MATH 6590, Homework 4

1. (10 points) Let Ω be a polygon and $\mathcal{T}_h$ be a quasi-uniform triangular mesh in Ω satisfying the minimum angle condition. Denote $\boldsymbol{x}_i$, $i = 1, \ldots, k$ to be the vertices in $\mathcal{T}_h$. We assume that each vertex is shared by at most N triangles. Let V_h be the P_1 conforming finite element space defined on $\mathcal{T}_h$.

   Using the scaling argument to prove that there exist positive constants c, C such that

$$c\|v_h\|_{L^2(\Omega)}^2 \le h^2 \sum_{i=1}^{k} |v_h(\boldsymbol{x}_i)|^2 \le C\|v_h\|_{L^2(\Omega)}^2 \qquad \text{for all } v_h \in V_h.$$

2. (10 points) Let $\Omega = (0,1) \times (0,1)$. Consider the following problem

$$\begin{cases} -\Delta u = 8\pi^2 \sin(2\pi x)\sin(2\pi y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

   The exact solution to this problem is $u = \sin(2\pi x)\sin(2\pi y)$.

   Solve the problem using the P_1 conforming finite element approximation on the $n \times n$ uniform mesh as shown in the graph below, for $n = 8, 16, 32, 64$. Compute $\|I_h u - u_h\|_{L^2(\Omega)}$ and $\left(h^2 \sum_{i=1}^{k} |(I_h u - u_h)(\boldsymbol{x}_i)|^2\right)^{1/2}$. Estimate their convergence rates.