
&\Big| \int_{\overline{\mathfrak{n}}_q^L} \int_{\mathfrak{n}'_{n-q}} h(\exp(v)(X+X')) \mathrm{e}^{-2\pi i \langle \exp(v)(X+X'), \exp(u)(Y+Y')\rangle} \nabla_q(X)^{2d(n-q)} dX' dv \Big| \\
&= \nabla_q(X)^{2d(n-q)} \Big| \int_{\overline{\mathfrak{n}}_q^L} \int_{\mathfrak{n}'_{n-q}} h(X + ad(v)X + X') \mathrm{e}^{-2\pi i \langle X', Y'\rangle} \mathrm{e}^{-2\pi i \langle ad(u)Y', ad(v)X\rangle} dX' dv \Big| \\
&= \Big| \int_{\widetilde{\mathfrak{n}}} \int_{\mathfrak{n}'_{n-q}} h(X + z + X')) \mathrm{e}^{-2\pi i \langle Y', X'\rangle} \mathrm{e}^{-2\pi i \langle ad(u)Y', z\rangle} dX' \Big| dz, \\
&\qquad \text{by applying the change of variables } z = -\mathrm{ad}(X)(v) \text{ and Lemma 3.14,} \\
&= \Big| \int_{\widetilde{\mathfrak{n}}} \mathcal{F}_{n-q}(h(X + z + \cdot))(Y') \mathrm{e}^{-2\pi i \langle ad(u)Y', z\rangle} dz \Big| \\
&= |(\mathcal{F}_z \mathcal{F}_{n-q} h(X + \cdot + \cdot)(\mathrm{ad}(u)Y', Y')|.
\end{aligned}$$

Where $\mathcal{F}_z$ is the Fourier transform in the $\widetilde{\mathfrak{n}}$ variable. Since h is Schwartz in each of the three variables so are the Fourier transforms. Therefore we may bound the above expression by

$$C'(1 + ||X||)^{-N}(1 + ||\mathrm{ad}(u)Y'||)^{-N},$$

which is bounded by

$$C(1 + ||X||)^{-N}(1 + ||u||)^{-N}$$

for some C (depending on Y') by (4.12). Now Fubini's Theorem allows us rewrite the expression (4.17):

$$\begin{aligned}
&\overline{T}_{\widehat{h}}(Y,Y') = \\
&= \int_{\mathfrak{n}_q} \nabla_q(X)^{2d(n-q)} \Big(\int_{\overline{\mathfrak{n}}_q^L} \int_{\overline{\mathfrak{n}}_q^L} \{ \int_{\mathfrak{n}'_{n-q}} h(\exp(v-u)(X+X')) \mathrm{e}^{2\pi i \langle X+X', \exp(u)(Y+Y')\rangle} dX' \} dv du \Big) dX = \\
&= \int_{\mathfrak{n}_q} \nabla_q(X)^{2d(n-q)} \mathrm{e}^{2\pi i \langle Y, X\rangle} \Big(\int_{\overline{\mathfrak{n}}_q^L} \int_{\overline{\mathfrak{n}}_q^L} \{ \int_{\mathfrak{n}'_{n-q}} h(\exp(v-u)(X+X')) \mathrm{e}^{2\pi i \langle X', Y'\rangle} \mathrm{e}^{2\pi i \langle X, \frac{1}{2} ad(u)^2 Y'\rangle} dX' \} dv du \Big) dX = \\
&= \int_{\mathfrak{n}_q} \nabla_q(X)^{2d(n-q)} \mathrm{e}^{2\pi i \langle Y, X\rangle} \Big(\int_{\overline{\mathfrak{n}}_q^L} \int_{\overline{\mathfrak{n}}_q^L} \phi_{X,Y'}(v-u) \mathrm{e}^{2\pi i \langle X, \frac{1}{2} ad(u)^2 Y'\rangle} dv du \Big) dX,
\end{aligned}$$

where

$$\phi_{X,Y'}(u) = \int_{\mathfrak{n}'_{n-q}} h(\exp(u)(X+X') \mathrm{e}^{2\pi i \langle Y', X'\rangle} dX'.$$

In order to apply Weil's formula (Prop. (4.9)) we need to know that for $X \in \mathcal{O}_q(q)$, $\phi_{X,Y'} \in \mathcal{S}(\overline{\mathfrak{n}}_q^L)$. To check this we apply (4.11):

$$\begin{aligned}
|\phi_{X,Y'}(u)| &= \Big| \int_{\mathfrak{n}'_{n-q}} h(X + \mathrm{ad}(u)X + (X' + \frac{1}{2}\mathrm{ad}(u)^2(X)))\mathrm{e}^{2\pi i\langle Y',X'\rangle} dX' \Big| \\
&= \Big| \int_{\mathfrak{n}'_{n-q}} h(X + \mathrm{ad}(u)X + X')\mathrm{e}^{2\pi i\langle Y',X' - \frac{1}{2}\mathrm{ad}(u)^2(X)\rangle} dX' \Big| \\
&\leq \int_{\mathfrak{n}'_{n-q}} |h(X + \mathrm{ad}(u)X + X')| dX' \\
&\leq C' \int_{\mathfrak{n}'_{n-q}} (1 + ||X'||)^{-M}(1 + ||\mathrm{ad}(u)X||)^{-N} dX' \\
&\leq C(1 + ||u||)^{-N}, \text{ by (4.12).}
\end{aligned}$$

Derivatives in the $\overline{\mathfrak{n}}_q^L$ directions also satisfy this type of estimate since these derivatives are just derivatives of h times polynomials in X and u.

Now we may apply Prop. 4.9. Suppose $\overline{\nabla}'_{n-q}(Y') \neq 0$.

$$\begin{aligned}
\overline{T}_{\widehat{h}}(Y,Y') &= \overline{\nabla}_{n-q}(Y')^{-dq} \int_{\mathfrak{n}_q} \nabla_q(X)^{d(n-q)} \mathrm{e}^{2\pi i\langle Y,X\rangle} \Big(\int_{\overline{\mathfrak{n}}_q^L} \int_{\mathfrak{n}'_{n-q}} h(\exp(u)(X + X'))\mathrm{e}^{2\pi i\langle Y',X'\rangle} dX' du \Big) dX \\
&= \overline{\nabla}_{n-q}(Y')^{-dq} \int_{\mathfrak{n}_q} \nabla_q(X)^{d(n-q)} \mathrm{e}^{2\pi i\langle Y,X\rangle} \Big(\int_{\mathfrak{n}'_{n-q}} \int_{\overline{\mathfrak{n}}_q^L} h(\exp(u)(X + X'))\mathrm{e}^{2\pi i\langle Y',X'\rangle} du dX' \Big) dX, \\
&\qquad \text{by Fubini's Theorem (using part (b) of Lemma 4.14),} \\
&= \overline{\nabla}_{n-q}(Y')^{-qd} \mathcal{F}_q(\nabla_q(\cdot)^{d(n-q)} \mathcal{F}_{n-q}(T_h))(Y,Y')
\end{aligned}$$

□

Now fix some q satisfying $1 \leq q \leq n-1$ and recall that $\mathcal{U}_q = \{X \in \mathfrak{n} : \nabla_q(\pi_q(X)) \neq 0\}$, an open set in $\mathfrak{n}$. Let $C_c^\infty(\mathfrak{n})$ be the space of smooth functions on $\mathfrak{n}$ having compact support. Suppose $h \in C_c^\infty$ and $\mathrm{supp}(h) \subset \mathcal{U}_q$. Then $h(\exp(u)(X + X') = h(X + \mathrm{ad}(u)X + (X' + \frac{1}{2}\mathrm{ad}(u)^2 X))$ is smooth and has compact support contained in $\mathcal{O}_q(q) \times \overline{\mathfrak{n}}_q^L \times \mathfrak{n}'_{n-q}$, by Lemma 3.14. This implies the following lemma.

Lemma 4.18. *If $h \in C_c^\infty$ and supp$(h) \subset \mathcal{U}_q$ then $T_h \in C_c^\infty(\mathfrak{n}_q \times \mathfrak{n}'_{n-q})$. The support of T_h is contained in $\mathcal{O}_q(q) \times \mathfrak{n}'_{n-q}$.*

Recall that $\mathcal{O}_n = \{X \in \mathfrak{n} : \nabla(X) \neq 0\}$.

Lemma 4.19. *Let $1 \leq q \leq n-1$ and suppose that the support of h is compact and contained in $\mathcal{U}_q \cap \mathcal{O}_n$. Then the following hold.*

(1) *$Z_{n-q}(T_h(X, \cdot), s)$ is a smooth function of compact support in the X-variable.*
(2) *$Z_n(h, s) = Z_q(\nabla_q^{2d(n-q)} Z_{n-q}(T_h, s), s)$, as meromorphic functions of s.*

Proof. (1) Note that if $\exp(u)(X + X') \in \mathcal{U}_q \cap \mathcal{O}_n$ then $\nabla_{n-q}(X') \neq 0$. Therefore

$$Z_{n-q}(T_h(X,\cdot), s) = \int_{\mathfrak{n}'_{n-q}} T_h(X,X')\nabla_{n-q}(X')^s \, dX'$$

for all $X \in \mathfrak{n}_q$ and all $s \in \mathbf{C}$. By differentiating inside the integral we see that $Z_{n-q}(T_h(X,\cdot), s)$ is smooth. By Lemma (4.18), $Z_{n-q}(T_h(X,\cdot), s)$ is of compact support.

(2) All integrals below converge for all $s \in \mathbf{C}$.

$$
\begin{aligned}
Z_n(h,s) &= \int_{\mathfrak{n}} h(X)\nabla(X)^s dX \\
&= \int_{\mathfrak{n}_q}\int_{\mathfrak{n}'_{n-q}}\int_{\overline{\mathfrak{n}}_q^L} h(\exp(u)(X+X'))\nabla(\exp(u)(X+X'))^s\nabla(X)^{2d(n-q)}du dX' dX, \\
&\qquad \text{by Lemma 3.13,} \\
&= \int_{\mathfrak{n}_q}\int_{\mathfrak{n}'_{n-q}} T_h(X,X')\nabla_q(X)^{s+2d(n-q)}\nabla_{n-q}(X')^s dX dX' \\
&= \int_{\mathfrak{n}_q} Z_{n-q}(T_h(X,\cdot),s)\nabla_q(X)^{s+2d(n-q)}dX \\
&= Z_q(\nabla(\cdot)^{2d(n-q)}Z_{n-q}(T_h,s),s).
\end{aligned}
\tag{4.20}
$$

The last line makes sense by (1). ☐

Remark 4.21. The following refinement of the above lemma holds and will be used in Section 5. Suppose that the hypothesis of the lemma is weakened to $\mathrm{supp}(h) \subset \mathcal{U}_q$. Then for $\mathrm{Re}(s) \gg 0$

$$
Z_n(h,s) = \int_{\mathfrak{n}_q} Z_{n-q}(T_h(X,\cdot),s)\nabla_q(X)^{s+2d(n-q)}dX \tag{4.22}
$$

as in (4.20). By formula (B.1) and meromorphic continuation $Z_{n-q}(T_h(X,\cdot),s)$ has compact support for $s \notin S$ (= set of potential poles). Furthermore, formula (B.1) shows that $Z_{n-q}(T_h(X,\cdot),s)$ is defined by an integral and is smooth in X. Therefore, the integrand in (4.22) is smooth with compact support away from $\nabla_q = 0$. Thus, the right hand side of (4.22) is

$$
Z_q(\nabla(\cdot)^{2d(n-q)}Z_{n-q}(T_h,s),s)
$$

for all s. Therefore, by meromorphic continuation statement (2) of the lemma holds as meromorphic functions.

