Math 5553, Homework 5, Due on 4/24/2014

1. (8 points) Consider the $m \times m$ tridiagonal matrix

$$T = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & b_{m-1} \\ 0 & 0 & 0 & 0 & \cdots & a_m \end{bmatrix}$$

where a_i, b_i, c_i are real numbers, and $b_i c_i > 0$ for all i. This matrix is not necessarily symmetric. Show that there exists a diagonal matrix D such that $D^{-1}TD$ is a symmetric tridiagonal matrix. In other words, T is similar to a symmetric tridiagonal matrix, and consequently all eigenvalues of T must be real.

2. (12 points) Consider a 20×20 symmetric positive definite, tridiagonal matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 2 & -1 & \dots & 0 \\ & & & \ddots & \ddots & \ddots & \\ 0 & \dots & & & -1 & 2 & -1 \\ 0 & \dots & & & 0 & -1 & 2 \end{bmatrix}$$

It is known that matrix A has eigenvalues $\lambda_i = 2 - 2\cos\frac{i\pi}{21}$ and corresponding eigenvectors

$$\mathbf{x}_i = \begin{bmatrix} \sin\frac{i\pi}{21} \\ \sin\frac{2i\pi}{21} \\ \sin\frac{3i\pi}{21} \\ \vdots \\ \sin\frac{20i\pi}{21} \end{bmatrix}$$

for $i = 1, 2, \dots, 20$.

(a) Use Gerschgorim's theorem to get a lower bound and an upper bound of the spectrum of A.

(b) Use the power method to compute the largest eigenvalue (you need to pick a suitable initial guess). Report the number of iterations needed to reach $\|A\mathbf{v}^{(k)} - \lambda^{(k)}\mathbf{v}^{(k)}\|_2 \leq 10^{-6}$. In each iteration step, compute $|\lambda^{(k)} - \lambda_{max}|$ and $\|\mathbf{v}^{(k)} - \mathbf{x}_{max}\|_2$ and plot them versus k. Which quantity converges faster?

(c) Use the inverse method to compute the smallest eigenvalue, with the same setting as in part (b) and initial μ be the lower bound of the spectrum from part (a). Repeat what you have reported in part (b).

(Remark: Is there any difference between the number of iterations needed for computing λ_{max} and λ_{min}? What causes this difference?)