COMPONENTS OF SPRINGER FIBERS ASSOCIATED TO CLOSED ORBITS FOR THE SYMMETRIC PAIRS $(Sp(n), GL(n))$ AND $(O(n), O(p) \times O(q))$ II

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Abstract. Let (G, K) be either $(Sp(2n), GL(n))$ or $(O(n), O(p)\times O(q))$, $p+q = n$. The results of [2] are used to show that certain components of Springer fibers are iterated bundles. Several consequences, including a vanishing theorem for sheaf cohomology, are given. It is shown that for $(Sp(2n), GL(n))$ a maximal torus of K acts on these components. This generalizes the results of [5].

1. Introduction

For the pairs

$$(G, K) = (Sp(n), GL(n)) \text{ and } (O(n), O(p) \times O(q)), p + q = n, \tag{1.1}$$

components of Springer fibers associated to closed K-orbits in the flag variety for G are described in [2]. We now give a geometric description of these components and some consequences. In particular, the components are shown to be iterated bundles involving generalized flag varieties for smaller groups. In addition, we show that for $Sp(n)$ a maximal torus acts on the components with a finite number of fixed points. These results extend the results of [5], where the pair $(GL(n), GL(p) \times GL(q))$ is considered. Certain components of Springer fibers for $GL(n)$ are also shown to be iterated bundles in [4].

Let (G, K) be one of the pairs (1.1) and suppose that $\mathcal{Q}$ is a closed K-orbit in the flag variety $\mathfrak{B}$ for G. Using the terminology and notation of [2], the components of a Springer fiber associated to $\mathcal{Q}$ are of the form

$$C_f = Q_{m,e} \cdots Q_{0,e} \cdot \mathfrak{b} = L_{m,e} \cdots L_{0,e} \cdot \mathfrak{b}. \tag{1.2}$$

This is the content of Theorem 4.8 and Lemma 4.9 of [2]. If we define

$$R_i = Q_{i,e} \cap Q_{i-1,e}, i = 1, 2, \ldots, m$$
$$R_0 = Q_{0,e} \cap B,$$

then we prove the following theorem in Section 2.

Theorem. *As algebraic varieties* $C_f \simeq Q_{m,e} \underset{R_m}{\times} Q_{m-1,e} \cdots Q_{1,e} \underset{R_1}{\times} Q_{0,e}/R_0$.

One consequence of this is that the Betti numbers of the components of C_f can be easily calculated.

In Section 3 we show that for the pairs $(Sp(n), GL(n))$ the diagonal torus H acts on C_f with a finite number of fixed points. Therefore, localization gives a formula for the H-character of

$$\sum_p (-1)^p H^p(C_f, \mathcal{O}(\tau)),$$

where $\mathcal{O}(\tau)$ is the sheaf of local sections of a homogeneous line bundle on $\mathfrak{B}$ pulled back to C_f. Euler characteristics of such cohomology are important in the representation theory of real reductive groups as they occur in formulas for the associated cycles of Harish-Chandra modules. See [3] and [8] for a discussion of this.

2. Preliminaries

As in [2] we consider the pairs (1.1). It is assumed that the reader is familiar with the construction and results of [2]. The notation used in this article will be as in [2] without further mention.

Let $\mathcal{Q} = K \cdot \mathfrak{b}$ be a closed K-orbit in $\mathfrak{B}$ defined by $\lambda \in \mathfrak{h}^*$. An important fact ([2, §2.3]) is that (G, K) embeds, in a nice, way into a pair $(\hat{G}, \hat{K})$ equal to either $(GL(2n), GL(n) \times GL(n))$ or $(GL(n), GL(p) \times GL(q))$. If $\hat{H}$ is the diagonal maximal torus, then there is a $\hat{\lambda} \in \hat{\mathfrak{h}}^*$ so that $\hat{\lambda}|_{\mathfrak{h}} = \lambda$ and the Borel subalgebra $\hat{\mathfrak{b}} = \hat{\mathfrak{h}} + \hat{\mathfrak{n}}^-$ defined by $\hat{\lambda}$ satisfies $\hat{\mathfrak{b}} \cap \mathfrak{g} = \mathfrak{b}$. In addition, the generic element f in $\mathfrak{n}^- \cap \mathfrak{p}$ is also generic in $\hat{\mathfrak{n}}^- \cap \hat{\mathfrak{p}}$. The subgroups L_i and Q_i are the intersections with K_i of the subgroups $\hat{L}_i$ and $\hat{Q}_i$ constructed from f for the pair $(\hat{G}, \hat{K})$ as in [1]. Note that $\hat{Q}_i$ is denoted by $Q_{i,K}$ in [1].