We now give the proof of Theorem 4.4. By [22][Thm. 1] there is a functional equation of the form

$$
\overline{Z}(\widehat{h}, s-\frac{m}{n}) = \beta_n(s)Z(h,-s), \quad h \in \mathcal{S}(\mathfrak{n})
$$

for some meromorphic function β_n. Therefore, to prove the explicit functional equation stated in Prop. 4.4 we need to show

$$
\beta_n(s) = \pi^{-ns+\frac{m}{2}}\frac{\Gamma_n(s)}{\Gamma_n(-s+\frac{m}{n})}. \tag{4.23}
$$

We proceed by induction on the rank of $\mathfrak{n}$. For $\mathrm{rank}(\mathfrak{n}) = 1$ the the formula (4.23) is contained in Prop. C.2.

Assume (4.23) holds for all $\mathfrak{n}$ of rank less than n. It suffices to assume that $h \in C_c^\infty(\mathfrak{n})$ and $\mathrm{supp}(h) \subset \mathcal{U}_q \cap \mathcal{O}_n$. We may take $q = 1$. Assume that $\mathrm{Re}(s) \gg 0$, the integrals below converge.

$$
\begin{aligned}
Z(\widehat{h}, s-\frac{m}{n}) &= \int_{\overline{\mathfrak{n}}} \widehat{h}(Y)\overline{\nabla}(Y)^{s-\frac{m}{n}}dY \\
&= \int_{\overline{\mathfrak{n}}_1}\int_{\overline{\mathfrak{n}}'_{n-1}}\int_{\overline{\mathfrak{n}}_1^L} \widehat{h}(\exp(u)(Y+Y'))\overline{\nabla}(\exp(u)(Y+Y'))^{s-\frac{m}{n}}\overline{\nabla}_{n-1}(Y')^{2d}dudY'dY \\
&\quad \text{by Lemma 3.13,} \\
&= \int_{\overline{\mathfrak{n}}_1}\int_{\overline{\mathfrak{n}}'_{n-1}} \overline{T}_{\widehat{h}}(Y+Y')\overline{\nabla}_1(Y)^{s-\frac{m}{n}}\overline{\nabla}_{n-1}(Y')^{s-\frac{m}{n}+2d}dY'dY \\
&= \int_{\overline{\mathfrak{n}}_1}\int_{\overline{\mathfrak{n}}'_{n-1}} \mathcal{F}_1(\nabla(\cdot)^{d(n-1)}\mathcal{F}_{n-1}(T_h))\overline{\nabla}(Y)_1^{s-\frac{m}{n}}\overline{\nabla}_{n-1}(Y')^{s-\frac{m}{n}+d}dY'dY, \\
&\quad \text{by Prop. 4.16,} \\
&= \overline{Z}_1\Big(\mathcal{F}_1(\nabla(\cdot)^{d(n-1)}\overline{Z}_{n-1}(\mathcal{F}_{n-1}(T_h), s-\frac{m_{n-1}}{n-1}), s-\frac{m}{n}\Big), \\
&\quad \text{since the order of integration over } \overline{\mathfrak{n}}'_{n-1} \text{ and the integral defining } \mathcal{F}_1 \\
&\quad \text{may be interchanged by Lemma 4.18,} \\
&= \beta_1(s)\beta_{n-1}(s-(n-1)d)Z_1\Big(\nabla_1(\cdot)^{d(n-1)}Z_{n-1}(T_h,-s), -(s-\frac{m}{n}+e+1)\Big) \\
&= \pi^{-ns+\frac{m}{2}}\frac{\Gamma_n(s)}{\Gamma_n(-s+\frac{m}{n})}Z_1\Big(Z_{n-1}(T_h,-s), -s+2d(n-1)\Big), \text{ by induction and (4.24)} \\
&= \pi^{-ns+\frac{m}{2}}\frac{\Gamma_n(s)}{\Gamma_n(-s+\frac{m}{n})}\mathbf{Z}_n(h,-s), \text{ by Lemma 4.19 (with } q=1).
\end{aligned}
$$

We have used the following fact, which is an easy calculation.

$$
\frac{\pi^{\frac{1}{2}(-s+(e+1))}\Gamma_1(s)}{\pi^{\frac{s}{2}}\Gamma_1(-s+(e+1))}\frac{\pi^{\frac{n-1}{2}(-(s-d)+\frac{m_{n-1}}{n-1})}\Gamma_{n-1}(s-d)}{\pi^{\frac{n-1}{2}(s-d)}\Gamma_{n-1}(-(s-d)+\frac{m_{n-1}}{n-1})} = \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}\Gamma_n(s)}{\pi^{\frac{ns}{2}}\Gamma_n(-s+\frac{m}{n})}. \tag{4.24}
$$

The formula for $\beta_n(s)$ now follows by meromorphic continuation and Theorem 4.4 is proved.

A few important consequences of the Theorem (and proof) follow.

Corollary 4.25. $b(s) = \prod_{j=0}^{n-1}(s+jd)(s+(e-1)+jd)$. *Therefore the set of potential poles of* $Z(h,s)$ *is* $S = S' \cup S'' = \{-jd+2l : l \in \mathbf{Z}_+\} \cup \{jd - \frac{m}{n} + 2l : l \in \mathbf{Z}_+\}$.

Proof. Use $\overline{P}(\partial_Y)f$ in place of f and $-s$ in place of s in the functional equation. The

$$
\begin{aligned}
\mathbf{Z}(\widehat{\overline{P}(\partial_Y)f}, -s-\frac{m}{n}) &= \mathbf{Z}((2\pi i)^{2n}P(X)\widehat{f}, -s-\frac{m}{n}) \\
&= (-1)^n(2\pi)^{2n}\mathbf{Z}(\widehat{f}, -s+2-\frac{m}{n}) \\
&= (-1)^n 2^{2n}\pi^{\frac{m}{2}+n(s)}\frac{\Gamma_n(-s+2)}{\Gamma_n(s-2+\frac{m}{n})}\overline{\mathbf{Z}}(f, s-2).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathbf{Z}(\widehat{\overline{P}(\partial_Y)}f, -s-\frac{m}{n}) &= \pi^{\frac{m}{2}+ns}\frac{\Gamma_n(-s)}{\Gamma_n(s+\frac{m}{n})}\overline{\mathbf{Z}}(\overline{P}(\partial_Y)f, s) \\
&= \pi^{\frac{m}{2}+ns}\frac{\Gamma_n(-s)}{\Gamma_n(s+\frac{m}{n})}b(s)\overline{\mathbf{Z}}(f, s-2).
\end{aligned}
$$

Therefore,

$$
b(s) = 2^{2n}\frac{\Gamma_n(s+\frac{m}{n})\Gamma_n(-s+2)}{\Gamma_n(s-2+\frac{m}{n})\Gamma_n(-s)}.
$$

Now the corollary follows from the formula

$$\frac{\Gamma_n(u+2)}{\Gamma_n(u)} = \prod_{j=0}^{n-1} \frac{u-dj}{2}.$$

□

Corollary 4.26. $\frac{1}{\Gamma_n(s+\frac{m}{n})} Z(h,s)$ *is an analytic function of* s *on the complement of* $[-(n-1)d, -(e+1)] \cap S$.

Proof. If $\mathrm{Re}(s) > -(e+1)$ then $Z(h,s)$ is defined by an integral and is therefore analytic in this range. Applying the functional equation we get analyticity for $\mathrm{Re}(s) < -(n-1)d$. □

5. Special values of the zeta distributions

In this section we prove Theorem 5.12 which states that the normalized zeta distribution (5.13) is an entire function of s and at certain special values of s is a quasi-invariant measure on an L-orbit in $\mathfrak{n}$.

Recall from (3.7) that a quasi-invariant measure on the L-orbit $\mathcal{O}_q$ satisfies

$$\int_{\mathcal{O}_q} (\ell \cdot h)(X) d\nu_q(X) = \chi(\ell)^q \int_{\mathcal{O}_q} h(X) d\nu_q(X)$$

where $(\ell \cdot h)(X) = h(\ell^{-1} \cdot X)$. Zeta distributions satisfy the similar homogeneity property

$$Z(\ell \cdot h, s) = \chi(\ell)^{\frac{1}{d}(s+\frac{m}{n})} Z(h,s). \tag{5.1}$$

Lemma 5.2. *For* $h \in \mathcal{S}(\mathfrak{n})$ *the following hold.*

(1) $\frac{1}{\Gamma_n(s+\frac{m}{n})} Z(h,s)\Big|_{s=-\frac{m}{n}} = \frac{\pi^{\frac{m}{2}}}{\Gamma_n(\frac{m}{n})} h(0).$

(2) *Let* $x_1, \dots, x_k$ *be* $\mathfrak{a}$*-root vectors in* $\mathfrak{n}$*. Write* $H = \sum H_i$ *(with* H_i *as in Section 2) and* $a_t = \exp(tH)$*. Then*

$$(x_1 \cdots x_k \cdot \delta_0)(a_t \cdot h) = \chi(a_t)^{-\frac{k}{dn}} (x_1 \cdots x_k \cdot) \delta_0(h)$$

Proof. (1) This follows immediately from the functional equation (4.5), Fourier inversion and the fact that $\overline{\mathbf{Z}}(\widehat{h}, s)$ is defined at $s=0$ by an integral.

$$\begin{aligned} \frac{1}{\Gamma_n(s+\frac{m}{n})} Z(h,s)\big|_{s=-\frac{m}{n}} &= \frac{\pi^{\frac{m}{2}}}{\Gamma_n(\frac{m}{n})} \overline{\mathbf{Z}}(\widehat{h}, 0) \\ &= \frac{\pi^{\frac{m}{2}}}{\Gamma_n(\frac{m}{n})} \int_{\overline{\mathfrak{n}}} \widehat{h}(Y)\, dY \\ &= \frac{\pi^{\frac{m}{2}}}{\Gamma_n(\frac{m}{n})} h(0). \end{aligned}$$

