MATH 6590, Homework 2

1. (5 points) Let $\Omega = (a,b) \times (c,d)$ be a rectangular domain.

   (a) Prove that for all $v \in C_0^\infty(\Omega)$, there exists a constant $c > 0$ such that
   $$(\nabla v, \nabla v) \geq c\|v\|_0^2, \tag{1}$$
   by using $v(x,y) = \int_a^x \frac{\partial v}{\partial x}(\xi, y)\, d\xi$ or $v(x,y) = \int_c^y \frac{\partial v}{\partial y}(x, \eta)\, d\eta$, and the Schwarz inequality. Try to get the best estimation you can, that is, find the largest possible value of c.

   (b) Show that Inequality (1) is also true for all $v \in H_0^1(\Omega)$, by using the fact that $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$.

   (c) Use the above result to prove the coercivity
   $$(\nabla v, \nabla v) \geq \alpha \|v\|_1^2 \qquad \text{for all } v \in H_0^1(\Omega).$$

2. (5 points) Textbook, 3.x.9.

3. (5 points) Textbook, 3.x.13.

4. (5 points) Given an arbitrary triangle K with vertices $v_i\,(x_i,\, y_i)$, $i = 1, 2, 3$. Write a Matlab function that computes the local stiffness matrix $\{(\nabla\phi_i, \nabla\phi_j)\}_{0 \leq i,j \leq 3}$ where ϕ_i is the linear basis function such that $\phi_i(v_j) = \delta_{ij}$. Your function should take the coordinates of v_i, $i = 1, 2, 3$, as inputs, and return the 3×3 local stiffness matrix. Test your function on the triangle with vertices $v_1(1,2)$, $v_2(2,1)$, $v_3(3,3)$ and print out the results.