The subgroups $Q_{0,e}, Q_{1,e}, \ldots, Q_{m,e}$ define an iterated bundle as follows.

Definition 2.1. Let $R_i = Q_{i,e} \cap Q_{i-1,e}$, for $i = 1, 2, \ldots, m$, and $R_0 = Q_{0,e} \cap B$.

The group $R_m \times \cdots \times R_0$ acts on $Q_{m,e} \times \cdots \times Q_{0,e}$ by

$$(r_m, \ldots, r_0) \cdot (q_m, \ldots, q_0) = (q_m r_m^{-1}, r_m q_{m-1} r_{m-1}^{-1}, \ldots, r_1 q_0 r_0^{-1}).$$

We refer to the quotient as an *iterated bundle* and write it as

$$X = Q_{m,e} \underset{R_m}{\times} Q_{m-1,e} \underset{R_{m-1}}{\times} \cdots \underset{R_1}{\times} Q_{0,e}/R_0.$$

Elements of this quotient are written as $[q_m, \ldots, q_0]$, for $q_i \in Q_{i,e}$.

The following is easily verified.

Lemma 2.2. *For each $i = 1, 2, \ldots, m$*

(1) *R_i is a parabolic subgroup of $Q_{i,e}$,*
(2) *$L_{i,e} \cap Q_{i-1,e}$ is a parabolic subgroup of $L_{i,e}$,*

(3) $L_{i,e} \cap Q_{i-1,e} = (L_{i,e} \cap L_{i-1,e})U_{i-1}^{-}$, *and*
(4) $Q_{i,e}/R_i \simeq L_{i,e}/L_{i,e} \cap Q_{i-1,e}$.

It follows that the iterated bundle X is a smooth projective variety; see [5, Prop. 2.5].

3. Geometric structure of the components

We continue with a closed K-orbit $\mathcal{Q} = K \cdot \mathfrak{b}$ in $\mathfrak{B}$ and a generic element f in $\mathfrak{n}^{-} \cap \mathfrak{p}$, as in Section 2. The components of the Springer fiber associated to $\mathcal{Q}$ (i.e., those components contained in the conormal bundle $T^*_{\mathcal{Q}}\mathfrak{B}$) are $A_K(f) := Z_K(f)/Z_K(f)_e$ translates of

$$C_f := Q_{m,e} \cdots Q_{o,e} \cdot \mathfrak{b} = L_{m,e} \cdots L_{0,e} \cdot \mathfrak{b}, \tag{3.1}$$

by, for example, [2, Prop. 2.1].

The map

$$\begin{aligned} F : X &\to C_f \\ [q_m, \ldots, q_0] &\mapsto q_m \ldots q_0 \mathfrak{b} \end{aligned}$$

is clearly a surjective morphism.

Theorem 3.2. *The morphism F is an isomorphism of varieties.*

Proof. As C_f is smooth (by, for example, [5, Lem. 2.9]) it suffices to prove that F is injective. (Recall that in characteristic 0, a bijective morphism of a variety onto a normal variety is an isomorphism ([10, Cor. 17.4.8]).)

Consider the corresponding morphism for $(\hat{G}, \hat{K})$:

$$\hat{F} : \hat{X} \to \hat{Q}_m \cdots \hat{Q}_0 \cdot \mathfrak{b},$$

where $\hat{X}$ is the iterated bundle constructed from $\hat{Q}_0, \ldots, \hat{Q}_m$ and $\hat{R}_i = \hat{Q}_i \cap \hat{Q}_{i-1}$. Note that the righthand side is a component $\hat{C}_f$ of a Springer fiber for f for the pair $(\hat{G}, \hat{K})$. Theorem 2.10 of [5] implies that $\hat{F}$ is an isomorphism of varieties. (We remark that the $\hat{Q}_i$ here are slightly different than the parabolics used to construct the iterated bundle in [5]. The parabolics in [5] have been enlarged to contain the maximal torus $\hat{H}$ of $\hat{K}$. One easily sees that this extra piece of the torus plays no role and $\hat{F}$ is an isomorphism.)