(2) Observe that $\mathrm{ad}(H)(x_j) = 2x_j$ for each root vector x_j. Also note that for any tempered distribution T, $(x_j \cdot T)(\ell \cdot h) = \big((\mathrm{Ad}(\ell^{-1})x_j) \cdot T\big)(h)$, for any $\ell \in L$. Thus,

$$\begin{aligned} (x_1 \cdots x_k \cdot \delta_0)(a_t \cdot h) &= (\mathrm{Ad}(a_t^{-1})x_1 \cdots \mathrm{Ad}(a_t^{-1})x_k \cdot \delta_0)(h) \\ &= \mathrm{e}^{-2kt}(x_1 \cdots x_k \cdot \delta_0)(h) \\ &= \chi(a_t)^{-\frac{k}{dn}} (x_1 \cdots x_k \cdot \delta_0)(h). \end{aligned}$$

□

At this point we normalize the quasi invariant measures on $\mathcal{O}_q \subset \mathfrak{n}$. Recall that the Lebesgue measures on $\mathfrak{n}$ and $\overline{\mathfrak{n}}$, and subspaces $\mathfrak{n}_q, \overline{\mathfrak{n}}_q^L$, etc., are normalized in Section 3 in terms of the inner product $\langle\, , \rangle_\theta$. Let

$$C_q^n = \frac{\pi^{\frac{m_{n-q}}{2}}}{\Gamma_q(dq)\Gamma_{n-q}(\frac{m_{n-q}}{n-q})}.$$

We take ν_q to be normalized so that

$$\nu_0 = C_0^n \delta_0 \tag{5.3}$$

$$\int_{\mathcal{O}_q} F(X)\, d\nu_q(X) = C_q^n \int_{\overline{\mathfrak{n}}_q^L} \int_{\mathfrak{n}_q} F(\exp(u)\cdot X)\nabla_q(X)^{nd-\frac{m_q}{q}}\, dX du, \text{ for } q = 1, 2, \ldots, n-1. \tag{5.4}$$

.

Proposition 5.5. *Let ν_q be the quasi-invariant measure on $\mathcal{O}_q$, for $q = 0, 1, \ldots, n-1$. For h a smooth function of compact support*

$$\frac{1}{\Gamma_n(s+\frac{m}{n})} Z(h,s)\Big|_{s=-\frac{m}{n}+qd} = \int_{\mathcal{O}_q} f d\nu_q.$$

Proof. Our proof is based on Remark 4.21 and the formula for the quasi-invariant measure given in Cor. 3.11. We proceed by induction on n, the rank of $\mathfrak{n}$. If $n = 1$ the statement is well-known (and follows from Prop. C.2 or Lemma 5.2 (2)).

Suppose first that $h \in \mathcal{S}(\mathfrak{n})$ and has compact support in

$$\mathcal{U}_1 = \{X \in \mathfrak{n} : \nabla_1(\pi_1(X)) \neq 0\}$$

where $\pi_1 : \mathfrak{n} \to \mathfrak{n}_1$ is the orthogonal projection (with respect to $\langle\, , \rangle_\theta$). Then Remark (4.21) applies (with $q = 1$) to give

$$\frac{1}{\Gamma_n(s+\frac{m}{n})} Z(h,s)\Big|_{-\frac{m}{n}+qd} = \frac{1}{\Gamma_1(s+\frac{m}{n})} \int_{\mathfrak{n}_1} \frac{1}{\Gamma_{n-1}(s+\frac{m_{n-1}}{n-1})} Z_{n-1}(T_h(X,\cdot), s)\nabla_1(X)^{2d(n-1)+s}\, dX\Big|_{-\frac{m}{n}+qd}$$

$$= \frac{1}{\Gamma_1(dq)} \int_{\mathfrak{n}_1} \int_{\mathcal{O}_{q-1}(n-1)} T_h(X,X')\, d\nu_{q-1}^{n-1}(X')\nabla_1(X)^{2d(n-1)-\frac{m}{n}+qd} dX. \tag{5.6}$$

The last equality follows from the inductive hypothesis (since $-\frac{m}{n}+qd = -\frac{m_{n-1}}{n-1}+(q-1)d$). Here, $\mathcal{O}_{q-1}(n-1)$ is the orbit of L_{n-1} in $\mathfrak{n}'_{n-1}$ through $E_2 + \cdots E_q$ and ν_{q-1}^{n-1} is the quasi-invariant measure on this orbit.

In order to apply Cor. 3.13 we will use the following temporary notation. (This notation is consistent with (4.6) with $\mathfrak{n}$ replaced by $\mathfrak{n}_q$ or $\mathfrak{n}_{n-1}$.)

$$\mathfrak{n}'_{n-1,q-1} = \mathfrak{n}'_{q,q-1} = \sum \mathfrak{g}^{(\epsilon_i+\epsilon_j)}, \quad 2 \le i,j \le q$$
$$\overline{\mathfrak{n}}'^L_{n-1,q-1} = \sum \mathfrak{g}^{-(\epsilon_i-\epsilon_j)}, \quad 2 \le i \le q < j \le n$$
$$\overline{\mathfrak{n}}^L_{q,1} = \sum \mathfrak{g}^{-(\epsilon_1-\epsilon_j)}, \quad 2 \le j \le q.$$

Note that

$$\overline{\mathfrak{n}}_1^L = (\overline{\mathfrak{n}}_1^L \cap \overline{\mathfrak{n}}_q^L) \oplus \overline{\mathfrak{n}}_{q,1}^L \tag{5.7}$$

$$\overline{\mathfrak{n}}_q^L = (\overline{\mathfrak{n}}_1^L \cap \overline{\mathfrak{n}}_q^L) \oplus \overline{\mathfrak{n}}'^L_{n-1,q-1}. \tag{5.8}$$

The integration formula of Cor. 3.13 applied to $\mathfrak{n}_q$ is

$$\int_{\mathfrak{n}_q} g(X)\, dX = \int_{\mathfrak{n}_1} \int_{\overline{\mathfrak{n}}_{q,1}^L} \int_{\mathfrak{n}'_{q,q-1}} g(\exp(u_1)(X_1+X_2))\nabla_1(X_1)^{2d(q-1)}\, dX_1 du_1 dX_2. \tag{5.9}$$

One more bit of notation used below is that $\nabla_{n-1,q-1}$ is the ∇ function for $\mathfrak{n}'_{n-1,q-1} = \mathfrak{n}'_{q,q-1}$.

Continuing with (5.6) we have

$$
\begin{aligned}
\frac{1}{\Gamma_n(s+\frac{m}{n})} & Z(h,s)\Big|_{-\frac{m}{n}+qd} \\
&= \frac{1}{\Gamma_1(qd)} \int_{\mathfrak{n}_1} C^{n-1}_{q-1} \int_{\overline{\mathfrak{n}}'^{L}_{n-1,q-1}} \int_{\mathfrak{n}'_{n-1,q-1}} T_h(X, \exp(v)X') \nabla_{n-1,q-1}(X')^{d(n-1)-\frac{m_{q-1}}{q-1}} \\
&\qquad \nabla_1(X)^{2d(n-1)-\frac{m}{n}+qd}\, dX' dv dX \text{ by Cor. 3.11} \\
&= C^n_q \int_{\overline{\mathfrak{n}}^L_1 \cap \overline{\mathfrak{n}}^L_q} \int_{\overline{\mathfrak{n}}'^L_{n-1,q-1}} \Big(\int_{\overline{\mathfrak{n}}^L_{q,1}} \int_{\mathfrak{n}_1} \int_{\mathfrak{n}'_{q,q-1}} h(\exp(u_1+u_2)(X+\exp(v)X')) \\
&\qquad \nabla_1(X)^{2d(q-1)} \nabla_q(X+X')^{nd-\frac{m_q}{q}}\, dX' dX du_1 \Big) dv du_2, \text{ by (5.8)} \\
&= C^n_q \int_{\overline{\mathfrak{n}}^L_q} \int_{\mathfrak{n}_q} h(\exp(u)X) \nabla_q(X)^{nd-\frac{m_q}{q}}\, dX du, \text{ see below for verification} \qquad (5.10) \\
&= \int_{\mathcal{O}_q} h\, d\nu_q.
\end{aligned}
$$

To verify (5.10) note that for $u_1 \in \overline{\mathfrak{n}}^L_{q,1}, u_2 \in \overline{\mathfrak{n}}^L_1 \cap \overline{\mathfrak{n}}^L_q, v \in \overline{\mathfrak{n}}'^L_{n-1,q-1}$ and $X \in \mathfrak{n}_1$

(i) $\mathrm{Ad}(\exp(v)^{-1})u_1 = u_1 - [v,u_1]$, and $[v,u_1] \in \overline{\mathfrak{n}}^L_1 \cap \overline{\mathfrak{n}}^L_q$,
(ii) u_1 and $[v,u_1]$ commute,
(iii) u_1 and u_2 commute,
(iv) $u_2, [v,u_1]$ and v mutually commute and
(v) v commutes with X.

Therefore,

$$
\begin{aligned}
\exp(u_1+u_2)(X+\exp(v)X') &= \exp(u_2)\exp(v)\exp(\mathrm{Ad}(\exp(v)^{-1})u_1)(X+X') \\
&= \exp(v)\exp(u_2)\exp(u_1-[v,u_1])(X+X') \\
&= \exp(v)\exp(u_2-[v,u_1])\exp(u_1)(X+X').
\end{aligned}
$$

By translation invariance of du_2 the line before (5.10) has integrand $h(\exp(v+u_2)\exp(u_1)(X+X'))$. Now, line (5.10) follows from (5.8) and (5.9).

We have proved that

$$
\frac{1}{\Gamma_n(s+\frac{m}{n})} Z(h,s)\Big|_{s=-\frac{m}{n}+qd} = \int_{\mathcal{O}_q} h(X) d\nu_q(X) \tag{5.11}
$$

for smooth functions h having compact support in $\mathcal{U}_1$. Now let h have compact support in $\mathfrak{n} \setminus \{0\}$. Since each positive dimensional L-orbit in $\mathfrak{n}$ meets $\mathcal{U}_1$, $\mathfrak{n} \setminus \{0\} = \cup_{\ell \in L} \ell \cdot \mathcal{U}_1$. Therefore, by compactness of the support of h we may find $\ell_1, \dots, \ell_k \in L$ so that $\mathrm{supp}(h) \subset \cup_{j=1}^k \ell_j \cdot \mathcal{U}_1$.

