MATH 6590, Homework 2

1. (5 points) Textbook, p.67, problem 2.x.15.

2. (15 points) Let $(H, (\cdot,\cdot))$ be a Hilbert space with norm $\|v\|_H = \sqrt{(v,v)}$, and V be a closed, convex subset of H. Here V is convex means that $(1-t)v_1 + tv_2 \in V$ for all v_1, v_2 in V and $0 \le t \le 1$. Consider the variational inequality problem: *Given $F \in H'$, find $u \in V$ such that*

$$a(u, v-u) \ge F(v-u) \qquad \text{for all } v \in V, \tag{1}$$

where $a(\cdot,\cdot)$ is a symmetric, continuous bilinear form and is coercive on H. Let α be the coercivity constant of $a(\cdot,\cdot)$.

(a) If u_1, u_2 are solutions to Problem (1) with right-hand side F_1 and F_2, respectively, prove

$$\|u_1 - u_2\|_H \le \frac{1}{\alpha}\|F_1 - F_2\|_{H'}$$

(Consequently, it gives the uniqueness of the solution to Problem (1).)

(b) Define $Q(v) = a(v,v) - 2F(v)$. Prove that Problem (1) is equivalent to the problem

$$\textit{find } u \in V \textit{ such that } Q(u) = \inf_{v\in V} Q(v). \tag{2}$$

(c) Show that

$$\inf_{v\in V} Q(v) \ge -\frac{1}{\alpha}\|F\|_{H'}^2 > -\infty.$$

(d) Given a sequence $\{v_n\} \subset V$ such that $\lim_{n\to\infty} Q(v_n) = \inf_{v\in V} Q(v)$. Prove that $\{v_n\}$ is a Cauchy sequence in V. Hence $\lim_{n\to\infty} v_n$ exists and is a solution to Problem (2). (Hint: show $a(v_m - v_n, v_m - v_n) = 2a(v_m, v_m) + 2a(v_n, v_n) - 4a(\frac{v_m+v_n}{2}, \frac{v_m+v_n}{2}) = 2Q(v_m) + 2Q(v_n) - 4Q(\frac{v_m+v_n}{2})$)

((b) and (d) together give the existence of the solution to Problem (1).)

(e) Finally, if V is a closed subspace of H, show that the variational inequality problem (2) is equivalent to: *find $u \in V$ such that*

$$a(u,v) = F(v) \qquad \text{for all } v \in V.$$