There is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & C_f \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle j} \\ \hat{X} & \xrightarrow[\hat{F}]{} & \hat{C}_f \end{array}$$

The map j is the restriction of the embedding $\mathfrak{B} \hookrightarrow \hat{\mathfrak{B}}$. The map ι is the map sending $[q_m, \dots, q_0]$ in X to $[q_m, \dots, q_0]$ in $\hat{X}$. Once we show that ι is injective, F will be injective and the theorem will be proved.

For the injectivity of ι we will apply the following lemma.

Lemma 3.3. (1) $Q_{i,e} = \hat{Q}_i \cap K_{i,e}$.
(2) $R_i = \hat{R}_i \cap K_{i,e}$.

It follows from our discussion in §2, that $\mathfrak{q}_i = \hat{\mathfrak{q}}_i \cap \mathfrak{k}_i$. This, along with the fact that $\hat{Q}_i$ is connected, implies (1). For (2) note that $R_i = Q_{i,e} \cap Q_{i-1,e} = \hat{Q}_i \cap \hat{Q}_{i-1} \cap K_i = \hat{R}_i \cap \hat{R}_{i-1} \cap K_{i,e}$.

Now suppose that $q_i, q_i' \in Q_{i,e}$ and $[q_m, \dots, q_0] = [q_m', \dots, q_0']$ in $\hat{X}$. Then there exist $\hat{r}_i \in \hat{R}_i$ so that

$$q_m \hat{r}_m^{-1} = q_m' \quad \text{and} \quad \hat{r}_i q_{i-1} \hat{r}_{i-1}^{-1} = q_{i-1}', i = 1, \dots, m.$$

It follows that $\hat{r}_m \in \hat{R}_m \cap Q_{m,e} \subset \hat{R}_m \cap K_{m,e} = R_m$ (by part (2) of the lemma). By downward induction, assuming that $\hat{r}_i \in \hat{R}_i$, we have

$$\hat{r}_{i-1} = {q'}_{i-1}^{-1} \hat{r}_i q_{i-1} \in \hat{R}_{i-1} \cap K_{i-1,e} = R_{i-1}.$$

Therefore $[q_m, \dots, q_0] = [q_m', \dots, q_0']$ in X, so ι is injective. □

When $(G, K) = (Sp(n), GL(n))$, the component group $A_K(f)$ is trivial and C_f is the only component of the Springer fiber for f associated to $\mathcal{Q}$. For $(G, K) = (O(p, q), O(p) \times O(q))$ the component group is not in general trivial. As mentioned above, the components associated to $\mathcal{Q}$ are translates of C_f. Such a translate $zC_f, z \in Z_K(f)$ is an iterated bundle for the parabolic subgroups $zQ_{i,e}z^{-1}$.

Now consider the map $\pi : X \to Q_{m,e}/R_m \simeq L_{m,e}/L_{m,e} \cap Q_{m-1}$ defined by $\pi([q_m, \dots, q_0]) = q_m R_m$. The fiber is $X_{m-1} := Q_{m-1,e} \underset{R_{m-1}}{\times} \cdots \underset{R_1}{\times} Q_{0,e}/R_0$. It is often the case that $L_{m,e}/L_{m,e} \cap Q_{m-1}$ is just a point; this happens when $L_m \subset L_{m-1}$. So, we suppose that m' is the greatest integer so that $L_{m'} \not\subseteq L_{m'-1}$. Then $X \simeq Q_{m',e} \underset{R_{m'}}{\times} \cdots \underset{R_1}{\times} Q_{0,e}/R_0$ and there is a fibration $X \to L_{m',e}/L_{m',e} \cap Q_{m'-1,e}$ with fiber $X_{m'-1} = Q_{m'-1,e} \underset{R_{m'-1}}{\times} \cdots \underset{R_1}{\times} Q_{0,e}/R_0$. Note that $X_{m'-1}$ is isomorphic to a component of a Springer fiber for $f' = f_0 + \cdots + f_{m'-1}$ for the pair $(G'_{m'}, K'_{m'})$, in the notation of [2, §3.2].