Choose a smooth partition of unity $\{\phi_j\}$ subordinate to $\{\ell_j \cdot \mathcal{U}_1\}_{j=1,\dots,k}$. Thus, $h = \sum_{j=1}^k \phi_j h$ and $\operatorname{supp}(\ell_j^{-1} \cdot (\phi_j h)) \subset \mathcal{U}_1$. By (5.1)

$$\begin{aligned}
\frac{1}{\Gamma_n(s+\frac{m}{n})} Z(h,s)\Big|_{s=-\frac{m}{n}+qd} &= \sum_{j=1}^k \frac{1}{\Gamma_n(s+\frac{m}{n})} Z(\phi_j h, s)\Big|_{s=-\frac{m}{n}+qd} \\
&= \sum_{j=1}^k \frac{\chi(\ell_j)^q}{\Gamma_n(s+\frac{m}{n})} Z(\ell_j^{-1}\cdot \phi_j h, s)\Big|_{s=-\frac{m}{n}+qd} \\
&= \sum_{j=1}^k \chi(\ell_j)^q \int_{\mathcal{O}_q} (\ell_j^{-1}\cdot \phi_j h)(X)\, d\nu_q(X), \text{ by (5.11)} \\
&= \sum_{j=1}^k \int_{\mathcal{O}_q} (\phi_j h)(X)\, d\nu_q(X) \\
&= \int_{\mathcal{O}_q} h(X)\, d\nu_q(X)
\end{aligned}$$

It now follows that

$$\frac{1}{\Gamma_n(s+\frac{m}{n})} Z(h,s)\Big|_{s=-\frac{m}{n}+qd} - \nu_q$$

is a distribution supported at $\{0\}$. Lemma 5.2 shows that no (nonzero) distribution supported at $\{0\}$ satisfies the necessary homogeneity property. This concludes the proof. $\square$

The main theorem of this section is the following.

Theorem 5.12. *For each $h \in \mathcal{S}(\mathfrak{n})$*

$$\frac{1}{\Gamma_n(s+\frac{m}{n})} Z(h,s) \tag{5.13}$$

is an entire function of s and defines a family of tempered distributions. For $q = 0, 1, 2, \dots, n-1$

$$\frac{1}{\Gamma_n(s+\frac{m}{n})} Z(h,s)\Big|_{s=qd-\frac{m}{n}} = \nu_q.$$

Proof. In place of (5.13) we consider

$$\frac{1}{\Gamma_n(s)} Z(h, s-\frac{m}{n}). \tag{5.14}$$

By the general discussion in Appendix B this is an analytic family of tempered distributions for $s \notin S \equiv \{jd-2l\} \cup \{jd+(e+1)-2l\}$ with $j = 0, 1, \dots, n-1$ and $l \in \mathbf{Z}_+$. Furthermore, for s outside the interval $[e+1, (n-1)d]$ (5.14) is analytic by Cor. 4.26. By

$$\frac{Z(\nabla^{2k} h, s-\frac{m}{n})}{\Gamma_n(s)} = \Big(\prod_{j=0}^{n-1}\prod_{l=0}^{k-1} \frac{s-jd+2l}{2}\Big) \cdot \frac{Z(h, s+2k-\frac{m}{n})}{\Gamma_n(s+2k)}$$

we have

$$\frac{Z(h, s-\frac{m}{n})}{\Gamma_n(s)} = \frac{1}{\Big(\prod_{j=0}^{n-1}\prod_{l=0}^{k-1}\frac{s-jd+2l}{2}\Big)} \frac{Z(\nabla^{2k} h, s-2k-\frac{m}{n})}{\Gamma_n(s-2k)}.$$

Choosing k big enough so that $s-2k < e+1$ we see that (5.14) is analytic away from $S' = \{jd-2l\}$. Therefore we need only check analyticity at points s in $S' \cap [e+1, (n-1)d]$.

By Cor. B.7 all we need to show is that

$$\frac{1}{\Gamma_n(s)} Z(h, s-\frac{m}{n})\big|_{s=s_0} < \infty$$

for $s_0 \in S \cap [e+1,(n-1)d]$ for $h \in C_0^\infty(\mathfrak{n})$. By Prop. 5.5 this is the case when $s_0 = jd, j = 0, 1, \dots, n-1$ i.e., when

$$\frac{1}{\Gamma_n(s)} Z(h, s - \frac{m}{n})\big|_{s=jd} = \nu_j.$$

For $d = 1$ or 2, $S \cap [e+1,(n-1)d] = \{jd : j = 0, 1, \dots, n-1\} \cap [e+1,(n-1)d]$ and we are finished.

For $d > 2$ note that

$$\prod_{j=0}^{n-1}\prod_{l=0}^{k-1} \frac{s - jd + 2l}{2}$$

has no multiple roots for $k = 1, 2, \dots, \frac{d}{2}$ (for d even) and $k = 1, 2, \dots, d$ (for d odd). Furthermore, $S' \cap [e+1,(n-1)d] = \{jd + 2k : j = 0, 1, \dots, n-1$ and $k = 1, \dots, \frac{d}{2}$ (or d)$\}$.

Case 1. Suppose d is even and greater than 2. Let $k = 0, 1, \dots, \frac{d}{2}$. Now consider

$$\frac{Z(\nabla^{2k}h, s - \frac{m}{n})}{\Gamma_n(s)} = \Big(\prod_{j=0}^{n-1}\prod_{l=0}^{k-1} \frac{s - jd + 2l}{2}\Big) \cdot \frac{Z(h, s + 2k - \frac{m}{n})}{\Gamma_n(s + 2k)}$$

evaluated at $s = jd$. The left hand side is zero as $\nabla^{2k}h = 0$ on $\mathcal{O}_j$. Since the multiplicities of the zeroes of the polynomial on the right hand side are one has

$$\frac{Z(h, s + 2k - \frac{m}{n})}{\Gamma_n(s + 2k)}\Big|_{s=jd} < \infty.$$

Therefore (5.14) is finite for $s \in S \cap [e+1,(n-1)d]$.

Case 2. When d is odd and greater than 2 the proof is exactly as above except k ranges from 0 to d. $\square$

6. Positivity

Define a family of distributions on $\mathfrak{n}$ by

$$R_s(h) = \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_n(-s+\frac{m}{n})} Z(h, -s), \qquad h \in \mathcal{S}(\mathfrak{n}). \tag{6.1}$$

The following theorem is mostly a restatement of what has been proved earlier.

Theorem 6.2. *For R_s as defined above, the following hold.*

(1) *R_s is defined by the convergent integral*

$$R_s(h) = \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_n(-s+\frac{m}{n})} \int_{\mathfrak{n}} h(X)\nabla(X)^{-s}\, dX$$

for $Re(s) < e + 1$.

(2) *R_s is a holomorphic function of s on all of $\mathbf{C}$.*

(3) $R_{\frac{m}{n}} = \dfrac{\pi^{\frac{m}{2}}}{\Gamma_n(\frac{m}{n})}\delta_0.$

(4) *For $k = 0, 1, 2, \dots,$*

$$P(\partial_X)^k \delta_0 = \frac{2^{2nk}}{\pi^{(\frac{m}{n}-2k)\frac{n}{2}}} \Gamma_n(\frac{m}{n} + 2k) R_{\frac{m}{n}+2k}.$$

(5) *For $q = 0, 1, 2, \dots, n-1$, $R_{\frac{m}{n}-qd} = \nu_q$, a χ_q-quasi-invariant measure on $\mathcal{O}_q$.*

(6) *For $h \in C_c^\infty(\mathfrak{n})$ with $supp(h) \subset \mathcal{U}_q$*

$$R_s(h) = R^q_{s+(n-q)d}\Big(\nabla^{(n-q)d} R^{n-q}_s(T_h)\Big). \tag{6.3}$$

Our goal is to see precisely when R_s is a positive distribution in the sense that $f \geq 0$ implies $R_s(f) \geq 0$. For a given $\mathfrak{n}$ we set

$$\Xi_n = (-\infty, e+1) \cup \{\frac{m}{n} - qd : q = 0, 1, 2, \dots, n-1\}.$$

By the theorem the distributions R_s are positive if $s \in \Xi_n$. The following includes a converse.

Theorem 6.4. *If R_s is a positive distribution then $s \in \Xi_n$. The distribution $-R_s$ is never positive.*

Proof. By Proposition C.4 the theorem holds for the $n = 1$ cases. Now assume $n > 1$ and the statement holds for $n-1$ in place of n.

Case 1. $s = \dfrac{m}{n} + 2k, k = 0, 1, 2, \dots$. By Thm. 6.2, part (4)

$$R_{\frac{m}{n}+2k} = C_k P(\partial_X)\delta_0, \quad \text{where } C_k \text{ is a positive constant.}$$

This is positive if and only if $k = 0$, i.e., $s = \frac{m}{n} \in \Xi_n$.

Case 2. $s \in \mathbf{R} \setminus \{\frac{m}{n} + 2k : k = 0, 1, 2, \dots\}$. Recall that Lemma 3.14 provides coordinates $\phi : \mathcal{O}_q(q) \times \overline{\mathfrak{n}}_q^L \times \mathfrak{n}'_{n-q} \to \mathcal{U}_q$ and take $q = 1$. Define a smooth function h as follows. Choose nonnegative (and not identically zero) functions

$$\begin{aligned}
&\varphi \in C_c^\infty(\mathfrak{n}_1) \ \text{ with } \operatorname{supp}(\varphi) \subset \mathcal{U}_1 \\
&\psi \in C_c^\infty(\overline{\mathfrak{n}}_1^L) \\
&\varphi' \in C_c^\infty(\mathfrak{n}'_1).
\end{aligned}$$

Then

$$h(x) = \begin{cases} \varphi(X)\psi(u)\varphi'(X'), & \text{if } x = \phi(X, u, X') \\ 0, & \text{if } x \notin \operatorname{image}(\phi) \end{cases}$$

defines a nonnegative smooth function on $\mathfrak{n}$ of compact support with $\operatorname{supp}(h) \subset \mathcal{U}_1$. Furthermore,

$$T_h(X, X') = c\varphi(X)\varphi'(X'), \quad c_\psi = \int \psi(u)du > 0. \tag{6.5}$$

It follows from Thm. 6.2, part (6) that

$$R_s(h) = c_\psi\, R^1_{s-(n-1)d}(\varphi \cdot \nabla_1^{d(n-1)}) R_s^{n-1}(\varphi').$$

Since $\operatorname{supp}(\varphi)$ is compact and away from $\{0\}$ $(= \{X \in \mathfrak{n}_1 : \nabla_1(X) = 0\})$,

$$R^1_{s-(n-1)d}(\varphi\nabla_1^{(n-1)d}) = \frac{\pi^{(-s+\frac{m}{n})\frac{n}{2}}}{\Gamma_1(-s+\frac{m}{n})} \int_{\mathfrak{n}_1} \varphi(X)\nabla_1(X)^{-s+2d(n-1)}\, dX$$

for all s. In particular this is nonzero and has the same sign as $\Gamma_1(-s + \frac{m}{n})$. Therefore, there are positive constants $C_+(s)$ so that

$$R_s^{n-1}(\varphi') = C_+(s)\Gamma_1(-s + \frac{m}{n})R_s(h).$$

Now suppose that R_s is a positive distribution. If $\Gamma_1(-s + \frac{m}{n}) > 0$ then $R_s^{n-1}(\varphi') > 0$ for all $\varphi' \in C_c^\infty(\mathfrak{n}'_{n-1})$. Therefore, R_s^{n-1} is a positive distribution, so $s \in \Xi_{n-1}$. Therefore, $s \in \Xi_n$. However, if $\Gamma_1(-s + \frac{m}{n}) < 0$ then $-R_s^{n-1}$ is positive which contradicts the inductive hypothesis.

