Math 5553, Homework 3, Due on 3/25/2014

1. (4 points) Let a, b, c, d, e, f be real numbers and $a > 0$, $d > 0$, $f > 0$. Define a symmetric matrix A as follows:

$$A = \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 + d^2 & bc + de \\ ac & bc + de & c^2 + e^2 + f^2 \end{bmatrix}$$

   Apply Algorithm 23.1 to this matrix, write down each step, and show that it computes the Cholesky factorization of this matrix.

2. (4 points) A well-known stationary iterative method is the Richardson iteration:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \omega(\mathbf{b} - A\mathbf{x}_{n-1}),$$

   where ω is a suitable positive number that ensures the iteration converges.

   (a) Find M and K so that the Richardson iteration can be expressed using the matrix splitting $A = M - K$.

   (b) Denote the residual $\mathbf{r}_n = \mathbf{b} - A\mathbf{x}_n$. Then the Richardson iteration can be written into

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \omega\mathbf{r}_{n-1}.$$

   Similarly, rewrite the Jacobi iteration and Gauss-Seidel iteration in terms of $\mathbf{x}_{n-1}$ and $\mathbf{r}_{n-1}$. That is, your final iteration formula should not contain $\mathbf{b}$.

3. (4 points) Given a linear system $A\mathbf{x} = \mathbf{b}$, the idea of preconditioning is to consider a new system $BA\mathbf{x} = B\mathbf{b}$ where B is an invertible matrix. Show that a stationary iterative method using matrix splitting $A = M - K$ is equivalent to solving a preconditioned system $M^{-1}A\mathbf{x} = M^{-1}\mathbf{b}$ using the Richardson iteration with $\omega = 1$.

4. (8 points) Consider the system $A\mathbf{x} = \mathbf{b}$ where the $m \times m$ symmetric tridiagonal matrix A and vector $\mathbf{b}$ are given by

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & 0 & \cdots & 0 & 0 \\ & & & \ddots & \ddots & & \\ & & & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

   It is not hard to see that the exact solution is $\mathbf{x} = [1\ 1\ \ldots\ 1]^t$. Set the initial guess $\mathbf{x}_0 = \mathbf{0}$.

   (a) Solve the system using the Jacobi and the Gauss-Seidel method with stopping criteria $\|\mathbf{r}_i\| < 10^{-6}\|\mathbf{r}_0\|$. Plot the number of iterations vs. m graph, for $m = 10, 20, 30, \ldots, 80$.

   (b) Let $m = 20$. Solve the system using the Jacobi and the Gauss-Seidel method with stopping criteria $\|\mathbf{r}_i\| < 10^{-6}\|\mathbf{r}_0\|$. Plot the 2-norm of the residual vs. iteration steps graph. For this particular problem, the error is computable because we do know the exact solution. On the same graph, also plot the 2-norm of error vs. iteration steps.