Corollary 3.4. *Either $C_f = L_{0,e} \cdot \mathfrak{b}$ or there is an m' ($1 \le m' \le m$) and a fibration*

$$C_f \to \mathcal{F}_{m'},$$

where $\mathcal{F}_{m'}$ is the nontrivial generalized flag variety $L_{m',e}/L_{m',e} \cap Q_{m'-1,e}$ and the fiber is a component of a Springer fiber associated to a closed K'-orbit for a smaller pair (G', K').

4. The topology of the components

As an application of Theorem 3.2 the Poincaré polynomials of the components C_f can be computed. Recall that the Poincaré polynomial of a topological space Z is defined by

$$P_t(Z) = \sum_k \dim(H^k(Z, \mathbf{Q}))t^k.$$

As described in [5, §2.2] the Poincaré polynomial of C_f is

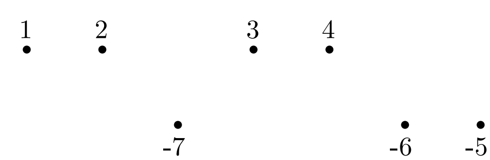
$$P_t(C_f) = \prod_{i=0}^{m} P_t(Q_{i,e}/R_i) = \prod_{i=1}^{m} P_t(L_{i,e}/L_{i,e} \cap R_i).$$

Since for each $i > 0$, $L_{i,e}/L_{i,e} \cap R_i$ is a generalized flag variety, and $L_{i,e} \cap R_i$ has Levi subgroup $L_{i,e} \cap L_{i-1,e}$, $P_t(L_{i,e}/L_{i,e} \cap R_i)$ is the quotient of the Poincaré polynomials of the flag varieties for $L_{i,e}$ and $L_{i,e} \cap L_{i-1,e}$. The structure of these groups is given in [2, Rem. 4.5]. Therefore we need only consider the general linear and special orthogonal groups. The flag varieties for these groups have Poincaré polynomials as follows:

$$\begin{aligned}
GL(k):& \quad \frac{(1-u^2)(1-u^3)\cdots(1-u^k)}{(1-u)^{k-1}},\\
SO(2k+1):& \quad \frac{(1-u^2)(1-u^4)\cdots(1-u^{2k})}{(1-u)^k},\\
SO(2k):& \quad \frac{(1-u^2)(1-u^4)\cdots(1-u^{2k-2})(1-u^k)}{(1-u)^k},
\end{aligned}$$

$u = t^2$. These equations follow from the fact that the Poincare polynomial of the flag variety of a semisimple complex group is $\sum_{w \in W} u^{\ell(w)}$ (where W is the Weyl group) together with [6] (in particular Theorem 3.15 and the list on page 59 of that book). Combining all these facts, we can easily compute the Poincare polynomial of C_f. We illustrate this with two examples.

Example 4.1. Type C. Consider the example of §3.1 of [2] for $Sp(7)$ with array given by



;

see [2, Eqn. 3.3]. Then

$$\begin{aligned}
&L_0 = GL(2) \times GL(1) \times GL(2) \times GL(2)\\
&L_1 = GL(2) \times GL(1)\\
&L_1 \cap L_0 = (GL(1) \times GL(1)) \times GL(1)\\
&L_2 = GL(1)\\
&L_2 \cap L_1 = GL(1).
\end{aligned}$$

Therefore,

$$P_t(C_f) = \frac{(1-u^2)}{(1-u)}\frac{(1-u^2)}{(1-u)}\frac{(1-u^2)}{(1-u)}\frac{(1-u^2)}{(1-u)} = (1+t^2)^4.$$

Example 4.2. Type D. Consider the pair $(G, K) = (O(20),\, O(12)\times O(8))$ with closed K-orbit determined by the array

$$\begin{array}{cccccccccc}
 & & & \overset{1}{\bullet} & \overset{2}{\bullet} & \overset{3}{\bullet} & & \overset{4}{\bullet} & \overset{5}{\bullet} & \overset{6}{\bullet} \\
 & & & & & & & & & \\
\underset{7}{\bullet} & \underset{8}{\bullet} & \underset{9}{\bullet} & & & & \underset{10}{\bullet} & & &
\end{array} .$$