Now suppose $-R_s$ is a positive distribution. When $\Gamma_1(-s+\frac{m}{n}) > 0$, $-R_s^{n-1}$ will be a positive distribution, a contradiction to the inductive hypothesis. When $\Gamma_1(-s + \frac{m}{n}) < 0$, R_s^{n-1} is a positive distribution, so $s \in \Xi_{n-1}$. However, $\Gamma_1(-s + \frac{m}{n}) > 0$ when $s \in \Xi_{n-1}$ (since $-s + \frac{m}{n} > 0$ for $s \in \Xi_{n-1}$).

Case 3. $s \in \mathbf{C} \setminus \mathbf{R}$. We choose nonnegative functions in a similar manner as above, however the roles of $\mathfrak{n}_1$ and $\mathfrak{n}_{n-1}$ are switched (that is, we take $q = n-1$). Let

$$\varphi \in C_c^\infty(\mathfrak{n}_{n-1}) \text{ with } \operatorname{supp}(\varphi) \subset \mathcal{O}_{n-1}(n-1),$$
$$\psi \in C_c^\infty(\overline{\mathfrak{n}}_q^L) \text{ with } c_\psi = \int \psi(u)du > 0,$$
$$\varphi_1, \varphi_2 \in C_c^\infty(\mathfrak{n}_1) \text{ such that } \frac{R_s^1(\varphi_1)}{R_s^1(\varphi_2)} = \alpha_s \in \mathbf{C} \setminus \mathbf{R}.$$

Note that since φ has compact support in $\mathcal{O}_{n-1}(n-1)$

$$R_{s-d}^{n-1}(\nabla_{n-1}\varphi) = \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_{n-1}(-s+\frac{m}{n})} \int_{\mathfrak{n}_{n-1}} \varphi(X)\nabla_{n-1}(X)^{-s+d}dX$$

for all $s \in \mathbf{C}$. Since we are in the case where $s \in \mathbf{C} \setminus \mathbf{R}$, $\Gamma_{n-1}(-s+\frac{m}{n})$ has no poles. Therefore, $c_\varphi \equiv R_{s-d}^{n-1}(\nabla_{n-1}^d \cdot \varphi) \neq 0$.

To see that functions φ_1 and φ_2 exist note that the functions $\tilde{\varphi}_1(X') = ||X'||^2 \mathrm{e}^{-||X'||^2}$ and $\tilde{\varphi}_2(X') = \mathrm{e}^{-||X'||^2}$, $X' \in \mathfrak{n}_1$ are Schwartz functions so that

$$\frac{R_s^1(\tilde{\varphi}_1)}{R_s^1(\tilde{\varphi}_2)} = \frac{1}{2}(-s+e+1)$$

by (C.5). We may take φ_1 and φ_2 to be compactly supported functions approximating $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ sufficiently closely.

Again using the coordinates $\phi : (X, u, X') \to \exp(u)(X + X')$ we define smooth compactly supported functions by

$$h_i(x) = \begin{cases} \varphi(X)\psi(u)\varphi_i(X'), & \text{if } x = \phi(X, u, X') \\ 0, & \text{if } x \notin \operatorname{image}(\phi) \end{cases}$$

for $i = 1, 2$. By Thm. 6.2, part (6) we have

$$R_s(h_i) = c_\psi c_\varphi R_s^1(\varphi_i).$$

Now $R_s(h_i)$ cannot be positive (or negative) for both $i = 1$ and $i = 2$ since

$$\frac{R_s(h_1)}{R_s(h_2)} = \frac{R_s^1(\varphi_1)}{R_s^1(\varphi_2)} = \alpha_s \in \mathbf{C} \setminus \mathbf{R}.$$

Therefore, $\pm R_s$ is not a positive distribution for $s \in \mathbf{C} \setminus \mathbf{R}$. □

7. Generalized principal series representations

For each $s \in \mathbf{C}$ there is a normalized principal series representation

$$\operatorname{Ind}_P^G(s) = \{\varphi : G \to \mathbf{C} \,|\, \varphi \text{ is smooth and } \varphi(gman) = \mathrm{e}^{-(s+\frac{m}{n})\Lambda_0(\log(a))}\varphi(g), man \in P = MAN\}.$$

The group G acts by left translation:

$$(g \cdot \varphi)(g_1) = \varphi(g^{-1}g_1).$$

This principal series representation may be realized as smooth functions on $\overline{\mathfrak{n}}$ as follows. Write $\overline{n}_Y \equiv \exp(Y)$ for $Y \in \overline{\mathfrak{n}}$. For $\varphi \in \operatorname{Ind}_P^G(s)$ set

$$f(Y) = \varphi(\overline{n}_Y), Y \in \overline{\mathfrak{n}}.$$

Then $\operatorname{Ind}_P^G(s)$ may be identified with

$$I(s) = \{f \in C^\infty(\overline{\mathfrak{n}}) : f(Y) = \varphi(\overline{n}_Y), \text{ for some } \varphi \in \operatorname{Ind}_P^G(s)\}.$$

Since $\overline{N}P$ is dense in G and any $g \in \overline{N}P$ has a unique decomposition as

$$g = \overline{n}(g)m(g)a(g)n(g) \in \overline{N}MAN,$$

the action is given by

$$(g \cdot f)(Y) = \mathrm{e}^{-(s+\frac{m}{n})\Lambda_0}(a(g^{-1}\overline{n}_Y))f(\log(\overline{n}(g^{-1}\overline{n}_Y))).$$

In particular

$$(\ell \cdot f)(Y_1) = \mathrm{e}^{-(s+\frac{m}{n})\Lambda_0}(a(\ell))f(\ell^{-1} \cdot Y_1)$$
$$(\overline{n}_Y \cdot f)(Y_1) = f(Y_1 - Y).$$

For each s with $\mathrm{Re}(s) > d(n-1)$ there is a G-intertwining operator

$$\tilde{A}_s : I(s) \to I(-s)$$

which is given by the convergent integral

$$(\tilde{A}_s f)(Y) = \int_{\overline{\mathfrak{n}}} \varphi(\overline{n}_Y w \overline{n}_{Y_1})dY_1.$$

The form we will use is easily derived from this ([10], pages 183 and 200):

$$\begin{aligned}(\tilde{A}_s f)(Y) &= \int_{\overline{\mathfrak{n}}} \overline{\nabla}(Y_1)^{s-\frac{m}{n}} f(Y+Y_1)\, dY_1 \\ &= \int_{\overline{\mathfrak{n}}} \overline{\nabla}(Y_1 - Y)^{s-\frac{m}{n}} f(Y_1)\, dY_1.\end{aligned}$$

The integral converges for $\mathrm{Re}(s) > d(n-1)$ (by Lemma 3.16 and by Lemma 3.15).

Note that the Schwartz space $\mathcal{S}(\overline{\mathfrak{n}})$ is contained in $I(s)$ for all s. Also, for each $f \in I(s)$ there is a constant C so that

$$|f(Y)| \le C\mathrm{e}^{-(\mathrm{Re}(s)+\frac{m}{n})\Lambda_0(H(\overline{n}_Y))},$$

for all $Y \in \overline{\mathfrak{n}}$. In particular,

$$I(s) \subset L^2(\overline{\mathfrak{n}}, \mathrm{e}^{2\mathrm{Re}(s)\Lambda_0(H(\overline{n}_Y))}dY). \tag{7.1}$$

The intertwining operators $\tilde{A}_s$ are complex analytic in s for $\mathrm{Re}(s) > d(n-1)$ and have meromorphic continuations to all of $\mathbf{C}$. This is a well-known general fact (see, for example, [10] and [26]). However one can see this directly for Schwartz functions as follows.

$$\tilde{A}_s(\overline{P}(\partial_Y)f)(Y) = \int_{\overline{\mathfrak{n}}} f(Y+Y_1)\overline{P}(\partial_Y)\overline{\nabla}(Y_1)^{s-\frac{m}{n}}\, dY_1 = b(s-\frac{m}{n})(\tilde{A}_{s-2}f)(Y),$$

so the argument of Section 4 applies. Define

$$A_s(f) = \frac{\pi^{\frac{ns}{2}}}{\Gamma_n(s)}\tilde{A}_s(f).$$

For $s \in \mathbf{R}$ there is a G-invariant hermitian form on $I(s)$ defined by

$$\langle f_1, f_2 \rangle = \int_{\overline{\mathfrak{n}}} f_1(Y)\overline{A_s f_2(Y)}dY. \tag{7.2}$$

See [10, Prop. 14.23], for example. Also, there is a well-defined invariant hermitian form on the image of A_s in $I(-s)$ given by

$$\langle A_s f_1, A_s f_2 \rangle_s = \langle f_1, f_2 \rangle. \tag{7.3}$$

Consider Schwartz functions f, f_1 and f_2 on $\mathcal{S}(\overline{\mathfrak{n}})$. Let $\tau_Y f = f(\,\cdot\, + Y)$. Then for $\mathrm{Re}(s) \gg 0$

$$(A_s f)(Y) = \frac{\pi^{\frac{ns}{2}}}{\Gamma_n(s)}\overline{\mathbf{Z}}(\tau_Y f, s - \frac{m}{n}),$$

by the definitions as integrals. This holds for all s since both sides are meromorphic in s. By Theorem 4.4

$$\begin{aligned}(A_s f)(Y) &= \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_n(-s+\frac{m}{n})}\mathbf{Z}((\tau_Y f)\widehat{\ }, -s) \\ &= R_s((\tau_Y f)\widehat{\ }).\end{aligned} \tag{7.4}$$

Therefore, for $s \ll 0$

$$\begin{aligned}\langle A_s f_1, A_s f_2\rangle_s &= \int_{\overline{\mathfrak{n}}} f_1(Y)\overline{R_s((\tau_Y f)\widehat{\ })}\,dY \\ &= \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_n(-s+\frac{m}{n})}\int_{\overline{\mathfrak{n}}}\int_{\mathfrak{n}} f_1(Y)\nabla(X)^{-s}e^{2\pi i\langle X,Y\rangle}\overline{\widehat{f_2}}(X)\,dX\,dY \\ &= \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_n(-s+\frac{m}{n})}\int_{\mathfrak{n}} \nabla(X)^{-s}\widehat{f_1}(X)\overline{\widehat{f_2}}(X)\,dX \\ &= R_s\big((f_1 * \overline{f_2})\widehat{\ }\big).\end{aligned}$$

Since both sides are real analytic in s we have proved the following proposition.

