MATH 6590, Homework 1

1. (6 pts) Let $\Omega = (0,1)$ and $v \in H^2(\Omega)$. Define a grid on Ω by $0 = x_0 < x_1 < \cdots < x_n = 1$ and denote $h = \max_{i=1}^{n}(x_i - x_{i-1})$. Let v_I be the nodal value interpolation of v using continuous piecewise linear polynomials. Prove that there exists a positive constant c independent of v such that
$$\|v - v_I\|_{L^2(\Omega)} \le ch^2 \|v''\|_{L^2(\Omega)}.$$

2. (6 pts) Given $\Omega = (0,1)$ and a uniform grid $x_k = kh$, where $k = 0, \ldots, n$ and $h = 1/n$ is the step size. Compute the local stiffness matrix and the local mass matrix
$$A_k = \left[\int_{x_{k-1}}^{x_k} \phi_i' \phi_j' \, dx\right]_{i,j}$$
$$M_k = \left[\int_{x_{k-1}}^{x_k} \phi_i \phi_j \, dx\right]_{i,j}$$
where $\{\phi_i\}$ are quadratic basis functions.

3. (8 pts) (Programming) Consider the problem
$$\begin{cases} -\varepsilon u'' + u' = 1 & \text{in } \Omega = (0,1), \\ u(0) = 0, \quad u(1) = 0, \end{cases}$$
where ε is a positive number. The exact solution to this problem is
$$u(x) = x - \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}.$$
Using a uniform grid $x_i = i/h$, where h is the step size, and piecewise linear polynomial finite element spaces to approximate this problem. Plot the graph of the exact solution together with the finite element solution, for the following cases:

   (a) Solve the problem for $\varepsilon = 1.0,\ 0.1,\ 0.05,\ 0.01$, with $h = 1/20$. Compare the solution.

   (b) For $\varepsilon = 0.01$, solve the problem with $h = 1/20, 1/40,\ 1/60$. Compare the solution.

   (c) For $\varepsilon = 0.01$, solve the problem using a non-uniform grid defined as following: $x_i = i/20$, for $i = 0, \ldots, 19$, $x_{20} = 1 - 1/40$, $x_{21} = 1$ (in other words, insert one more grid point in the middle of the last subinterval of a uniform grid with step size 1/20). Plot the solution and compare it with the result from (b).

   (d) Repeat (b) and (c) for $\varepsilon = 0.005$. Let u be the exact solution, u_1 be the finite element solution on the uniform grid with $h = 1/20$, and u_2 be the finite element solution on the grid defined in (c). Compare the sign of $u(x_i) - u_1(x_i)$ and $u(x_i) - u_2(x_i)$, are they the same on each grid node x_i, for $i = 0, \ldots, 19$?