Then

$$\begin{aligned}
&L_0 = GL(3) \times GL(3) \times GL(1) \times SO(6)\\
&L_1 = GL(2) \times SO(9)\\
&L_1 \cap L_0 = GL(2) \times GL(2) \times SO(5)\\
&L_2 = GL(1) \times SO(8)\\
&L_2 \cap L_1 = GL(1) \times SO(8)\\
&L_3 = SO(7)\\
&L_3 \cap L_2 = SO(7).
\end{aligned}$$

Therefore,

$$\begin{aligned}
P_t(C_f) &= \frac{(1-u^2)^2(1-u^3)^3(1-u^4)(1-u^6)(1-u^8)}{(1-u)^8}\\
&= (1+u)^4(1+u^2)^2(1+u^3)(1+u^4)(1+u+u^2)^4\\
&= (1+t^2)^4(1+t^4)^2(1+t^6)(1+t^8)(1+t^2+t^4)^4.
\end{aligned}$$

5. A vanishing theorem

Suppose χ_τ is a character of H with differential $\tau \in \mathfrak{h}^*$. Then χ_τ extends to a character of $B \cap K$, and therefore defines a holomorphic homogeneous line bundle $\mathcal{L}_\tau$ on the flag variety $\mathfrak{B}_K$. For any subvariety $Z \subset \mathfrak{B}_K$ we may pull back $\mathcal{L}_\tau$ to a line bundle on Z. Denote the sheaf of local sections of this line bundle by $\mathcal{O}_Z(\tau)$. When $Z = C_f$ the polynomial

$$p(\tau) := \sum_r (-1)^r \dim(H^r(Z, \mathcal{O}_Z(\tau)))$$

is of particular interest. The significance of this Euler characteristic is that it is a polynomial in whose leading homogeneous term is the multiplicity of $\overline{K \cdot f}$ in the associated cycle of the coherent family of discrete series representations associated to the closed K-orbit $\mathfrak{Q}$ in $\mathfrak{B}$. For a discussion of this see, for example, [3], [8] and [1, §6]. See [7] for a such an Euler characteristic arising in a slightly different context. See Section 6 for more on this alternating sum. The main result of this section is the following vanishing theorem.

Theorem 5.1. *If $\tau \in \mathfrak{h}^*$ is Δ_c^+-dominant, then $H^r(C_f, \mathcal{O}_{C_f}(\tau)) = 0$, for $r > 0$.*

Proof. Suppose that $\tau \in \mathfrak{h}^*$ is Δ_c^+-dominant. Let us use the notation

$$X_k := Q_{k,e} \underset{R_k}{\times} \cdots \underset{R_1}{\times} Q_{0,e}/R_0, \text{ for } k = 1, \dots, m,$$

and $X_0 := Q_{0,e}/R_0$. The map

$$\begin{aligned} \psi_k &: X_k \to \mathfrak{B}_K \\ [q_k, \dots, q_0] &\mapsto q_k \cdots q_0 \cdot \mathfrak{b} \end{aligned}$$

is an isomorphism onto its image (as X_k is a closed subvariety of X via the embedding $[q_k, ..., q_0] \mapsto [1, 1, .., 1, q_k, ..., q_0]$ and ψ_k is the restriction of the morphism F of Theorem 3.2). Denote by $\mathcal{O}_k(\tau)$ the sheaf of local sections of the pullback of $\mathcal{L}_\tau$ to X_k. Then the theorem is equivalent to the statement $H^r(X_m, \mathcal{O}_m(\tau)) = 0$, for $r > 0$. This will be proved using the Leray spectral sequence and induction.

The first observation is that for each $k = 1, \dots, m$ there is a fibration

$$\pi_k : X_k \to \mathfrak{F}_k := Q_{k,e}/R_k$$

with fibers isomorphic to X_{k-1}. The Leray spectral sequence has E_2 terms

$$H^p(\mathfrak{F}_k, R^q\pi_k(\mathcal{O}_k(\tau))). \tag{5.2}$$

Furthermore, $R^q\pi_k(\mathcal{O}_k(\tau))$ is the sheaf of sections of the homogeneous bundle associated to the R_k-representation $H^q(X_{k-1}, \mathcal{O}_{k-1}(\tau))$.