Proposition 7.5. *For $s \in \mathbf{R}$ and $f_1, f_2 \in \mathcal{S}(\overline{\mathfrak{n}})$*

$$\langle A_s f_1, A_s f_2\rangle_s = R_s\big((f_1 * \overline{f_2})\widehat{\ }\big) = R_s\big(\widehat{f_1}\overline{\widehat{f_2}}\big). \tag{7.6}$$

Proposition 7.7. *When $s \notin \Xi_n$ the hermitian form $\langle\ ,\ \rangle_s$ on $Im(A_s)$ is not positive definite (so the $(\mathfrak{g}, K)$-module of $Im(A_s)$ is not unitarizable).*

Proof. The distribution R_s is positive if and only if the right hand side of (7.6) is positive when $f_1 = f_2$. □

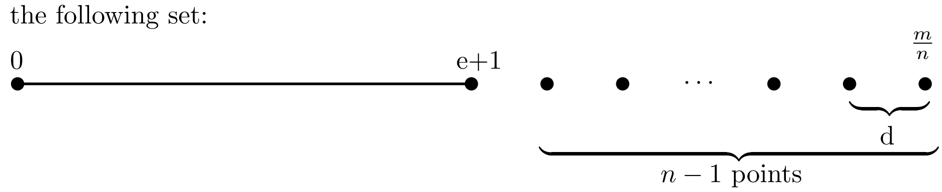
Proposition 7.8. *(*[20, Theorems 4A and 4B]*) For $G \neq SO(p,q)$ (case 4 on tables 1 and 2) the principal series representation $I(s)$ satisfies*

(1) *$I(s)$ is irreducible and unitarizable for $|s| < e+1$, and*
(2) *$I(s)$ contains a unitarizable quotient for $s = \frac{m}{n} - dq$ for $q = 0, 1, \ldots, n-1$.*

Remark 7.9. Considering $s \geq 0$, the points of possible unitarizability of $Im(A_s)$ are

$$\Xi_n \cap [0,\infty) = [0, e+1) \cup \{\frac{m}{n} - qd : q = 0, 1, \ldots, n-1\}$$

pictured as the following set:



8. Unitary Realization

For this section we assume the $G \neq SO(p,q)$ and we consider only $s \geq 0$. Consider the image of A_s in $I(-s)$. $Im(A_s)$ is G-invariant and (7.3) defines a G-invariant hermitian form on $Im(A_s)$. By Prop. 7.8, this form is positive definite (on the K-finite vectors) for

$$s \in \Xi_n. \tag{8.1}$$

For s satisfying (8.1) define

$$\mathcal{H}_s^0 = Im(A_s),$$

a pre-hilbert space with inner product $\langle\ ,\ \rangle_s$. Let $\mathcal{H}_s$ be the completion. Then $\mathcal{H}_s$ is a unitary representation of G.

Lemma 8.2. *$A_s(\mathcal{S}(\overline{\mathfrak{n}})) \subset \mathcal{H}_s$ is dense.*

Proof. We check that $A_s(\mathcal{S}(\overline{\mathfrak{n}}))^{\perp} = 0$. Suppose that $f_2 \in I(s)$ and $A_s f_2 \perp A_s(\mathcal{S}(\overline{\mathfrak{n}}))$ i.e., for all $f_1 \in \mathcal{S}(\overline{\mathfrak{n}})$

$$0 = \langle A_s f_1, A_s f_2\rangle_s = \int_{\overline{\mathfrak{n}}} f_1(Y)\overline{A_s f_2(Y)}dY.$$

By (7.1)

$$f_1 \in L^2(\overline{\mathfrak{n}}, \mathrm{e}^{2\mathrm{Re}(s)\Lambda_0(H(\overline{n}_Y))}dY) \text{ and } A_s f_2 \in L^2(\overline{\mathfrak{n}}, \mathrm{e}^{-2\mathrm{Re}(s)\Lambda_0(H(\overline{n}_Y))}dY).$$

Therefore,

$$0 = \int_{\overline{\mathfrak{n}}} \big(f_1(Y)\mathrm{e}^{-s\Lambda_0(H(\overline{n}_Y))}\big)\big(A_s f_2(Y)\mathrm{e}^{s\Lambda_0(H(\overline{n}_Y))}\big)\, dY,$$

where the integrand is now written as a product of two functions in $L^2(\overline{\mathfrak{n}}, dY)$. Since $\mathrm{e}^{-s\Lambda_0(H(\overline{n}_Y))}$ is a power of a nonzero polynomial $\{f_1(Y)\mathrm{e}^{-s\Lambda_0(H(\overline{n}_Y))} : f_1 \in \mathcal{S}(\overline{\mathfrak{n}})\}$ is dense in $L^2(\overline{\mathfrak{n}}, dY)$. We now conclude that $A_s f_2 = 0$. The lemma follows from the fact that $\mathcal{H}_s^0$ is dense in $\mathcal{H}_s$. $\square$

For the unitary realizations of the I_s we treat the cases of $s \in [0, e+1)$ and $s = \frac{m}{n} - qd$, $d = 0, 1, \ldots, n-1$ separately because the distributions R_s have very different forms in these two cases.

<u>Case 1:</u> $s \in [0, e+1)$. For $s \in [0, e+1)$, $R_{\frac{m}{n}-s}$ is given by a convergent integral. Therefore by (7.4) and Prop. 7.5

$$\begin{aligned} A_s f(Y) &= \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_n(-s+\frac{m}{n})}\int_{\mathfrak{n}} \mathrm{e}^{-s\pi i\langle X,Y\rangle}\widehat{f}(X)\nabla(X)^{-s}dX \\ &= \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_n(-s+\frac{m}{n})}\big(\widehat{f}\,\nabla(\,\cdot\,)^{-s}\big)\widehat{\ }(Y) \end{aligned} \tag{8.3}$$

and

$$\langle A_s f_1, A_s f_2\rangle_s = \frac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_n(-s+\frac{m}{n})}\int_{\mathfrak{n}} \widehat{f}_1(X)\overline{\widehat{f}_2(X)}\nabla(X)^{-s}dX \tag{8.4}$$

for all Schwartz functions f, f_1 and f_2 on $\overline{\mathfrak{n}}$.

The Schwartz space $\mathcal{S}(\mathfrak{n})$ has a natural action by $\overline{P} = L\overline{N}$ via the Fourier transform as follows. Let $^{\vee}$ denote the inverse Fourier transform. For $h \in \mathcal{S}(\mathfrak{n})$ define

$$(\overline{p}\cdot h)(X) = (\overline{p}\cdot h^{\vee})\widehat{\ }(X).$$

Then

$$\begin{aligned} (\ell\cdot h)(X) &= \mathrm{e}^{(s-\frac{m}{n})\Lambda_0}(a(\ell))h(\ell\cdot X) \\ (\overline{n}_Y\cdot h)(X) &= \mathrm{e}^{-2\pi i\langle X,Y\rangle}h(X). \end{aligned} \tag{8.5}$$

Theorem 8.6. *For $s \in [0, e+1)$, let $d_sX = \dfrac{\pi^{\frac{n}{2}(-s+\frac{m}{n})}}{\Gamma_n(-s+\frac{m}{n})}dX$. Then $\mathcal{H}_s$ is unitarily equivalent to $L^2(\mathfrak{n}, \nabla(X)^{-s}d_sX)$ as $\overline{P}$-representations. This representation is irreducible.*

Proof. Temporarily set $V_s = A_s(\mathcal{S}(\overline{\mathfrak{n}}))$. Then

$$V_s = \{\big(h\nabla(X)^{-s}dX\big)\widehat{\ } : h \in \mathcal{S}(\mathfrak{n})\}$$

by (7.4). We have $\mathcal{S}(\mathfrak{n}) \subset L^2(\mathfrak{n}, \nabla(X)^{-s}d_sX)$ and $V_s \subset \mathcal{H}_s$ as dense subspaces (by Lemma 8.2). Consider

$$T : \mathcal{S}(\mathfrak{n}) \to V_s$$

defined by

$$T(h) = \big(h \cdot \nabla(\,\cdot\,)^{-s}\big)\widehat{\ }.$$

We will check that T is an isometry onto a dense subset of $\mathcal{H}_s$ and is $\overline{P}$-equivariant, then T will extend to a unitary equivalence of unitary $\overline{P}$-representations.

Let $h \in \mathcal{S}(\mathfrak{n})$.

$$\begin{aligned}\langle T(h), T(h)\rangle_s &= \langle A_s(h^\vee), A_s(h^\vee)\rangle_s, \text{ by (8.3)}\\ &= \int_{\mathfrak{n}} |h(X)|^2 \nabla(X)^{-s} d_s X, \text{ by (8.4).}\end{aligned}$$

(Note that R_s is given by a convergent integral for $s \in [0, e+1)$.) The equivariance follows easily:

$$T(\overline{p} \cdot h) = T((\overline{p} \cdot h^\vee)\widehat{\ }) = A_s(\overline{p} \cdot h^\vee) = \overline{p} \cdot A_s(h^\vee) = \overline{p} \cdot T(h).$$

The irreducibility is a standard application of Schur's Lemma as follows. Any $\overline{N}$-intertwining operator of $L^2(\mathfrak{n}, \nabla(X)^{-s} d_s X)$ must be multiplication by a bounded function. Transitivity (up to measure zero) of L on $\mathfrak{n}$ shows that if the operator is also L-intertwining then the bounded function must be a constant (a.e.). $\square$

Corollary 8.7. *The $\overline{P}$-representation $L^2(\mathfrak{n}, \nabla(X)^{-s} d_s X)$ extends to an irreducible unitary representation of G.*

<u>Case 2:</u> $s = \frac{m}{n} - qd$, $q = 0, 1, \ldots, n-1$. Define

$$\mathcal{F}_R(f) = \widehat{f}|_{\mathcal{O}_q}, \text{ for } f \in \mathcal{S}(\overline{\mathfrak{n}})$$

and

$$\mathcal{F}_E(h) = \big(h\, d\nu_q\big)\widehat{\ }, \text{ for } h \in \mathcal{F}_R\mathcal{S}(\mathfrak{n}).$$

