MATH 6590, Homework 4

1. (10 points) Let $\Omega$ be a polygon and $\mathcal{T}_h$ be a quasi-uniform triangular mesh in $\Omega$ satisfying the minimum angle condition. Denote $x_i, \ i = 1, \ldots, k$ to be the vertices in $\mathcal{T}_h$. We assume that each vertex is shared by at most $N$ triangles. Let $V_h$ be the $P_1$ conforming finite element space defined on $\mathcal{T}_h$.

Using the scaling argument to prove that there exist positive constants $c, C$ such that

$$c \|v_h\|_{L^2(\Omega)}^2 \leq h^2 \sum_{i=1}^k |v_h(x_i)|^2 \leq C \|v_h\|_{L^2(\Omega)}^2$$

for all $v_h \in V_h$.

2. (10 points) Let $\Omega = (0,1) \times (0,1)$. Consider the following problem

$$\begin{cases}
-\Delta u = 8\pi^2 \sin(2\pi x) \sin(2\pi y) & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega.
\end{cases}$$

The exact solution to this problem is $u = \sin(2\pi x) \sin(2\pi y)$.

Solve the problem using the $P_1$ conforming finite element approximation on the $n \times n$ uniform mesh as shown in the graph below, for $n = 8, 16, 32, 64$. Compute $\|I_h u - u_h\|_{L^2(\Omega)}$ and $\left( h^2 \sum_{i=1}^k |(I_h u - u_h)(x_i)|^2 \right)^{1/2}$. Estimate their convergence rates.

\[\text{Diagram of the mesh}\]