We use induction to prove the following claim for $k = 0, 1, \dots, m$.
Claim: $H^r(X_k, \mathcal{O}_k(\tau)) = 0, r > 0$, and $H^0(X_k, \mathcal{O}_k(\tau))$ is a direct sum of irreducible $Q_{k,e}$-representations.

Note that since $Q_{k,e}$ acts on X_k, $H^0(X_k, \mathcal{O}_k(\tau))$ is a $Q_{k,e}$-representation. Typically a finite dimensional representation of $Q_{k,e}$ does not decompose into a direct sum of irreducible subrepresentations. The statement of the claim is that $H^0(X_k, \mathcal{O}_k(\tau))$

does in fact decompose under $Q_{k,e}$ into a direct sum of irreducibles. It follows that U_k^- acts trivially on $H^0(X_k, \mathcal{O}_k(\tau))$.

If $k = 0$ then the Borel-Weil Theorem implies that the claim holds, because τ is dominant, and $H^0(X_0, \mathcal{O}_k(\tau))$ is already irreducible under $L_{0,e}$.

Now assume that $1 \leq k \leq m$ and the claim holds for $k-1$ in place of k. Decompose $H^0(X_{k-1}, \mathcal{O}_{k-1}(\tau))$ into irreducible $Q_{k-1,e}$-representations (as we may by the inductive hypothesis):

$$H^0(X_{k-1}, \mathcal{O}_{k-1}(\tau)) = \bigoplus_i E_{-\tau_i},$$

where $E_{-\tau_i}$ has lowest weight $-\tau_i$ as $L_{k-1,e}$-representation. We observe that $-\tau_i$ is antidominant for $\Delta^+_{c,k-1} := \Delta^+(\mathfrak{k}_{k-1})$, since the lowest weight vector is annihilated by $\mathfrak{u}_{k-1}^-$ (and $\mathfrak{q}_{k-1}$ is a parabolic subalgebra of $\mathfrak{k}_{k-1}$). Now decompose each $E_{-\tau_i}$ under $L_{k,e} \cap L_{k-1,e}$:

$$E_{-\tau_i}|_{L_{k,e} \cap L_{k-1,e}} = \bigoplus_j F_{-\tau_{ij}},$$

where $-\tau_{ij}$ is the lowest weight of $F_{-\tau_{ij}}$. Since U_{k-1}^- acts trivially, as noted above, this is in fact a decomposition as R_k-representations. It follows that

$$R^0\pi_k(\mathcal{O}_k(\tau)) = \bigoplus_{i,j} \mathcal{O}_k(F_{-\tau_{ij}}),$$

where $\mathcal{O}_k(F_{-\tau_{ij}})$ is the sheaf of sections of the homogeneous bundle on $\mathcal{F}_k$ associated to the R_k representation $F_{-\tau_{ij}}$. Our inductive hypothesis implies that $R^q\pi_k(\mathcal{O}_k(\tau)) = 0$, for $q > 0$. Therefore (5.2) is zero for $q > 0$ and when $q = 0$ is

$$\bigoplus_{i,j} H^p(\mathcal{F}_k, \mathcal{O}_k(F_{-\tau_{ij}})). \tag{5.3}$$

This will be easy to compute once we know that each τ_{ij} is dominant for $\Delta^+(\mathfrak{l}_k)$. We now prove this. As noted above each τ_i is dominant for $\Delta^+(\mathfrak{k}_{k-1})$. Let $W_{-\tau_i}$ be the irreducible $K_{k-1,e}$ representation having lowest weight $-\tau_i$ and let $w_{-\tau_i}$ be a lowest weight vector. It follows that

$$F_{-\tau_{ij}} \subset E_{-\tau_i} = \operatorname{span}_{\mathbf{C}}\{L_{k-1,e} \cdot w_{-\tau_i}\} \subset W_{-\tau_i}.$$

Since $L_{k-1,e}$ normalizes $\mathfrak{u}_{k-1}^-$ (and $w_{-\tau_i}$ is annihilated by $\mathfrak{u}_{k-1}^-$) we conclude that $F_{-\tau_{ij}}$ is annihilated by $\mathfrak{u}_{k-1}^-$. Letting $w_{-\tau_{ij}}$ be a lowest weight vector of $F_{-\tau_{ij}}$, the negative root vectors in $\mathfrak{l}_k \cap \mathfrak{l}_{k-1}$ (as well as the root vectors in $\mathfrak{u}_{k-1}^-$) annihilate $w_{-\tau_{ij}}$. By Lemma 2.2, (2) and (3), $w_{-\tau_{ij}}$ is annihilated by all negative root vectors for $\mathfrak{l}_k$, so τ_{ij} is dominant for $\Delta^+(\mathfrak{l}_k)$.