Then by (7.4) and (7.6)

$$A_s(f) = \mathcal{F}_E\mathcal{F}_R(f), f \in \mathcal{S}(\overline{\mathfrak{n}}), \tag{8.8}$$

and

$$\langle A_s f_1, A_s f_2\rangle_s = \int_{\mathcal{O}_q} \widehat{f}_1 \overline{\widehat{f}_2} d\nu_q. \tag{8.9}$$

As in Case 1, we may define a $\overline{P}$-action on $L^2(\mathcal{O}_q, d\nu_q)$ via the Fourier transform. Letting $h \in L^2(\mathcal{O}_q, d\nu_q)$ and $s = \frac{m}{n} - dq$ we define

$$\begin{aligned}(\ell \cdot h)(X) &= e^{-dq\Lambda_0}(a(\ell)) h(\ell^{-1} \cdot X) = \chi_d(\ell)^{-\frac{1}{2}} h(\ell^{-1} \cdot X)\\ (\overline{n}_Y \cdot h)(X) &= \mathrm{e}^{-2\pi i \langle X, Y\rangle} h(X).\end{aligned}$$

Lemma 8.10. *Let $q = 0, 1, \ldots, n-1$.*

(1) *$L^2(\mathcal{O}_q, d\nu_q)$ is an irreducible unitary representation of $\overline{P}$.*
(2) *$\mathcal{F}_R : \mathcal{S}(\overline{\mathfrak{n}}) \to L^2(\mathcal{O}_q, d\nu_q)$ is $\overline{P}$-equivariant and has dense image.*

Proof. For unitarity note that $||\overline{n}_Y \cdot h||^2_{\mathcal{O}_q} = ||h||^2$ is clear from (8.5) and

$$\begin{aligned} ||\ell \cdot h||^2_{\mathcal{O}_q} &= \int_{\mathcal{O}_q} ||\ell \cdot h||^2 d\nu_q \\ &= \int_{\mathcal{O}_q} ||\chi_d(\ell)^{-\frac{1}{2}} h(\ell^{-1} \cdot X)||^2 d\nu_q(X) \\ &= \int_{\mathcal{O}_q} ||h(X)||^2 d\nu_q(X), \text{ by Cor. 3.7.} \end{aligned}$$

Irreducibility is as in the proof of Theorem 8.6. The $\overline{P}$-equivariance of $\mathcal{F}_R$ is also as in the proof of Theorem 8.6. The image of $\mathcal{F}_R$ is dense by irreducibility of $L^2(\mathcal{O}_q, d\nu_q)$. □

Theorem 8.11. *$\mathcal{F}_E$ extends to a $\overline{P}$-equivariant unitary equivalence between $L^2(\mathcal{O}_q, d\nu_q)$ and $\mathcal{H}_{\frac{m}{n}-qd}$, $q = 0, 1, \ldots, n-1$.*

Proof. $\mathcal{F}_E : \mathcal{F}_R(\mathcal{S}(\overline{\mathfrak{n}})) \to \mathcal{H}^0_{\frac{m}{n}-qd}$ is an isometry by (8.8) and (8.9). By part (2) of Lemma 8.10, $\mathcal{F}_E$ extends to a unitary equivalence of $L^2(\mathcal{O}_q, d\nu_q)$ and $\mathcal{H}_{\frac{m}{n}-dq} = \overline{\mathcal{H}^0_{\frac{m}{n}-dq}}$. We now check the $\overline{P}$-equivariance of $\mathcal{F}_E$ on $\mathcal{F}_R(\mathcal{S}(\overline{\mathfrak{n}}))$. Let $h = \mathcal{F}_R(f) \in \mathcal{F}_R(\mathcal{S}(\mathfrak{n}))$.

$$\mathcal{F}_E(\overline{p} \cdot h) = \mathcal{F}_E(\overline{p} \cdot \mathcal{F}_R f) = \mathcal{F}_E \mathcal{F}_R(\overline{p} \cdot f) A_s(\overline{p} \cdot f) = \overline{p} \cdot A_s f = \mathcal{F}_E(\mathcal{F}_R f) = \mathcal{F}_E(h).$$

□

Corollary 8.12. *The $\overline{P}$-representations $L^2(\mathcal{O}_q, d\nu_q)$ extend to irreducible unitary representations of G.*

Appendix A. Tables

The following two tables give information on the groups under consideration in this paper.

	G	$n = \text{rank}(\mathfrak{n})$	$m = \dim(\mathfrak{n})$	d	e
1.	$GL(2n, \mathbf{R}), n \geq 2$	n	n^2	1	0
2.	$O(2n, 2n), n \geq 2$	n	$n(2n-1)$	2	0
3.	$E_7(7)$	3	27	4	0
4.	$O(p, q),\ p, q \geq 3$	2	$p+q-2$	$(p+q-4)/2$	0
5.	$Sp(n, \mathbf{C})$	n	$n(n+1)$	1	1
6.	$SL(2n, \mathbf{C})$	n	$2n^2$	2	1
7.	$SO(4n, \mathbf{C})$	n	$2n(2n-1)$	4	1
8.	$E_{7,\mathbf{C}}$	3	54	8	1
9.	$SO(p, \mathbf{C})$	2	$2(p-2)$	$p-4$	1
10.	$Sp(n, n)$	n	$n(2n+1)$	2	2
11.	$SL(2n, \mathbf{H})$	n	$4n^2$	4	3
12.	$SO(p, 1)$	1	p	0	$p-1$

Table 1

	$V \simeq \mathfrak{n}$	L	∇
1.	$M(n\times n,\mathbf{R})$	$GL(n,\mathbf{R})\times GL(n,\mathbf{R})$	$\lvert\det\rvert$
2.	$Skew(2n:\mathbf{R})$	$GL(2n,\,\mathbf{R})$	Pfaffian
3.	$Herm(3,\mathbf{O}_{\text{split}})$	$E_6(6)\times\mathbf{R}^\times$	degree 3 real polynomial
4.	$\mathbf{R}^{p-1,q-1}$	$\mathbf{R}^\times SO(p-1,q-1)$	(X,X)
5.	$Sym(n,\mathbf{C})$	$GL(n:\mathbf{C})$	$\lvert\det\rvert$
6.	$M(n\times n,\mathbf{C})$	$S(GL(n,\mathbf{C})\times GL(n,\mathbf{C}))$	$\lvert\det\rvert$
7.	$Skew(2n,\mathbf{C})$	$GL(2n,\mathbf{C})$	|Pfaffian|
8.	$Herm(3,\mathbf{O})_{\mathbf{C}}$	$E_{6,\mathbf{C}}\mathbf{C}^\times$	|degree 3 complex poly|
9.	$\mathbf{C}^{p-1}$	$O(p-2:\mathbf{C})\times\mathbf{C}^\times$	$\lvert(Z,Z)\rvert$
10.	$Sym(2n,\mathbf{C})\cap M(n\times n,\mathbf{H})$	$GL(n,\mathbf{H})$	$\lvert\det_{\mathbf{C}}(Z)\rvert^{\frac{1}{2}}$
11.	$M(n\times n,\mathbf{H})$	$GL(n,\mathbf{H})\times GL(n,\mathbf{H})$	$\lvert\det_{\mathbf{C}}(Z)\rvert^{\frac{1}{2}}$
12.	$\mathbf{R}^{p,1}$	$O(p-1)\times\mathbf{R}^\times$	$\lvert\lvert\cdot\rvert\rvert$

Table 2. Jordan algebras for the groups in Table 1.

Remark A.1. For Cases 10 and 11 we view the quaternionic matrices as complex matrices of the form $Z=\begin{pmatrix} A & B\\ -\overline{B} & \overline{A}\end{pmatrix}$. Then $\det_{\mathbf{C}}$ refers to the determinant of the complex matrix.

Appendix B. Meromorphic families of distributions

We give a few (well-known) general facts about meromorphic families of distributions. Suppose $m(x)$ is a positive polynomial of degree m on $\mathbf{R}^n$ with Bernstein polynomial $b(s)$ defined by $m(\partial_x)m(x)^{\frac{s}{2}} = b(s)m(x)^{\frac{s}{2}-1}$. Let $\{\alpha_j\}$ be the roots of $b(s)$ and set $S=\{\alpha_j-2l : l\in\mathbf{Z}_+\}$. Define, for $h\in\mathcal{S}(\mathbf{R}^n)$,

$$T_s(h)=\int_{\mathbf{R}^n} h(x)m(x)^{\frac{s}{2}}\,dx,$$

a convergent integral for $\mathrm{Re}(s)>0$. For $\mathrm{Re}(s)>0$, $T_s(h)$ is a complex analytic function of s. From the formula

$$T_s(h)=\frac{1}{b(s+2)b(s+4)\cdots b(s+2k)}T_{s+k}(m(\partial_x)^k h(x)) \tag{B.1}$$

it follows that $T_s(h)$ has a meromorphic continuation to all of $\mathbf{C}$ with possible poles in S.

Lemma B.2. *Consider the Laurent expansion of $T_s(h)$ for an arbitrary $s_0\in\mathbf{C}$:*

$$T_s(h)=\sum_{m\geq -d}^{\infty} T^m(h)(s-s_0)^m$$

for some $d=0,1,2,\dots$. Then each T^m is a tempered distribution.

Proof. Let $N\in\mathbf{Z}_+$ and let $||\cdot||_N$ be the Schwartz norm

$$||h||_N=\sup\{(1+|x|)^N|h(x)|\}.$$

In particular, $|h(x)|\leq ||h||_N(1+|x|)^{-N}$ for all $x\in\mathbf{R}^n$. Choose a constant C' so that

$$|m(x)|\leq C'(1+|x|)^M,\quad x\in\mathbf{R}^N.$$

Therefore,

$$|h(x)m(x)^{\frac{s}{2}}|\leq C'||h||_N(1+|x|)^{-(N-\mathrm{Re}(\frac{s}{2})M)} \text{ for } \mathrm{Re}s>0.$$

For any s_1 with $\mathrm{Re}(s_1) > 0$ we may choose a neighborhood $U_\delta = \{s : |\mathrm{Re}(s) - \mathrm{Re}(s_1)| < \delta\}$ in the right half-plane so that $N - \mathrm{Re}(\frac{s}{2})M \geq n+1$, for all $s \in U_\delta$. Therefore, for $s \in U_\delta$

$$|h(x)m(x)^{\frac{s}{2}}| \leq C'||h||_N(1+|x|)^{-(n+1)} \tag{B.3}$$

and

$$|T_s(h)| \leq C'||h||_N\Big(\int_{\mathbf{R}^N}(1+|x|)^{-(n+1)}\,dx\Big) = C||h||_N. \tag{B.4}$$

Therefore, T_s is a tempered distribution for $\mathrm{Re}(s) > 0$. Furthermore, there is a convergent power series expansion $T_s(h) = \sum_{m=0}^{\infty} T^m(h)(s-s_1)^m$. We see that each T^m is a tempered distribution as follows. Let $h_j \to h$ in $\mathcal{S}(\mathbf{R}^n)$. Then by (B.4) $T_s(h_j) \to T_s(h)$ uniformly on compact sets (in the right half-plane). Therefore, $T^m(h_j) \to T^m(h)$ (as the coefficients in the expansion are derivatives of $T_s(h)$).