We may now conclude that (5.3) is zero when $p > 0$ and is a direct sum of irreducible $Q_{k,e}$-representations (by the Borel-Weil Theorem) when $p = 0$.

Therefore, the claim holds for k. This completes the proof of the theorem. $\square$

6. Torus action

First suppose that $(G, K) = (Sp(n), GL(n))$. Let H be the diagonal torus in G. Then H is a maximal torus for both G and K. We prove the following.

Theorem 6.1. *The component C_f is stable under the action of H on $\mathfrak{B}$.*

This follows immediately from (1.2) and the next lemma.

Lemma 6.2. *There is a decomposition $H = T_m \cdots T_1 T_0$, where each T_i is a torus in L_i that commutes with all L_j, with $j > i$.*

Proof. Set $T_m = H \cap L_m$. For $i = 0, 1, \ldots, m-1$ set

$$T_i = \{diag(z_1, \ldots, z_{2n}) \in Sp(n) : z_l = 1 \text{ unless } l \text{ is in the } i^{th} \text{ (doubled) string}\}.$$

□

Since $C_f \subset \mathfrak{B}$, this H-action has a finite number of fixed points. Therefore, the H-character of the Euler characteristic of the cohomology of a line bundle on C_f can be computed using localization. In particular, if $\tau \in \mathfrak{h}^*$ is an integral weight, then the formula of [5, Thm. 4.6] computes the H-character of

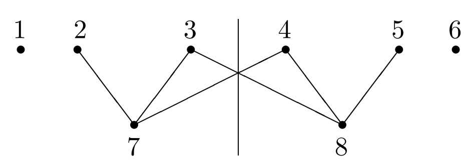
$$\sum (-1)^p \, H^p(C_f, \mathcal{O}_{C_f}(\tau))$$

as a sum over the fixed points.

The discussion surrounding Theorems 4.8 and 4.9 of [5] applies without change to the components C_f for the pair $(G, K) = (Sp(n), GL(n))$. Thus, formulas may be given for the classes of $[C_f]$ in equivariant cohomology and K-theory in terms of the classes of Schubert varieties. This answers a question of T. A. Springer ([9]) for the components C_f.

When $(G, K) = (O(p+q), O(p) \times O(q))$ the diagonal maximal torus does not act on all components C_f. It is also the case that the diagonal maximal torus of K need not act on a component C_f. A simple example is the following.

Suppose $(G, K) = (O(8), O(6) \times O(2))$ and $\lambda = (4, 3, 1, 2)$ defines the positive system Δ^+ that determines the closed K-orbit $\mathcal{Q}$. Then the doubled array is



and $f = X_{4-2} + X_{3-4} + X_{-(3+4)}$. Let $l_1 = \exp(-(X_{1-3} + X_{1+3}))$ and $h = diag(1, 1, z^{-1}, z, 1, 1, 1, 1)$. Then one easily computes

$$l_1^{-1} h^{-1} \cdot f = h \cdot f + (z - z^{-1}) X_{1-4}.$$

This lies in $\mathfrak{n}^- \cap \mathfrak{p}$ if and only if $z = \pm 1$. Therefore, $l_1 \cdot \mathfrak{b} \in \gamma_{_Q}^{-1}(f)$, but $hl_1 \cdot f \notin \gamma_{_Q}^{-1}(f)$ (for $z \neq \pm 1$). Therefore $\gamma_{_Q}^{-1}(f)$, so C_f, is not stable under the diagonal maximal torus of K.

We note that the argument for Type C fails because H cannot be decomposed as in Lemma 6.2. We have not excluded the possibility of a different maximal torus of K acting on C_f.

References

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