Now consider an arbitrary $s_1 \in \mathbf{C}$. Choose k in (B.1) large enough so that $\mathrm{Re}(s_1) + k > 0$. Write $b(s+2)\cdots b(s+2k) = b_1(s)(s-s_1)^d$ with $b_1(s_1) \neq 0$ and d a nonnegative integer. Then

$$T_s(h) = \frac{1}{b_1(s)}\frac{1}{(s-s_1)^d}T_{s+k}(m(\partial_x)h(x)).$$

Expanding T_{s+k} about $s_1 + k$ (as above) gives

$$T_s(h) = \frac{1}{b_1(s)}\frac{1}{(s-s_1)^d}\sum_{m=0}^{\infty} T^m(m(\partial_x)h(x))(s-s_1)^m \tag{B.5}$$

$$= \frac{1}{b_1(s)}\sum_{m=-d}^{\infty} T^{m+d}(m(\partial_x)h(x))(s-s_1)^m. \tag{B.6}$$

As each T^m in (B.5) is tempered the Lemma is proved. □

Now suppose $g(s)$ is an entire function and set $\tilde{T}_s(h) = g(s)T_s(h)$ for each $h \in \mathcal{S}(\mathfrak{n})$. By the above Lemma $\tilde{T}_s$ is a meromorphic family of tempered distributions and we may write

$$\tilde{T}_s(h) = g_1(s)\sum_{m \geq -d_1} T^m(h)(s-s_1)^m$$

where $g(s) = g_1(s)(s-s_1)^{d_1}, g_1(s_1) \neq 0$.

Corollary B.7. *Suppose s_1 is a possible pole of $\tilde{T}_s(h)$, for some $h \in \mathcal{S}(\mathbf{R}^n)$. If $\tilde{T}_{s_1}(h) < \infty$ for all $h \in C_0^\infty(\mathbf{R}^n)$ (=compactly supported functions) then $\tilde{T}_{s_1}(h)$ is finite for all $h \in \mathcal{S}(\mathbf{R}^n)$ and defines a tempered distribution.*

Proof. If $\tilde{T}_{s_1}(h) < \infty$ for all $h \in C_0^\infty(\mathbf{R}^n)$ then $T^m(h) = 0$ for all $m = -1, -2, \ldots, -d_1$ and $h \in C_0^\infty(\mathbf{R}^n)$. As each T^m is tempered, $T^m(h) = 0$ for all $m = -1, -2, \ldots, -d_1$ and $h \in \mathcal{S}(\mathbf{R}^n)$. Now

$$T_{s_1}(h) = g_1(s_1)T^0(h),$$

a tempered distribution. □

Appendix C. Rank one case

Many of the arguments given in this article use induction on the rank of $\mathfrak{n}$. In this appendix we collect the facts about the rank one case which are used. First we make some observations about the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{n}_1$, when $\mathfrak{g}$ is on Table 1.

Recall that $\mathfrak{g}_1$ is the Lie algebra generated by the root spaces for $\pm 2\epsilon_1$. This is a simple Lie algebra and, although $\mathfrak{g}_1$ does not in general satisfy 2.1, the integer $e+1$ (the dimension of the root space for the long root) is the same as for $\mathfrak{g}$. From Table 1 we see that $\mathfrak{g}_1$ is

$\mathfrak{gl}(2,\mathbf{R})$ in cases 1-4,
$\mathfrak{sl}(2,\mathbf{C})$ in cases 5-9,
$\mathfrak{sp}(1,1)$ in case 10,
$\mathfrak{sl}(2,\mathbf{H})$ in case 11 and
$\mathfrak{so}(p,1)$ in case 12.

Note that $\dim(\mathfrak{n}_1)=e+1$. The following lemma computes $\nabla(X)$ and $\overline{\nabla}(Y)$ for each $\mathfrak{g}_1$. We express ∇ and $\overline{\nabla}$ in terms of $\langle\ ,\ \rangle = -\frac{1}{4m}B_{\mathfrak{g}}$.

Lemma C.1. *Suppose that* $\mathfrak{n}$ *has rank* 1*. Then* $\nabla(X)=||X||$ *and* $\overline{\nabla}(Y)=||Y||$.

Proof. It is enough to compute just $\overline{\nabla}(Y)=||Y||$ since for $X=\theta(Y)$, $||X||^2=\langle\theta(X),X\rangle=\langle Y,\theta(Y)\rangle=||Y||^2$. We do a case-by-case calculation. The cases of $\mathfrak{sl}(2,\mathbf{R})$ and $\mathfrak{sl}(2,\mathbf{C})$ may be done simultaneously. Here

$$\overline{\mathfrak{n}}=\Big\{Y=\begin{pmatrix}0&0\\y&0\end{pmatrix}\Big\}\text{ with } y\in\mathbf{R}\text{ or }\mathbf{C}$$

and $\mathfrak{n}=\theta(\overline{\mathfrak{n}})$. Also

$$\mathfrak{a}=\Big\{\begin{pmatrix}a&0\\0&-a\end{pmatrix}:a\in\mathbf{R}\Big\},$$

and the Weyl group element is represented by

$$w=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

We need to compute $\overline{\nabla}(Y)=e^{\epsilon_1(a(w\overline{n}_Y))}$. Consider the standard representation on $\mathbf{C}^2$ with the usual hermitian metric $(\ ,\ )$. Then

$$\begin{aligned}e^{\epsilon_1(a(w\overline{n}_Y))}&=\left|\left(\begin{pmatrix}1\\0\end{pmatrix},w\overline{n}_Y\cdot\begin{pmatrix}1\\0\end{pmatrix}\right)\right|\\&=\left|\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}y&1\\-1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)\right|\\&=|y|.\end{aligned}$$

Now we need to compare $|y|$ with $||Y||$. Since $\frac{1}{4m}B(\xi,\eta)=\mathrm{Re}(\mathrm{Tr}(\xi\eta)$ we have $||Y||^2=\mathrm{Tr}(\overline{Y}^tY)=|y|^2$. Therefore, $\overline{\nabla}(Y)=||Y||$. The other cases are done similarly. □

For the cases when $\mathfrak{n}$ is of rank one, the functional equation is simply a statement about the distribution $|x|^{-1}$ in euclidean space $\mathbf{R}^m$ and is well-known. A good reference is [8, vol. 1, Ch. II.3].

Proposition C.2. *Consider* $\mathbf{R}^m$ *for any integer* m *and* $f\in\mathcal{S}(\mathbf{R}^m)$*. With Fourier transform defined by*

$$\widehat{f}(y)=\int_{\mathbf{R}^m}f(x)e^{-2\pi i\langle y,x\rangle}dx,$$

the functional equation takes the form

$$\frac{\pi^{\frac{m-s}{2}}}{\Gamma(\frac{m-s}{2})}\int_{\mathbf{R}^m}f(x)|x|^{-s}dx=\frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}\int_{\mathbf{R}^m}\widehat{f}(y)|y|^{s-m}dy. \tag{C.3}$$

The following Proposition is also well-known, see for example [8, vol 4, Ch. II.3.6]. We give a short proof here since our proof of Theorem 6.4 refers to formula (C.5) occurring below.

Proposition C.4. *The distributions* $R_s^1(f) = \dfrac{\pi^{\frac{-s+m}{2}}}{\Gamma(\frac{s}{2})}|x|^{-s}$ *are positive if and only if* $s \leq m$.

Proof. When $s < m$ the function $|x|^{-s}$ is a locally L_1 positive function (and $\Gamma(\frac{-s+m}{2}) > 0$) so the distribution is clearly positive. For $s = m$, $R_m^1 = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} \cdot \delta_0$, a positive distribution.

To see that R_s^1 is not positive for $s \in \mathbf{C} \setminus (-\infty, m]$ consider the positive Schwartz functions

$$\varphi_1(x) = |x|^2 \mathrm{e}^{-|x|^2} \text{ and } \varphi_2(x) = \mathrm{e}^{-|x|^2}.$$

We claim that

$$\begin{aligned} R_s^1(\varphi_1) &= \frac{1}{4}\pi^{\frac{-s+m}{2}} \mathrm{Vol}(S^{m-1}) \cdot (-s+m) \\ R_s^1(\varphi_2) &= \frac{1}{2}\pi^{\frac{-s+m}{2}} \mathrm{Vol}(S^{m-1}). \end{aligned}$$

Then

$$\frac{R_s^1(\varphi_1)}{R_s^1(\varphi_2)} = \frac{-s+m}{2}, \tag{C.5}$$

so both $R_s^1(\varphi_i), i = 1, 2$ cannot be positive for $s \in \mathbf{C} \setminus (-\infty, m)$.

By analytic continuation it suffices to check the claim for $s \ll 0$.

$$\begin{aligned} R_s^1(\varphi_1) &= \frac{\pi^{\frac{-s+m}{2}}}{\Gamma(\frac{-s+m}{2})} \int_{\mathbf{R}^m} \mathrm{e}^{-|x|^2} |x|^{-s+2} dx \\ &= \frac{\pi^{\frac{-s+m}{2}}}{\Gamma(\frac{-s+m}{2})} \int_{S^{m-1}} \int_0^\infty \mathrm{e}^{-r^2} r^{-s+m+1} dr d\sigma \\ &= \frac{\pi^{\frac{-s+m}{2}}}{\Gamma(\frac{-s+m}{2})} \mathrm{Vol}(S^{m-1}) \int_0^\infty \mathrm{e}^{-t} t^{\frac{-s+m+1}{2}} \frac{dt}{2\sqrt{t}} dt \\ &= \frac{1}{2}\pi^{\frac{-s+m}{2}} \mathrm{Vol}(S^{m-1}) \frac{\Gamma(\frac{-s+m}{2}+1)}{\Gamma(\frac{-s+m}{2})} = \frac{1}{4}\pi^{\frac{-s+m}{2}} \mathrm{Vol}(S^{m-1}) \cdot (-s+m). \end{aligned}$$

The second part of the claim has a similar proof. □

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